FROM BINARY GROUPS TO TERMINAL RINGS

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Communicated by Erhard Aichinger

Original Research Paper

Received: Oct 3, 2019 · Accepted: Jul 5, 2021

First published online: October 8, 2021

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ABSTRACT

Binary groups are a meaningful step up from non-associative rings and nearrings. It makes sense to study them in terms of their nearrings of zero-fixing polynomial maps. As this involves algebras of a more specialized nature these are looked into in sections three and four. One of the main theorems of this paper occurs in section five where it is shown that a binary group V is a $P_0(V)$ ring module if, and only if, it is a rather restricted form of non-associative ring. Properties of these non-associative rings (called terminal rings) are investigated in sections six and seven. The finite case is of special interest since here terminal rings of odd order really are quite restricted. Sections eight to thirteen are taken up with the study of terminal rings of order p^n (p an odd prime and $n \geq 1$ an integer ≤ 7).

KEYWORDS

nearring, binary group, anti-associative ring, anti-commutative ring, inverse ring, terminal ring

MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 08A05; Secondary 17D99, 16Y30

1. INTRODUCING BINARY GROUPS

One of the more fundamental notions of abstract algebra is that of a group. Groups tend to arise in nearly all branches of this subject. In this paper we are involved in the study of groups with an additional operation. In these investigations certain types of non-associative rings will arise. One in particular will be studied. It will be these non-associative rings that allow us to determine when the group structure is a ring module relative to the nearring of zero-fixing polynomial maps.

Traditionally in the study of groups the binary operation has been multiplication (ie. that denoted by juxtaposition). As is the case with nearrings this paper uses additive notation. Here the notation used is not meant to imply the groups involved have to be commutative. In accordance with our additive convention the identity (ie. additive identity) of a group V is denoted by 0. Also if S is a subset of V , $S \setminus \{0\}$ is written as S^* and $\langle S \rangle$ is taken as the subgroup of V generated by S .

The reason additive convention for groups is useful is that it streamlines binary group (now defined) studies. A binary group is a group with a further binary operation for which multiplicative notation is used. Thus, if V is such a group and γ its further operation, then $(u, v)\gamma$ ($u \in V$ and $v \in V$) is denoted by uv .

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With a slight exception in the case of certain nearrings the binary groups we study will be assumed to be zero-symmetric. Thus in general a binary group V will be considered to have the property that $0u = u0 = 0$ for all u in V . Sometimes V may have an identity (a particular element denoted by 1 and such that $u1 = 1u = u$ for all u in V). When this is the case the 1 of V is unique. We indicate V has such an element by calling it a binary group with 1. Following on fairly naturally from the definition of a binary group comes the notion of a binary subgroup. U is such a subobject of the binary group V when it is an additive subgroup of V with the property that $u_1 u_2$ is in U for all $u_i, i = 1, 2$, in U .

Looking into matters to do with subsets of binary groups is perhaps the next step in their investigation. Here, for two non-empty subsets S_1 and S_2 of such a group V , $S_1 + S_2$ and $S_1 S_2$ are respectively defined to be $\{s_1 + s_2 : s_1 \in S_1, s_2 \in S_2\}$ and $\{s_1 s_2 : s_1 \in S_1, s_2 \in S_2\}$. As the addition and multiplication of V cannot be expected to be commutative, it is not necessarily the case that $S_1 + S_2 = S_2 + S_1$ or that $S_1 S_2 = S_2 S_1$. Furthermore, for S_3 another non-empty subset of V , it is not necessarily the case that $(S_1 S_2) S_3 = S_1 (S_2 S_3)$ although addition of non-empty subsets is certainly associative. It appears that particular subsets of V are of greater interest. These are the right (left) V -subsets. Here a non-empty subset S of V is a right (left) V -subset when SV (when VS) is contained in S . When S is also an additive subgroup of V it is called a right (left) V -subgroup. In the case of it being both a right and left V -subset (V -subgroup) it is simply called a V -subset (V -subgroup).

Two things about binary groups deserving of attention are a natural relationship between them and what this relationship throws up (ie. ideals). These matters are approached here from an Ω -group (that is an expanded group) perspective. Ω -groups have been about for a long time (see [2]) and it is assumed that in that direction the reader is familiar with the basics. Ω -group definitions present us with homomorphisms between binary groups. Also isomorphism theorems are available. The ideals of the binary group V are just the kernels of homomorphisms on V . Alternatively an ideal of V is a normal subgroup U of V (ie. of $(V, +)$) for which $(v_1 + u_1)(v_2 + u_2) - v_1 v_2$ is in U for all $v_i, i = 1, 2$, in V and $u_i, i = 1, 2$, in U . Such an ideal A of V supplies us with the binary group V/A for which $(w_1 + A) + (w_2 + A) = w_1 + w_2 + A$ and $(w_1 + A)(w_2 + A) = w_1 w_2 + A$ ($w_i, i = 1, 2$, in V). In the above definition of ideals using elements of V , we do not need to assume the subgroup U is normal when V is binary with 1. This is because with $v_2 = 1$ and $u_2 = 0$ we see $v_1 + u_1 - v_1$ is in U . Also, because zero-symmetry is being assumed, we have:

PROPOSITION 1.1. If V is a binary group, then an ideal U of V is a V -subgroup of U .

Proof. In the above definition of ideals using elements of V , we see on taking $u_1 = 0$ and $v_2 = 0$, that $v_1 u_2$ is in U . On taking $v_1 = 0$ and $u_2 = 0$ it follows that $u_1 v_2$ is in U . Proposition 1.1 is proved. $\square$

From Ω -group considerations it follows that any sum of ideals of a binary group is an ideal. A direct sum of ideals of such a group is defined in the standard manner. We also have that sums of two subobjects where only one is an ideal tend to fall into line. Here, if U is taken respectively to be a binary subgroup, right (left) V -subgroup or just a V -subgroup of the binary group V , then the same holds for $U + A$ (A an ideal of V).

If V is a binary group and k an integer ≥ 1 and $v_i, i = 1, \dots, k$, elements of V then, when associativity is present, the product $v_1 v_2 \dots v_k$ has a precise meaning. This may not be the case for general V and overcoming the problem involved here means defining product sets. This is done inductively. For a single element a of V the product set $[a]$ of V is taken as $\{a\}$. If for some integer $n \geq 1$ all product sets $[a_1, \dots, a_m]$ ($1 \leq m \leq n$ and $a_i, i = 1, \dots, n$, in V) are defined, then the product set $[v_1, v_2, \dots, v_{n+1}]$ ($v_i, i = 1, \dots, n+1$, in V) is taken as $\cup_{m=1}^n ([v_1, \dots, v_m] \cdot [v_{m+1}, \dots, v_{n+1}])$. This definition will be of help latter. Developments of section three and four will make quite real use of it.

In this paper a number of different algebraic structures arise. The more general are binary groups and the more specific are terminal rings (see section five). Making the transition from one extreme to the other is, in part, what this paper is about. It is investigations into polynomial maps on binary groups that find us within a terminal ring environment. The question of when polynomial maps distribute over an ideal is central to this.

Nearrings of polynomial maps are a powerful tool in binary group (in Ω -group) investigations. As has been indicated above they lead to rather meaningful research possibilities. The next section will introduce the nearrings $P(V)$ and $P_0(V)$ (V a binary group) of polynomial maps.

2. NEARRINGS $P(V)$ AND $P_0(V)$ (V A BINARY GROUP)

If maps of a group into itself are being considered, then there often arises another structure. In such situations nearrings often come into play. First attempts to investigate them go back a long way (see [3]) and this paper assumes the reader has some knowledge of both nearrings and nearring groups. A basic text on them is [4]. Here, unlike [4], we will be using the left distributive law (ie. $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$) rather than the right. This of course means functions are written on the right.

An N -group V (N zero-symmetric) is called tame (see [8]) if V is a unitary N -group with the property that all N -subgroups are submodules (ie. N -ideals). This is characterized by requiring that for each v and w in V and α in N , we can find a β in N such that $(v + w)\alpha - v\alpha = w\beta$. In the situation where β can be found independently of w , V is called a compatible N -group. Although compatible N -groups are considerably more restricted than tame ones, they are still very plentiful (see §5 of [10]). When a nearring N has a faithful tame (compatible) N -group (V say) it is said to be tame (compatible) or more precisely tame (compatible) on V .

As is common in nearring theory the symbols $M(V)$ ($M_0(V)$) are used to denote the nearring of all (all zero fixing) functions of a group V into itself. $M_0(V)$ is a zero symmetric nearring while $M(V)$ is only zero symmetric when $V = \{0\}$. $M(V)$ plays a somewhat further role when V is a binary group. Here it is not only a nearring under pointwise addition and composition but a binary group under a rather natural multiplication. This multiplication is that given by taking the product $\alpha.\beta$ (α and β in $M(V)$) to be the function of $M(V)$ taking v in V to $(v\alpha)(v\beta)$. It is the binary subgroup $P(V)$ of $M(V)$ generated by $\{1\} \cup C$ (C the subnearring of constant maps of $M(V)$) that will be of interest and in order to study it we inductively define certain subsets $\Gamma_n(V)$, $n = 1, 2, \dots$. Here $\Gamma_1(V) = \{1\} \cup C$ and for $n \geq 2$, $\Gamma_n(V)$ is taken as

$$\Gamma_{n-1}(V) \cup \{\alpha.\beta, \alpha - \beta : \alpha, \beta \in \Gamma_{n-1}(V)\}.$$

The notation $\Gamma_n(V)$, $n = 1, 2, \dots$, has been introduced because it occurs again in section four. Clearly $\Gamma_1(V) \subseteq \Gamma_2(V) \subseteq \dots$ and $P(V) = \cup_{n=1}^{\infty} \Gamma_n(V)$. Certainly $P(V)$ is a binary subgroup of $M(V)$ but something more is true.

In the more general context of V being an Ω -group the Ω -subgroup $P(V)$ of $M(V)$ is readily defined (see Section 7.118 onwards of [4]). Moreover, it is a subnearring of $M(V)$ called the nearring of polynomial maps of V into V . This nearring is of considerable use in the study of V . Perhaps even more useful is the zero-symmetric part $P_0(V)$ of $P(V)$. Either of these N ($P(V)$ or $P_0(V)$) have V as a faithful N -group. In the case of the $P_0(V)$ -group V an important fact emerges. This is that $P_0(V)$ is compatible on V . Thus, in accordance with what was set out above, $P_0(V)$ -subgroups of V are submodules. However, more is true.

It is of considerable help in investigating an Ω -group V to view it as a $P_0(V)$ -group. Taken on the surface the fact that V is a compatible $P_0(V)$ -group does not seem to say a great deal. A reason for this is that first impressions are that the $P_0(V)$ -subgroups of V may not be related in any way much to Ω -group structure. The fact that they are actually the ideals (see Section 7.123 of [4]) means much more becomes possible. On restricting to binary groups Section 7.123 of [4] becomes:

THEOREM 2.1. Suppose V is a binary group. U is a $P_0(V)$ -subgroup of V if, and only if, it is an ideal.

It would seem that due to Theorem 2.1 any serious attempt to understand binary groups must mean viewing them as $P_0(V)$ -groups. An important aspect of this paper is that of investigating the U (as in Theorem 2.1) which are $P_0(V)$ ring modules.

3. ANTI-ASSOCIATIVE RINGS

In this paper the term ring will in general mean non-associative ring. On the whole only when $P_0(V)$ -ring modules are being considered or the word associative is used is the associative law operating.

Binary groups can satisfy laws other than zero-symmetry. These more restricted algebras can be of very real interest. One such class is that of non-associative rings. These are just additively abelian binary groups satisfying both distributive laws. The past has seen considerable attention paid to particular subclasses. Here interest has focused on such things as Lie rings, Jordan rings and alternative rings (see [5]). In the study of them the additional laws give them their unique character. This is certainly the case for associative rings resting so much as they do on associativity. Associativity can seem fundamental but when studying binary groups another law sometimes tends to take its place. A subclass of the class of non-associative rings is that of anti-associative rings. Here, a non-associative ring V is an anti-associative ring if $u(vw) = -(uv)w$ for all u, v and w in V . The purpose of the present section is to prove a result about the subset $[v_1, v_2, \dots, v_k]$ ($k \geq 3$ an integer and $v_i, i = 1, \dots, k$, in V) of the anti-associative ring V . To do this we introduce certain specialized products of a binary group.

Suppose V is a binary group and $u_1, u_2, u_3, \dots$ a sequence of elements of V . We let $\xi_l(u_1)$ (l stands for left) be u_1 and with $n \geq 1$ an integer and $\xi_l(u_1, \dots, u_n)$ defined, $\xi_l(u_1, \dots, u_{n+1})$ is taken as $\xi_l(u_1, \dots, u_n)u_{n+1}$. Thus the sequence, $\xi_l(u_1), \xi_l(u_1, u_2), \xi_l(u_1, u_2, u_3), \xi_l(u_1, u_2, u_3, u_4), \dots$ is just the sequence, $u_1, u_1 u_2, (u_1 u_2)u_3, ((u_1 u_2)u_3)u_4, \dots$ of elements of V . A right handed version of ξ_l will also be of use. Here $\xi_r(u_1) = u_1$ and for $n \geq 2$ an integer $\xi_r(u_1, \dots, u_n)$ is defined inductively as $u_n \xi_r(u_1, \dots, u_{n-1})$. Thus the sequence $\xi_r(u_1), \xi_r(u_1, u_2), \xi_r(u_1, u_2, u_3), \xi_r(u_1, u_2, u_3, u_4), \dots$ is just the sequence, $u_1, u_2 u_1, u_3(u_2 u_1), u_4(u_3(u_2 u_1)), \dots$ of elements of V .

If V is an anti-associative ring, $k \geq 1$ an integer and $v_1, \dots, v_k$, elements of V , then much can be said about the subset $[v_1, \dots, v_k]$ of V . Clearly for $k = 1$ or 2 , $[v_1, \dots, v_k]$ is respectively v_1 and $v_1 v_2$ but for $k \geq 3$ things are also simple enough. The theorem this section proves is:

THEOREM 3.1. Suppose V is an anti-associative ring and $k \geq 3$ an integer. If $v_1, \dots, v_k$, are elements of V , then $[v_1, \dots, v_k]$ has the form $\{d, -d\}$ where d is an element of V .

Proof. It will first be shown that $[v_1, \dots, v_k] \subseteq \{d, -d\}$ where d is an element of V . To do this suppose $n \geq 1$ is an integer minimal such that there exists $u_1, \dots, u_n$ in V with $[u_1, \dots, u_n]$ not contained in such a subset of V . If this assumption leads to a contradiction the first part of the proof will be complete.

Because for $n = 1$ or 2 , $[u_1, \dots, u_n]$ is a single element of V , $n \geq 3$. Now c in $[u_1, \dots, u_n]$ is of the form $c_1 c_2$ with c_1 in $[u_1, \dots, u_a]$ and c_2 in $[u_{a+1}, \dots, u_n]$ (a an integer ≥ 1 and $< n$). The minimality of n forces $[u_1, \dots, u_a] \subseteq \{d_1, -d_1\}$ and $[u_{a+1}, \dots, u_n] \subseteq \{d_2, -d_2\}$ where d_1 and d_2 are in V . Thus c is in $\{\pm \xi_l(u_1, \dots, u_a) \cdot \xi_r(u_n, u_{n-1}, \dots, u_{a+1})\}$. However, for $a + 1 = n$

$$\xi_l(u_1, \dots, u_a) \cdot \xi_r(u_n, \dots, u_{a+1}) = \xi_l(u_1, \dots, u_a) \cdot u_n = \xi_l(u_1, \dots, u_n)$$

and for $a + 1 < n$

$$\xi_l(u_1, \dots, u_a) \cdot \xi_r(u_n, \dots, u_{a+1}) = \xi_l(u_1, \dots, u_a)(u_{a+1} \cdot \xi_r(u_n, \dots, u_{a+2})),$$

where the right hand side is $-\xi_l(u_1, \dots, u_{a+1}) \cdot \xi_r(u_n, \dots, u_{a+2})$. Continuing in this manner we see $\{\xi_l(u_1, \dots, u_a) \cdot \xi_r(u_n, \dots, u_{a+1})\}$ is necessarily contained in $\{\pm \xi_l(u_1, \dots, u_n)\}$. Thus c is contained in $\{\pm \xi_l(u_1, \dots, u_n)\}$ and, by contradiction, the first part of the proof is complete.

To complete the proof we must show that if $k \geq 3$ and g is in $[v_1, \dots, v_k]$, then $-g$ is in $[v_1, \dots, v_k]$. Here g is of the form $g_1 g_2$ with g_1 in $[v_1, \dots, v_b]$ and g_2 in $[v_{b+1}, \dots, v_k]$ (b an integer ≥ 1 and $< k$). Two cases arise. These are where $b = 1$ or $b \geq 2$. In the first we have $k - b \geq 2$ and this situation is in many ways like the second. Therefore we deal with the second case only. Here g_1 is of the form $g_3 g_4$ with g_3 in $[v_1, \dots, v_e]$ and g_4 in $[v_{e+1}, \dots, v_b]$ ($e \geq 1$ an integer $< b$). Thus $g = (g_3 g_4) g_2$ and $g_3(g_4 g_2)$ is also in $[v_1, \dots, v_k]$. Since $g_3(g_4 g_2) = -g$ the theorem is proved. $\square$

4. POWER ASSOCIATIVITY AND INVERSE RINGS

The last section dealt with anti-associative rings. It was proved that in such a non-associative ring V a product set $[v_1, \dots, v_k]$ ($k \geq 1$ an integer and $v_i, i = 1, \dots, k$, in V) has at most two elements. In the proof of this the anti-associative law was used and also the distributive laws. When it comes to associativity rather than anti-associativity a well known much more general result is possible. Here, in a binary group with a multiplicative associative law (called an associative binary group), product sets consist of a single element. Also such a condition on product sets implies the binary group is associative. However, moving away from all product sets being singletons, to requiring this for some of a select nature means much more general binary groups can occur. A binary group V will be called power associative if any product set $[v_1, \dots, v_k]$ in which all the v_i are equal (equal to v say) is a singleton. In this case the natural notation v^k will be used to denote this element of V .

Power associative binary groups are defined. Clearly an anti-associative ring V will be power associative (see Theorem 3.1) if all product sets $[v_1, v_2, \dots, v_k]$ ($k \geq 2$) of V with the v_i equal have an element of additive order ≤ 2 . There is a reasonably natural (quite strong) condition on V that ensures this. A non-associative ring V which is anti-commutative (ie. satisfies $uv = -vu$ for all u and v in V) will be called an anti-commutative ring. If an anti-associative ring is also anti-commutative it will be called an inverse ring.

PROPOSITION 4.1. Suppose V is an anti-commutative ring and $[v_1, \dots, v_k]$ a product set of V where all the v_i are equal. If $k \geq 2$, then all elements of $[v_1, \dots, v_k]$ have additive order ≤ 2 .

Proof. There exists v in V such that $v_i = v$ for $i = 1, \dots, k$. When $k = 2$, $[v_1, \dots, v_k]$ consists of the single element vv and since, by anti-commutativity $vv = -vv$, Proposition 4.1 holds. Suppose it has been shown that for all integers $a \geq 2$ and $< k$, $[v_1, \dots, v_a]$ consists of elements of additive order ≤ 2 . If $k \geq 3$, then an element c of $[v_1, \dots, v_k]$ is of the form $c_1 c_2$ where c_1 is in $[v_1, \dots, v_b]$ and c_2 in $[v_{b+1}, \dots, v_k]$ (b an integer ≥ 1 and $< k$) so that either c_1 or c_2 is in $[v_1, \dots, v_e]$ (e an integer ≥ 2 and $< k$). Thus either c_1 or c_2 has additive order ≤ 2 . Since distributive laws hold $|c| \leq 2$ and Proposition 4.1 has been inductively proved. $\square$

COROLLARY 4.2. An inverse ring is power associative.

Proof. By Proposition 4.1, this follows from Theorem 3.1. $\square$

For a binary group V elements of $P(V)$ can, in a sense, be thought of as defined inductively in terms of the $\Gamma_n(V)$, $n = 1, 2, \dots$. Here we have $P(V) = \cup_{n=1}^{\infty} \Gamma_n(V)$ where $\Gamma_1(V)$ is specified and $\Gamma_n(V)$ ($n \geq 2$) is given in terms of $\Gamma_{n-1}(V)$. While very useful, this definition of $P(V)$ does not help much with visualizing $P(V)$'s elements. Indeed, in general, it is hard to know what elements of $P(V)$ look like. However, when V is an inverse ring things are different. Here, elements of $P(V)$ can be specified in terms of those of type (6.1), (8.1) or (10.1) (to be defined). Type (6.1), (8.1) and (10.1) maps also exist in power associative binary groups. Although they tell us little in general about elements of $P(V)$ (V a power associative binary group) it is in this wider context they will be defined.

Let V be a power associative binary group. Type (6.1) elements of $P(V)$ are taken as the constant maps (ie. those mapping all of V to a single element of V). Type (8.1) elements of $P(V)$ are taken as those maps mapping v in V to v^n (to $-v^n$) where $n \geq 1$ is an integer. Type (10.1) elements of $P(V)$ are taken as maps taking v in V to $u.v^n$ ($u \in V$ and $n \geq 1$ an integer). Type (6.1), (8.1) and (10.1) elements of $P(V)$ are called elementary and $E(V)$ is defined as $\{\alpha \in P(V) : \alpha \text{ of type (6.1), (8.1) or (10.1)}\}$ while the zero symmetric part $E(V) \setminus \{\alpha \in P(V) : \alpha \text{ of type (6.1)}\}$ of $E(V)$ will be denoted by $E_0(V)$.

Let V be a binary group. $P(V)$ is not only a nearring under substitution and pointwise addition but there is another naturally defined multiplication on $P(V)$ extending that of V and making $P(V)$ into a binary group. In the case of V being an inverse ring this multiplication makes $P(V)$ into an inverse ring. In this situation the product $E(V).E(V)$ of the subset $E(V)$ with itself is $\subseteq E(V)$. We also have $-E(V) = \{-\alpha : \alpha \in E(V)\}$ is $E(V)$. This second statement is easily proved. All that really needs to be observed is that $-(u.v^n)$ ($u \in V$ and $n \geq 1$ an integer) is $(-u).v^n$. The first statement's

proof is omitted. It is easy enough to check out. For example part of the proof consists of showing $(u_1 v^{n_1}) \cdot (u_2 v^{n_2}) (u_i, i = 1, 2, \text{ in } V \text{ and } n_i \geq 1 \text{ integers})$ is equal to

$$-(u_1(v^{n_1}(u_2 v^{n_2}))) = u_1((v^{n_1} u_2) v^{n_2}) = -(u_1((u_2 v^{n_1}) v^{n_2})) = u_1(u_2 v^{n_1+n_2})$$

and this is just $-((u_1 u_2) \cdot v^{n_1+n_2}) = (-u_1 u_2) v^{n_1+n_2}$.

A proposition formalizes what has been indicated.

PROPOSITION 4.3. If V is an inverse ring, then $-E(V) = E(V)$ and $E(V) \cdot E(V)$ is contained in $E(V)$.

The time has come to see just what elements of $P(V)$ (V an inverse ring) look like.

THEOREM 4.4. An element α of $P(V)$ (V an inverse ring) is of the form $\sum_{i=1}^n \alpha_i$ where the α_i are in $E(V)$.

Proof. Suppose there exist elements of $P(V)$ not of the desired form. Furthermore, suppose the integer n is chosen minimal such that $\Gamma_n(V)$ contains such an element (β say). Since all elements of $\Gamma_1(V)$ are in $E(V)$ it follows that $n \geq 2$. The nature of β and n imply $\beta = \beta_1 - \beta_2$ or $\beta = \beta_1 \beta_2$ where $\beta_1 = \gamma_1 + \dots + \gamma_{n_1}$ ($n_1 \geq 1$ an integer and $\gamma_i, i = 1, \dots, n_1$, in $E(V)$) and $\beta_2 = \lambda_1 + \dots, \lambda_{n_2}$ ($n_2 \geq 1$ an integer and $\lambda_i, i = 1, \dots, n_2$, in $E(V)$). However, as

$$\beta_1 - \beta_2 = \gamma_1 + \dots + \gamma_{n_1} - \lambda_{n_2} - \dots - \lambda_1 \quad \text{and} \quad \beta_1 \beta_2 = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \gamma_i \lambda_j,$$

Proposition 4.3 supplies us with the contradiction that $\beta_1 - \beta_2$ and $\beta_1 \beta_2$ are a finite sum of elements of $E(V)$. Theorem 4.4 has been proved. $\square$

COROLLARY 4.5. An element α of $P_0(V)$ (V an inverse ring) is of the form $\sum_{i=1}^n \alpha_i$ where the α_i are in $E_0(V)$.

Proof. By Theorem 4.4, $\alpha = \sum_{i=1}^n \alpha_i$ where the α_i are in $E(V)$. If all the α_i are of type (8.1) or (10.1), then α has the expression indicated. If none of the α_i are of type (8.1) or (10.1), then α , being constant, is zero. In this situation $\alpha = 1 + (-1)$ where 1 and -1 are of type (8.1). If just some of the α_i are of type (8.1) or type (10.1), then $\alpha = \beta_1 + \dots + \beta_{n_1} + \gamma_1 + \dots + \gamma_{n_2}$ ($n_i, i = 1, 2$ integers ≥ 1 and the β_j of type (6.1) and the γ_j of type (8.1) or (10.1)). From the fact that $0\alpha = 0$ we conclude that $\beta_1 + \dots + \beta_{n_1} = 0$ and α has the expression indicated. Corollary 4.5 has been proved. $\square$

5. TERMINAL RINGS AND $P_0(V)$ RING SUBMODULES

Up until now there has been increasing specialization in the structures being considered. We have moved from binary groups to non-associative rings and then to anti-associative and anti-commutative rings. These have been followed (see the last section) by the introduction of inverse rings. We now take a step further in this direction with the introduction of terminal rings (to be defined). The name would seem appropriate as they allow one of the main goals of this paper to be reached. The goal here is the determination of when a binary group V is a $P_0(V)$ -ring module.

An inverse ring has been defined as a non-associative ring which is both anti-associative and anti-commutative. Such non-associative rings are considerably restricted. For example, in an inverse ring V , elements of $P(V)$ have a rather basic form (see Theorem 4.4). However, if the goal is to specify when V is a $P_0(V)$ ring module a further important condition is necessary. We shall call an inverse ring V a terminal ring if $v^2 u + u^2 v = 0$ for all u and v in V .

Anti-associativity and anti-commutativity are, in some sense, quite natural conditions. By comparison the law $v^2 u + u^2 v = 0$ seems strange. It might be expected to have a very limiting effect. This is the case. It implies powers of elements of V are quite strongly restricted.

PROPOSITION 5.1. Any element v of an terminal ring must be such that $v^4 = v^5 = v^6 = \dots$

Proof. By Proposition 4.1 and Corollary 4.2, it follows that when the u of $v^2 u + u^2 v = 0$ is replaced by v^2 we obtain $v^4 = v^5$. Thus, $v^4 v = v^5 v$ ($v^5 = v^6$) etc. Proposition 5.1 is proved. $\square$

If V is a binary group and U a $P_0(V)$ -subgroup of V , then it is a V -subgroup of V (see Proposition 1.1 and Theorem 2.1) and, of course, a binary subgroup. It would appear that in general not much can be said about such binary subgroups. However, when U is a $P_0(V)$ ring submodule, things are different. This fact is largely a consequence of a proposition about how elements of U and V interact.

PROPOSITION 5.2. If V is a binary group and U a $P_0(V)$ ring submodule of V , then $(u_1 + u_2)w = u_1w + u_2w$, $w(u_1 + u_2) = wu_1 + wu_2$ and $(u_1w)u_2 = -(u_2w)u_1$, for all u_i , $i = 1, 2$, in U and w in V .

Proof. $(U, +)$ is an abelian group and an element α of $P_0(V)$ has the property that $(u_1 + u_2)\alpha = u_1\alpha + u_2\alpha$. Taking α as the map of $P_0(V)$ taking v of V to vw we see $(u_1 + u_2)w = u_1w + u_2w$. When α is taken as the map mapping v in V to wv the fact that $w(u_1 + u_2) = wu_1 + wu_2$ is obtained. With α the map taking v in V to $(vw)v$ we see $((u_1 + u_2)w)(u_1 + u_2) = (u_1w)u_1 + (u_2w)u_2$. Because for all u_i , $i = 1, 2$, in U and w in V , $(u_1 + u_2)w = u_1w + u_2w$, $w(u_1 + u_2) = wu_1 + wu_2$ and the u_iw are in U , the left hand side of this expression is

$$(u_1w + u_2w)(u_1 + u_2) = (u_1w)u_1 + (u_2w)u_1 + (u_1w)u_2 + (u_2w)u_2$$

so that $(u_2w)u_1 + (u_1w)u_2 = 0$. This means the third equation of Proposition 5.2's statement holds. Proposition 5.2 is proved. $\square$

Proposition 5.2 allows the proof that U must be a terminal ring.

THEOREM 5.3. Suppose U is a $P_0(V)$ ring submodule of the binary group V . When U is viewed as a binary group it is a terminal ring.

Proof. $(U, +)$ is abelian and, by Proposition 5.2, the binary group U satisfies both distributive laws. To show it is a terminal ring we must prove it is anti-commutative, anti-associative (the inverse ring conditions) and that $v^2u + u^2v = 0$ for all u and v in V .

To prove U is an inverse ring note, because the map taking x in V to xx is in $P_0(V)$, $(u_1 + u_2)(u_1 + u_2) = u_1u_1 + u_2u_2$ for all u_i , $i = 1, 2$, in U . Here, distributive laws imply $(u_1 + u_2)(u_1 + u_2) = u_1u_1 + u_1u_2 + u_2u_1 + u_2u_2$ so that $u_1u_2 + u_2u_1 = 0$. With u_3 also in U , it follows from Proposition 5.2 that $(u_1u_2)u_3 = -(u_3u_2)u_1$ where, by anti-commutativity, $-(u_3u_2)u_1 = (u_2u_3)u_1 = -u_1(u_2u_3)$. Thus U is an inverse ring and is power associative.

To prove $v^2u + u^2v = 0$ for all u and v in V we use the distributivity over U of the maps taking x in V to x^2 and x in V to x^3 . This implies

$$(v + u)^3 = v^3 + u^3 = (v + u)^2(v + u) = (v^2 + u^2)(v + u) = v^3 + v^2u + u^2v + u^3$$

from which the terminal nature of U follows. Theorem 5.3 is proved. $\square$

When the U of Theorem 5.3 can be taken as V we have an 'if and only if' result.

THEOREM 5.4. The binary group V is a $P_0(V)$ ring module if, and only if, it is a terminal ring.

Proof. By Theorem 5.3, when V is a $P_0(V)$ ring module it has to be a terminal ring. Suppose V is a terminal ring. Certainly it is an inverse ring. $(V, +)$ is abelian and Corollary 4.5 ensures that an α of $P_0(V)$ is of the form $\sum_{i=1}^n \alpha_i$ where the elements α_i of $P_0(V)$ are of type (8.1) or type (10.1). Clearly, if it is shown that for any given i in $\{1, \dots, n\}$ and v_j , $j = 1, 2$, in V , $(v_1 + v_2)\alpha_i = v_1\alpha_i + v_2\alpha_i$, then Theorem 5.4 will be proved. Each α_i is a map taking v in V to v^r (to $-v^r$) or to $u.v^r$ ($u \in V$) where r is an integer ≥ 1 and ≤ 4 (see Proposition 5.1). It follows readily from this that α_i will have the desired property if, for such an r , $(v_1 + v_2)^r = v_1^r + v_2^r$. For $r = 1$, this is trivial. For $r = 2$, we have $(v_1 + v_2)^2 = v_1^2 + v_1v_2 + v_2v_1 + v_2^2 = v_1^2 + v_2^2$ by anti-commutativity. For $r = 3$, it follows that

$$(v_1 + v_2)^3 = (v_1 + v_2)^2(v_1 + v_2) = (v_1^2 + v_2^2)(v_1 + v_2) = v_1^3 + v_1^2v_2 + v_2^2v_1 + v_2^3$$

where the terminal nature of V ensures the right hand side is $v_1^3 + v_2^3$. For $r = 4$, it follows that

$$(v_1 + v_2)^4 = (v_1 + v_2)^2(v_1 + v_2)^2 = (v_1^2 + v_2^2)^2 = v_1^4 + v_1^2v_2^2 + v_2^2v_1^2 + v_2^4$$

where the anti-commutativity of V ensures the right hand side is $v_1^4 + v_2^4$. The theorem is proved. $\square$

6. DECOMPOSITION OF TERMINAL RINGS

There is a certain amount of overlap between the class of rings (associative) and anti-associative rings. Here, any ring in which all elements are of additive order ≤ 2 is anti-associative. However, rings in which all elements have order ≤ 2 are not necessarily anti-commutative and certainly may not be terminal.

When it comes to studying terminal rings a subclass of rings stands out. Boolean rings are easily shown to be terminal rings and can, when an identity is present, be the dominant feature.

PROPOSITION 6.1. A binary group V with 1 is a $P_0(V)$ ring module if, and only if, V is a Boolean ring.

Proof. Since a Boolean ring is a terminal ring it follows from Theorem 5.4 that, when V is Boolean, it is a $P_0(V)$ ring module. On the other hand when V is a $P_0(V)$ ring module Theorem 5.4 implies V is terminal. If in the expression $v^2u + u^2v = 0$ (u and v in V) we take $u = 1$, then it follows that $v^2 + v = 0$. But now Proposition 4.1 implies $v^2 = v$ for all v in V and all v have additive order ≤ 2 . This last fact implies the associative law holds. Thus V is a Boolean ring and Proposition 6.1 is completely proved. $\square$

At least one special case of Proposition 6.1 is already known. Section 7.128 of [4] deals with V being a ring with identity.

Proposition 6.1 seems to say a $P_0(V)$ ring module V is rare. However, an alternative view is that the existence of an identity is a restrictive requirement. This viewpoint tends to be reinforced by more general binary group studies. In fact Boolean rings often only enter into things as part of the picture. We now set about making this plain.

If V is a terminal ring, then we let $\mathcal{E}(V)$ be the set of all e in V such that $e^2 = e$. We let $\mathcal{N}(V)$ be all v in V , such that $v^n = 0$ for some integer $n \geq 1$. $\mathcal{E}(V)$ and $\mathcal{N}(V)$ supply a decomposition of the terminal ring V .

THEOREM 6.2. If V is a terminal ring, then $\mathcal{E}(V)$ and $\mathcal{N}(V)$ are ideals of V . Furthermore, $V = \mathcal{E}(V) \oplus \mathcal{N}(V)$ and $v^4 = 0$ for all v in $\mathcal{N}(V)$.

Proof. In a non-associative ring R an R -subgroup A is an ideal. This follows because $(A, +)$ is a normal subgroup of $(R, +)$ and for $r_i, i = 1, 2$, in R and a_i in A , $(r_1 + a_1)(r_2 + a_2) - r_1 r_2 = r_1 a_2 + r_2 a_1 + a_1 a_2$. Thus, to prove the first statement of the theorem it is sufficient to show $\mathcal{E}(V)$ and $\mathcal{N}(V)$ are V -subgroups of V .

If $e_i, i = 1, 2$, are in $\mathcal{E}(V)$, then as $e_2^2 = e_2$, e_2 has order ≤ 2 and by anti-commutativity

$$(e_1 - e_2)^2 = (e_1 + e_2)^2 = e_1^2 + e_1 e_2 + e_2 e_1 + e_2^2 = e_1 + e_2 = e_1 - e_2.$$

Thus $\mathcal{E}(V)$ is a subgroup of V . For e in $\mathcal{E}(V)$ and u in V we have $e^2 \cdot u + u^2 \cdot e = 0$. Here $e^2 \cdot u = eu$ while anti-associativity and anti-commutativity imply

$$u^2 e = (uu)(ee) = -u(u(ee)) = u((ue)e) = -u((eu)e) = u(e(ue)) = -(ue)(ue)$$

so that the right hand side is $-(eu)^2$ and $(eu)^2 = eu$. This means $\mathcal{E}(V)$ is a right V -subgroup of V and by anti-commutativity it is a V -subgroup.

To prove $\mathcal{N}(V)$ is a subgroup of V it is first indicated how $v^4 = 0$ for all v in $\mathcal{N}(V)$. Because v is in $\mathcal{N}(V)$, $v^n = 0$ for some integer $n \geq 1$. If $n = 1, 2$ or 3 reasonably elementary considerations imply $v^4 = 0$. If $n \geq 4$ then, by Proposition 5.1, $v^4 = 0$. We now note that if $u_i, i = 1, 2$, are in $\mathcal{N}(V)$, then $(u_1 - u_2)^2 = u_1^2 + (-u_2)^2 = u_1^2 + u_2^2$ so that $(u_1 - u_2)^4 = (u_1^2 + u_2^2)^2 = u_1^4 + u_2^4 = 0$. Thus $\mathcal{N}(V)$ is a subgroup of V . If a is in V , then anti-commutativity and anti-associativity imply $(au_1)^2 = -a^2 \cdot u_1^2 = (-a^2) \cdot u_1^2$. As u_1^2 is in $\mathcal{N}(V)$ this in turn yields $(au_1)^4 = ((-a^2) \cdot u_1^2)^2 = [-(-a^2)^2] \cdot u_1^4 = 0$. Thus $\mathcal{N}(V)$ is a left V -subgroup of V and, by anti-commutativity, a V -subgroup.

It has been shown that $\mathcal{E}(V)$ and $\mathcal{N}(V)$ are ideals of V and $v^4 = 0$ for all v in $\mathcal{N}(V)$. To prove $V = \mathcal{E}(V) \oplus \mathcal{N}(V)$ we note that $\mathcal{E}(V) \cap \mathcal{N}(V) = \{0\}$ and, by Proposition 5.1 (Proposition 4.1 also), a w in V is of the form $w^4 + (w^4 + w)$ where w^4 is in $\mathcal{E}(V)$. However, the map α taking x in V to x^4 is in $P_0(V)$ so that, by Theorem 5.4, $(w^4 + w)\alpha = (w^4 + w)^4 = w^{16} + w^4$ where, by Proposition 5.1 and

Proposition 4.1, the last expression is 0. Thus $w^4 + w$ is in $\mathcal{N}(V)$ and the sum of the ideals $\mathcal{E}(V)$ and $\mathcal{N}(V)$ is V . Theorem 6.2 is proved. $\square$

If V is a binary group and U a $P_0(V)$ ring submodule of V , then Theorem 6.2 is applicable to the terminal ring U . A real enough question is whether $\mathcal{E}(U)$ and $\mathcal{N}(U)$ are ideals of V . It appears that for $\mathcal{E}(U)$ this may not be the case. For $\mathcal{N}(U)$ things are different.

THEOREM 6.3. If V is a binary group and U a $P_0(V)$ ring submodule of V , then $\mathcal{N}(U)$ is an ideal of V .

Proof. Suppose e is in $\mathcal{E}(U)$, u in $\mathcal{N}(U)$ and α in $P_0(V)$. Consider the map β of $P_0(V)$ taking v of V to $(ev)(v\alpha)$. Since U is a $P_0(V)$ ring submodule,

$$(e(e+u))((e+u)\alpha) = (e+u)\beta = e\beta + u\beta = (ee)(e\alpha) + (eu)(u\alpha). \quad (6.1)$$

As U is a $P_0(V)$ ring submodule and a non-associative ring the left hand side of (6.1) is $(ee+eu)(e\alpha+u\alpha)$. Here, because $ee, eu, e\alpha$ and $u\alpha$ are all in the non-associative ring U , the left hand side of (6.1) is $(ee)(e\alpha) + (ee)(u\alpha) + (eu)(e\alpha) + (eu)(u\alpha)$. Thus, from (6.1), $(ee)(u\alpha) + (eu)(e\alpha) = 0$ and since, eu is in $\mathcal{E}(U) \cap \mathcal{N}(U) (= 0)$ it follows that $e(u\alpha) = 0$. It will be shown this is enough to prove U is a $P_0(V)$ -subgroup of V so that, by Theorem 2.1, the theorem will be established.

Suppose there exists a u_1 in U such that $u_1\gamma$ is not in U for some γ in $P_0(V)$. By Theorem 6.2, $u_1\gamma = u_2 + u_3$, where $u_2 (\neq 0)$ is in $\mathcal{E}(U)$ and u_3 in $\mathcal{N}(U)$. It follows readily that $u_2(u_1\gamma) = u_2$. But now with $e = u_2$, $u = u_1$ and $\alpha = \gamma$, we have a contradiction to what was proved above. It has been shown U must be an ideal of V . Theorem 6.3 is proved. $\square$

Theorems 6.2 and 6.3 give considerable insight into $P_0(V)$ ring submodules of a binary group V . A lot more can be said when the ring submodule U is minimal. The fact that elements of $P_0(V)$ distribute over U can be used to show (not undertaken in this paper) either $U \cdot U = \{0\}$ or U is a simple Boolean ring.

7. TERMINAL RINGS WITH NO ELEMENTS OF ORDER 2

The definitions of solubility and nilpotency common to group settings have, after suitable generalization, been used more widely. In essence these concepts arise in the case of nearring groups (see [7]) and elsewhere. Indeed, Ω -group solubility and nilpotency have gained importance (see for example [9] and [1]). Many questions can be asked. Certainly the Ω -group solubility of V implies its $P_0(V)$ solubility but finding conditions under which a converse holds is not easy. However, in spite of there being many elementary type Ω -groups which are exceptions, reasonably general conditions supply us with a converse.

If V is a terminal ring then, as observed in Theorem 6.2, $v^4 = 0$ for all v in $\mathcal{N}(V)$. In view of this it seems natural to ask if there is a finite sequence $\{0\} = U_0 \leq U_1 \leq \dots \leq U_k = \mathcal{N}(V)$ of ideals of $\mathcal{N}(V)$ (of V) such that $U_i \cdot U_i \subseteq U_{i-1}$ for i in $\{1, \dots, k\}$? Questions of this sort have bearing on binary groups generally. Indeed such a sequence often allows us to obtain Ω -group solubility (nilpotency) from the $P_0(V)$ solubility (nilpotency) of the binary group V . However, the question of when there is a series of this type is not always easily answered. In the case where $\mathcal{N}(V)$ has elements of order two it is not yet known if we can find such a series. On the other hand, when V has no such element (here $V = \mathcal{N}(V)$) much can be said. In fact, in this situation, something much stronger than the existence of such a series can be supplied. The main step in proving the theorem involved is a lemma.

LEMMA 7.1. Suppose V is a binary group and U a $P_0(V)$ ring submodule of V without elements of additive order two. If a, b, c and d are in U and α and β in $P_0(V)$, then $[(ab)\alpha][(cd)\beta] = 0$ and $((ab)\alpha)c)\beta)d = 0$.

Proof. The map taking v in V to $[(vb)\alpha][(vd)\beta]$ is in $P_0(V)$ and, since U is a $P_0(V)$ ring submodule of V , it follows that

$$[((a+c)b)\alpha][((a+c)d)\beta] = [(ab)\alpha][(ad)\beta] + [(cb)\alpha][(cd)\beta].$$

Because U is a $P_0(V)$ ring submodule and a non-associative ring the right hand side of this expression is

$$[(ab)\alpha][(ad)\beta] + [(ab)\alpha][(cd)\beta] + [(cb)\alpha][(ad)\beta] + [(cb)\alpha][(cd)\beta],$$

from which it follows that $[(ab)\alpha][(cd)\beta] = -[(cb)\alpha][(ad)\beta]$. In a similar fashion to that just employed (use the map taking v in V to $[(cv)\alpha][(av)\beta]$) it can be shown that $[(cb)\alpha][(ad)\beta] = -[(cd)\alpha][(ab)\beta]$. Hence, $[(ab)\alpha][(cd)\beta] = [(cd)\alpha][(ab)\beta]$ which, on using the map taking v in V to $(v\alpha)(v\beta)$ supplies the fact that $[(ab)\alpha][(cd)\beta] = -[(ab)\alpha][(cd)\beta]$. The first statement of the lemma has been proved.

The second statement of the lemma is easily proved. In much the same way as was accomplished above (use the map taking v in V to $(vb)\alpha.v$) it can be shown that $0 = [(ab)\alpha][(cd)\beta] = -(((cd)\beta)b)\alpha)a$, so that with c going to a , d to b , b to c , a to d , β to α and α to β , the desired conclusion is obtained. The lemma has been proved. $\square$

In the second paragraph of this section it was asked if a terminal ring had a certain series of ideals. It is now shown that when no elements of order two are involved something much stronger is possible.

THEOREM 7.2. If V is a binary group and U a $P_0(V)$ ring submodule of V without elements of order two, then there exists a series $\{0\} = U_0 \leq U_1 \leq U_2 \leq U_3 = U$ of ideals of V such that $U_2.U_2 = \{0\}$ and $(U.U_i) \cup (U_i.U) \subseteq U_{i-1}$ for $i = 1, 2$ and 3 .

Proof. For v_1 and v_2 in U , $(v_1 v_2)P_0(V)$ is, by Theorem 2.1, an ideal of V contained in U . Let U_2 be the sum $\sum (v_1 v_2)P_0(V)$ over all ideals of this kind. Certainly U_2 is contained in $U_3 (= U)$. Now, for any such v_1 and v_2 , it is the case that $((v_1 v_2)\alpha)v_3$ (v_3 in U and α in $P_0(V)$) is in U_2 . Thus, by Theorem 2.1, $((v_1 v_2)\alpha)v_3)P_0(V)$ is an ideal of V contained in U_2 . Let U_1 be the sum $\sum (((v_1 v_2)\alpha)v_3)P_0(V)$ over all ideals of this kind (ie. with the v_i , $i = 1, 2, 3$, taking values in U and the α taking values in $P_0(V)$). Certainly U_1 is contained in U_2 but, as is now shown, U_1 and U_2 can be taken as those of the statement of Theorem 7.2.

In proving this we first note that, from U 's distributive laws, an element of $U_2.U_2$, is a sum of elements of the form $((a_1 a_2)\alpha_1)((a_3 a_4)\alpha_2)$ where the a_i are in U and the α_i in $P_0(V)$. Since, by Lemma 7.1, such an element of U is 0, it follows that $U_2.U_2 = \{0\}$. Secondly, we note that, an element of $U.U_3$ (of $U_3.U$) is of the form $(b_1 b_2)1$ and is necessarily in U_2 . Thirdly, we note that, from U 's distributive laws, an element of $U_2.U$ is a sum of elements of the form $((c_1 c_2)\gamma)c_3)1$ where the c_i are in U and the γ in $P_0(V)$. Thus $U_2.U$ is contained in U_1 and by anti-commutativity the same is true for $U.U_2$. Finally, we note that, from U 's distributive laws, an element of $U_1.U$ is a sum of elements of the form $((d_1 d_2)\gamma_1)d_3)\gamma_3)d_4$ where the d_i are in U and the γ_i in $P_0(V)$. By Lemma 7.1, such an element is 0. Thus $U_1.U$ has to be in $\{0\}$ and by anti-commutativity the same is true for $U.U_1$. Theorem 7.2 is proved. $\square$

Theorem 7.2 tells us that the terminal ring U (see Theorem 5.3) is rather restricted. In the next section it will be seen that finite U are often even more basic (see Theorem 8.3). A definition along these lines needs to be made. A basic terminal ring V is taken as one with an ideal W such that $V.V \subseteq W$ and $(W.V) \cup (V.W) = \{0\}$. A characterization of anti-commutative rings, which are basic terminal rings and without elements of order two, follows. This result will be used in sections eleven, twelve and thirteen.

THEOREM 7.3. If V is an anti-commutative ring without elements of order two, then the following are equivalent:

- (i) V is associative,
- (ii) $(ab)c = 0$ for all a, b and c in V ,
- (iii) V is an associative terminal ring and
- (iv) V is a basic terminal ring.

Proof. Proving the equivalence of (ii) to (iv) and showing (ii) implies (i) are rather elementary matters. Their proofs are omitted. It will be shown that (i) implies (ii). Anti-commutativity, the right

distributive law and associativity imply $(vb)v = -(bv)v = -b(vv) = 0$ for all v in V . Thus,

$$((a+c)b)(a+c) = 0 = (ab)a + (ab)c + (cb)a + (cb)c = (ab)c + (cb)a$$

so that $(ab)c = -(cb)a = (bc)a = -a(bc)$. Because $(ab)c = a(bc)$, it can be concluded that $(ab)c = 0$. Theorem 7.3 is proved. $\square$

A feature of a basic $P_0(V)$ ring submodule W of a binary group V is that an ideal U of W , having $WW \subseteq U$ and $UW = \{0\}$, can in fact be taken as an ideal of V .

PROPOSITION 7.4. If W is a basic $P_0(V)$ ring submodule of the binary group V , then there exists an ideal $U (\subseteq W)$ of V such that $WW \subseteq U$ and $UW = \{0\}$.

Proof. As in the proof of Theorem 7.2, we see that the sum $\sum (w_1 w_2) P_0(V)$ over all $w_i, i = 1, 2$, in W is an ideal of V contained in W . We take this to be U . Clearly, $WW \subseteq U$. With α in $P_0(V)$, the map taking v in V to $(v\alpha)v$ is in $P_0(V)$. Because W is a $P_0(V)$ ring submodule of V , this element of $P_0(V)$ supplies the fact that $(u_1 u_2)\alpha u_3 + (u_3\alpha)(u_1 u_2) = 0$ ($u_i, i = 1, 2, 3$, in W). As $(u_3\alpha)(u_1 u_2) = -(u_1 u_2)(u_3\alpha) = 0$, it follows that $(u_1 u_2)\alpha u_3 = 0$. The distributive laws imply $UW = \{0\}$. Proposition 7.4 is proved. $\square$

In the next section it will be seen that many finite terminal rings are basic.

8. CERTAIN FINITE TERMINAL RINGS CANNOT OCCUR

Finite basic terminal rings of odd order are in reality rather rudimentary associative rings (see Theorem 7.3). However, if Theorem 7.2 is taken into consideration, it seems natural to ask if there are any finite terminal rings of odd order which are not basic? The present section does not answer this. Instead results are supplied that tend to justify the need to look into this. What is proved here (Corollary 8.4) is that finite terminal rings of order $p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$ (the p_i distinct odd primes and the n_i integers ≥ 1 and ≤ 6) are in fact basic. In section ten it will be shown there is in fact a non-basic terminal ring of order p^7 (p an odd prime). Things start with a rather elementary proposition. Its corollary will help with the proof of the theorem that follows.

PROPOSITION 8.1. Suppose V is a binary group and U a $P_0(V)$ ring submodule of V without elements of order two. If u is in U , then the ideal $Id(u)$ of V generated by u is such that $Id(u).Id(u) = \{0\}$.

Proof. It readily follows from Theorem 2.1 that $uP_0(V) = Id(u)$. Thus an element of $Id(u).Id(u)$ is of the form $(u\alpha)(u\beta)$ (α and β in $P_0(V)$). Consider the element of $P_0(V)$ taking v in V to $(v\alpha)(v\beta)$. The ring module nature of U supplies the fact that

$$((u+u)\alpha)((u+u)\beta) = (u\alpha)(u\beta) + (u\alpha)(u\beta) = (u\alpha+u\alpha)(u\beta+u\beta),$$

where the right hand side is $[(u\alpha)(u\beta)]4$. Thus $[(u\alpha)(u\beta)]4 = [(u\alpha)(u\beta)]2$ and $(u\alpha)(u\beta) = \{0\}$ so that $Id(u).Id(u) = \{0\}$. Proposition 8.1 is proved. $\square$

From Theorem 5.4 and Proposition 8.1 we have:

COROLLARY 8.2. Suppose V is a terminal ring without elements of order two. If u is in V , then $Id(u).Id(u) = \{0\}$.

A real enough impression conveyed by the theorem that follows is that finite non-basic terminal rings of odd order must be rare.

THEOREM 8.3. A terminal ring V having order p^n (p an odd prime) is basic whenever the integer $n (\geq 1)$ is ≤ 6 .

Proof. Suppose V of the type indicated is not basic. Elements u, v and w of V can be found such that $(uv)w \neq 0$ (see Theorem 7.3). As V is anti-commutative, anti-associative and such that $bb = 0$ for all b in V , there is no real difficulty in showing the subgroup $\langle u, v, w, uv, uw, vw, (uv)w \rangle (= W$ say) of V is a terminal subring. Since $(uv)w \neq 0$ this terminal subring is non-basic. Also anti-commutativity, anti-associativity and the fact that $bb = 0$ for all b in W , imply $\langle uv, uw, vw, (uv)w \rangle (= A$ say) is an ideal of W .

It is now shown that W/A cannot be generated by a subset of $\{u+A, v+A, w+A\}$ of less than three elements. From the symmetry of the situation proving $\langle u+A, v+A \rangle$ is $\neq W/A$ is sufficient for this. If $\langle u+A, v+A \rangle$ were to be $= W/A$ then, as the ideals $Id(u)$ and $Id(v)$ of W generated by u and v are such that $A \leq Id(u) + Id(v)$, we would have $Id(u) + Id(v) = W$. Here, with $Id(u) \cap Id(v) = B$, W/B is a direct sum $(Id(u)/B) \oplus (Id(v)/B)$ of two null groups (see Corollary 8.2) so that $WW \subseteq B$. However, by Corollary 8.2, the fact that $BW \subseteq B.Id(u) + B.Id(v) = \{0\}$ yields the contradiction, W must be basic. Thus W/A is generated by $u+A, v+A$ and $w+A$ and not by a two or one element subset. This means $|W/A| \geq p^3$ and, as $|W| \leq p^6$, it therefore follows that $|A| \leq p^3$.

The fact that $|A| \leq p^3$ means a subset that additively generates A contains a subset S with $|S| \leq 3$ and $\langle S \rangle = (A, +)$. Thus, even if some of the elements of $\{uv, uw, vw, (uv)w\}$ coincide, there is a subset S_1 of this set with $|S_1| \leq 3$ and $\langle S_1 \rangle = (A, +)$. A consequence of this is that one of the subsets

$$\{uv, uw, (uv)w\}, \{uv, vw, (uv)w\}, \{uw, vw, (uv)w\} \quad (8.1)$$

or $\{uv, uw, vw\}$ additively generate A . Suppose $\{uv, uw, vw\}$ is such a subset. Here, either two elements of it additively generate A or, because A has order p^3 , $(A, +)$ is of the form $C_p \oplus C_p \oplus C_p$ (here C_k with $k \geq 1$ an integer is the cyclic group of order k). In the $C_p \oplus C_p \oplus C_p$ case $(uv)w$ is a non-zero element of A and, from this it is easy enough to see, that one of the subsets of (8.1) generate $(A, +)$. It is now shown how this leads to a contradiction. As is not hard to see, the symmetry of the situation allows us to assume $\{uv, uw, (uv)w\}$ additively generates A . Thus vw is in $\langle uv, uw, (uv)w \rangle$ and $u\langle uv, uw, (uv)w \rangle \neq \{0\}$. This is contrary to the fact that $u(uv) = u(uw) = u((uv)w) = 0$ (use the terminal nature of V). A contradiction has been arrived at and Theorem 8.3 is proved. $\square$

In a finite non-associative ring V , the distributive laws imply the Sylow subgroups of $(V, +)$ are ideals. As a consequence V is a direct sum of these ideals. By Theorem 8.3, a proof of the corollary announced at the outset of this section has been achieved. We have:

COROLLARY 8.4. A terminal ring V having order $p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$ ($k \geq 1$ an integer and the p_i distinct odd primes with n_i integers ≥ 1 and ≤ 6) is basic.

9. SAC-DECOMPOSITION OF ANTI-COMMUTATIVE RINGS

In order to proceed further in the study of terminal rings some general theory is required. That is what this section supplies. The last section raised the question of whether there are finite non-basic terminal rings of odd order. Theory obtained here will allow us to answer this. It will also be of considerable help in classifying terminal rings of order p^n (p an odd prime and $n \geq 1$ an integer ≤ 4). What is now undertaken is supplying a way in which an anti-commutative ring V can be specified. For this we need the subgroup $\ker_2(V)$ of all elements of V of order two (kernel of the map taking v in V to v^2).

We come now to defining what might be called a sub-anti-commutative decomposition of a finite abelian group V . Such a decomposition will simply be called an sac-decomposition of V . Here if $n \geq 1$ is an integer, then a sac-decomposition of V (more precisely a $(\text{sac})_n$ -decomposition of V) is a pair (Λ, μ) where Λ is a certain type of subset of V^* and μ a certain type of function of $\Lambda \times \Lambda$ into V . Λ is required to be n distinct elements $a_1, \dots, a_n$ of V^* and $V = \langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \dots \oplus \langle a_n \rangle$ ($\langle a_i \rangle$ is the cyclic subgroup of V generated by a_i). μ is required to be a function of $\Lambda \times \Lambda$ into V with the following properties:

- (i) (a_i, a_j) ($i > j$ in $\{1, \dots, n\}$) is mapped by μ to $(a_i, a_j)\mu$ where the only requirement is that the order $|(a_i, a_j)\mu|$ of $(a_i, a_j)\mu$ divides $|a_i|$ and divides $|a_j|$,
- (ii) $(a_i, a_i)\mu$ is in $\ker_2(V)$ for $i = 1, \dots, n$ and
- (iii) $(a_i, a_j)\mu$ is taken as $-(a_j, a_i)\mu$ for $i < j$ in $\{1, \dots, n\}$.

sac-decompositions of a finite abelian group V have been defined and with (Λ, μ) such a decomposition our intension now becomes to show that the function μ of $\Lambda \times \Lambda$ into V can be extended in a unique way to one $e(\mu)$ of $V \times V$ into V in such a way that, under $e(\mu)$, V is an anti-commutative ring.

Suppose V is a finite abelian group, (Λ, μ) a $(\text{sac})_n$ -decomposition of V , $\Lambda = \{a_1, \dots, a_n\}$ and b and c elements of V . If b (if c) can be written as $am_1 + \dots + a_nm_n$ (as $a_1k_1 + \dots + a_nk_n$) where m_i and k_i , $i = 1, \dots, n$, are integers, then the product $(b, c)e(\mu)$ of b and c is defined to be $\sum_{i=1}^n \sum_{j=1}^n (a_i, a_j)\mu m_i k_j$. There is no ambiguity in this definition since if $b = a_1m'_1 + \dots + a_nm'_n$ and $c = a_1k'_1 + \dots + a_nk'_n$ where the m'_i and k'_j are integers, then $m_i \equiv m'_i \pmod{|a_i|}$ and $k_j \equiv k'_j \pmod{|a_j|}$ so that $m_i k_j \equiv m'_i k'_j \pmod{g.c.d(|a_i|, |a_j|)}$ and $m_i k_j \equiv m'_i k'_j \pmod{|(a_i, a_j)\mu|}$.

PROPOSITION 9.1. If V is a finite abelian group and (Λ, μ) a $(\text{sac})_n$ -decomposition of V , then $e(\mu)$ is the unique extension of μ from $\Lambda \times \Lambda$ to $V \times V$ making V into an anti-commutative ring.

Proof. Suppose $\Lambda = \{a_1, \dots, a_n\}$ and $b = a_1m_1 + \dots + a_nm_n$ and $c = a_1k_1 + \dots + a_nk_n$ are in V with the m_i and k_i being integers. If μ' is an extension of μ to $V \times V$ making V into an anti-commutative ring, then the distributive laws imply that $(b, c)\mu' = \sum_{i=1}^n \sum_{j=1}^n (a_i, a_j)\mu m_i k_j$. Thus $\mu' = e(\mu)$. It only remains to show that, under $e(\mu)$, V is an anti-commutative ring. If $d = a_1l_1 + \dots + a_nl_n$ (l_i to l_n integers), then $(d, b+c)e(\mu) = \sum_{i=1}^n \sum_{j=1}^n (a_i, a_j)\mu l_i(m_j + k_j)$ where the right hand side is

$$\sum_{i=1}^n \sum_{j=1}^n (a_i, a_j)\mu l_i m_j + \sum_{i=1}^n \sum_{j=1}^n (a_i, a_j)\mu l_i k_j = (d, b)e(\mu) + (d, c)e(\mu).$$

As a similar argument shows that $(b+c, d)e(\mu) = (b, d)e(\mu) + (c, d)e(\mu)$, it only remains to show $e(\mu)$ satisfies the anti-commutative law. To see this we note the definition of μ supplies the fact that

$$(b, c)e(\mu) = \sum_{i=1}^n \sum_{j=1}^n (a_i, a_j)\mu m_i k_j = \sum_{j=1}^n \sum_{i=1}^n [-(a_j, a_i)\mu] k_j m_i,$$

where the right hand side is $-(c, b)e(\mu)$. Proposition 9.1 is proved. $\square$

The proof of Proposition 9.1 was very straightforward. Two further results of an elementary nature are now given. The first is a proposition showing that for finite non-zero anti-commutative rings there are in general many ways in which the binary operation is an $e(\mu)$ (μ as in Proposition 9.1). The second is also a proposition. It supplies information as to when a group isomorphism between finite abelian groups is in fact an isomorphism of anti-commutative rings.

PROPOSITION 9.2. Suppose V is a non-zero finite anti-commutative ring with additive group a direct sum $\langle a_1 \rangle \oplus \dots \oplus \langle a_n \rangle$ of the non-zero cyclic subgroups $\langle a_i \rangle$ ($a_i \in V^*$). If $\Lambda = \{a_1, \dots, a_n\}$ and μ the function of $\Lambda \times \Lambda$ into V defined by $(a_i, a_j)\mu = a_i a_j$ ($(i, j) \in \{1, \dots, n\} \times \{1, \dots, n\}$), then (Λ, μ) is a $(\text{sac})_n$ -decomposition of V and $e(\mu)$ is the binary operation of V .

Proof. It is not difficult to see μ satisfies (i), (ii) and (iii) of the definition of $(\text{sac})_n$ -decomposition. It is now clear from Proposition 9.1 that $e(\mu)$ is the binary operation of V . Proposition 9.2 holds. $\square$

PROPOSITION 9.3. Suppose V_i , $i = 1, 2$, are two finite abelian groups, $V_1 = \langle a_1 \rangle \oplus \dots \oplus \langle a_n \rangle$ ($n \geq 1$ and a_i , $i = 1, \dots, n$, elements of V_1^*), $V_2 = \langle b_1 \rangle \oplus \dots \oplus \langle b_n \rangle$ ($b_1, \dots, b_n$, elements of V_2^*) and τ is an isomorphism of V_1 onto V_2 with the property that $a_i \tau = b_i$ for $i = 1, \dots, n$. If (Λ_i, μ_i) are $(\text{sac})_n$ -decompositions of V_i , $i = 1, 2$, with $\Lambda_1 = \{a_1, \dots, a_n\}$, $\Lambda_2 = \{b_1, \dots, b_n\}$ and $(a_j, a_k)\mu_1 \tau = (b_j, b_k)\mu_2$ for all j and k in $\{1, \dots, n\}$, then τ is an anti-commutative ring isomorphism of V_1 (multiplication given by $e(\mu_1)$) onto V_2 (multiplication given by $e(\mu_2)$).

Proof. Proving τ is as indicated amounts to showing $(c, d)e(\mu_1)\tau$ (c and d in V_1) is $(c\tau, d\tau)e(\mu_2)$. Without going into detail it can be seen that, with $c = a_1m_1 + \dots + a_nm_n$ and $d = a_1l_1 + \dots + a_nl_n$ (the m_i and l_j integers), we have

$$(c, d)e(\mu_1)\tau = \left(\sum_{i=1}^n \sum_{j=1}^n (a_i, a_j)\mu_1 m_i l_j \right) \tau = \sum_{i=1}^n \sum_{j=1}^n (b_i, b_j)\mu_2 m_i l_j = (c\tau, d\tau)e(\mu_2)$$

and Proposition 9.3 holds. $\square$

Results of this section are now made use of to supply a rather noteworthy terminal ring.

10. A NON-BASIC TERMINAL RING OF ORDER P^7 (P ODD)

Theorem 8.3 raises a question. Does there exist a finite non-basic terminal ring of order p^n (p an odd prime and $n \geq 1$ an integer)? The answer to this question is that such a terminal ring V can be found. We now move into providing results affirming this. The material supplied is more illuminating than just the proof that V exists. This is because of Theorem 8.3. What is supplied is a V of order p^7 .

Things start by specifying the $(V, +)$ we will be considering. It is determined by being able to take within it seven elements u_i , $i = 1, \dots, 7$, of additive order p with $(V, +) = \oplus_{i=1}^7 \langle u_i \rangle$. Such a $(V, +)$ can be considered as the additive group of the seven dimensional vector space over the field Z_p of p elements. It will make sense to denote it by $\mathcal{V}_p(7)$. The idea now becomes to define a sac_7 -decomposition $(\{u_1, \dots, u_7\}, \mu)$ on V having the property that V is, with respect to $e(\mu)$, a non-basic terminal ring. Thus, in accordance with the fact that $(u_i, u_i)\mu$ must be 0 for $i = 1, \dots, 7$ and the fact that $(u_j, u_i)\mu$ must be $-(u_i, u_j)\mu$ for $i < j$ ($\{i, j\} \subseteq \{1, \dots, 7\}$) we need only indicate the value of $(u_i, u_j)\mu$ for $i < j$. There are twenty one different $\{i, j\}$ needing consideration. In listing these, five categories that respectively deal with $i = 1, 2, 3, 4$ and i in $\{5, 6\}$, are provided. The first three categories will in turn introduce integers k_1, k_2 and k_3 (each ≥ 1 and coprime to p). Values for these integers will soon be given that make our anti-commutative ring V (multiplication $e(\mu)$) non-basic and terminal. A sac_7 -decomposition $(\{u_1, \dots, u_7\}, \mu)$ of $\mathcal{V}_p(7)$ of the type now supplied will be called a $\text{sac}(k_1, k_2, k_3)$ -decomposition of $(V, +)$ (ie. of $\mathcal{V}_p(7)$). The listing of the $(u_i, u_j)\mu$ is as follows:

The six $(u_i, u_j)\mu$ with $i = 1$ are given by,

$$(u_1, u_j)\mu = 0 \quad \text{for } j = 2, \dots, 6 \quad \text{and} \quad (u_1, u_7)\mu = u_4 k_1. \quad (10.1)$$

The five $(u_i, u_j)\mu$ with $i = 2$ are given by,

$$(u_2, u_3)\mu = (u_2, u_4)\mu = (u_2, u_5)\mu = (u_2, u_7)\mu = 0 \quad \text{and} \quad (u_2, u_6)\mu = u_4 k_2. \quad (10.2)$$

The four $(u_i, u_j)\mu$ with $i = 3$ are given by

$$(u_3, u_4)\mu = (u_3, u_6)\mu = (u_3, u_7)\mu = 0 \quad \text{and} \quad (u_3, u_5)\mu = u_4 k_3. \quad (10.3)$$

The three $(u_i, u_j)\mu$ with $i = 4$ are given by

$$(u_4, u_5)\mu = (u_4, u_6)\mu = (u_4, u_7)\mu = 0. \quad (10.4)$$

The two (one) $(u_i, u_j)\mu$ where $i = 5$ (where $i = 6$) are given by

$$(u_5, u_6)\mu = u_1, (u_5, u_7)\mu = u_2 \quad \text{and} \quad (u_6, u_7)\mu = u_3. \quad (10.5)$$

With this definition of $\text{sac}(k_1, k_2, k_3)$ -decomposition behind us, something seeming rather meaningful can be supplied.

THEOREM 10.1. If V is an anti-commutative ring with additive group $\mathcal{V}_p(7)$, then V is a non-basic terminal ring if, and only if, $(V, +)$ has a $\text{sac}(1, p-1, 1)$ -decomposition $(\{u_1, \dots, u_7\}, \mu)$ with $e(\mu)$ the multiplication in V .

Proof. It is first shown that if V is a non-basic terminal ring, then the $\text{sac}(1, p-1, 1)$ -decomposition $(\{u_1, \dots, u_7\}, \mu)$ of $(V, +)$ exists.

According to Theorem 7.3, there must exist u_5, u_6 , and u_7 in V such that $(u_5 u_6)u_7 \neq 0$. It will be shown that with $u_1 = u_5 u_6$, $u_2 = u_5 u_7$, $u_3 = u_6 u_7$, $u_4 = (u_5 u_6)u_7$ and μ the function of $\{u_1, \dots, u_7\} \times \{u_1, \dots, u_7\}$ into V given by $(u_i, u_j)\mu = u_i u_j$ ($\{i, j\} \subseteq \{1, \dots, 7\}$), $(\{u_1, \dots, u_7\}, \mu)$ is a $\text{sac}(1, p-1, 1)$ -decomposition of $(V, +)$ where multiplication in V is $e(\mu)$. In this direction things start by showing $|u_i| = p$, $i = 1, \dots, 7$, and $(V, +) = \oplus_{i=1}^7 \langle u_i \rangle$. An aspect of this (details omitted) is to note that the definition of the u_i , $i = 1, \dots, 7$, and the terminal nature of V imply the subgroup $\langle u_1, \dots, u_7 \rangle$ of $(V, +)$ is a terminal subring of V . It cannot be basic as $(u_5 u_6)u_7 \neq 0$. A consequence of Theorem 8.3 and the fact that $|(V, +)| = p^7$ is that $\langle u_1, \dots, u_7 \rangle = (V, +)$. As $(V, +) = \mathcal{V}_p(7)$ and $(V, +)$ is generated by the seven elements u_i , $i = 1, \dots, 7$, we see $|u_i| = p$ and $(V, +) = \oplus_{i=1}^7 \langle u_i \rangle$. Things now move into showing that if in (10.1) to (10.5), $k_1 = 1$, $k_2 = p-1$ and $k_3 = 1$, then these equations are satisfied.

The manner in which u_1 , u_2 and u_3 are defined ensure (10.5) holds. The terminal nature of V along with the definition of u_2 , u_3 and u_4 imply $(u_1, u_j)\mu = 0$ for j in $\{2, 3, 4\}$. Also the definition of u_1 and the terminal nature of V implies $(u_1, u_5)\mu = (u_1, u_6)\mu = 0$. On also taking into consideration the definition of u_4 we see that (10.1) holds with $k_1 = 1$. Without going into unnecessary detail, definitions of the u_i , $i = 1, \dots, 7$, are seen to yield the fact that $(u_2, u_j)\mu = 0$ for $j \in \{3, 4, 5, 7\}$. Also $u_2 u_6 = (u_5 u_7)u_6 = -(u_5 u_6)u_7$ (V is terminal) so that (10.2) holds with $k_2 = p - 1$. Again the nature of the u_i and V implies $(u_3, u_j)\mu = 0$ for $j \in \{4, 6, 7\}$ while $u_3 u_5 = (u_6 u_7)u_5 = (u_5 u_6)u_7$. Thus (10.3) holds when k_3 is taken as 1. Furthermore $(u_4, u_j)\mu = 0$ for j in $\{5, 6, 7\}$. It has been shown that with $k_1 = 1$, $k_2 = p - 1$ and $k_3 = 1$, (10.1) to (10.5) hold. Thus $(\{u_1, \dots, u_7\}, \mu)$ is a $\text{sac}(1, p - 1, 1)$ -decomposition of $(V, +)$. The multiplication in V is $e(\mu)$ (see Proposition 9.2). It remains to show that when such a decomposition exists, the V (multiplication $e(\mu)$) is a non-basic terminal ring.

Suppose the $\text{sac}(1, p - 1, 1)$ decomposition $(\{u_1, \dots, u_7\}, \mu)$ of $(V, +)$ exists and is such that the product $w_1 w_2$ of w_i , $i = 1, 2$, of V is $(w_1, w_2)e(\mu)$. From Proposition 9.1, it follows that V has become an anti-commutative ring. We now set about showing it is anti-associative. A substantial part of this will be to show that $(u_i u_j)u_k = -u_i(u_j u_k)$ for all i, j and k in $\{1, \dots, 7\}$. Three cases occur.

CASE 1. With $S = \{u_1, u_2, u_3, u_4\}$ it is shown the assumption that $\{u_i, u_j, u_k\} \cap S \neq \emptyset$ implies $(u_i u_j)u_k = 0 = u_i(u_j u_k)$.

Here reasonably careful examination of (10.1) to (10.5) along with the anti-commutative ring nature of V implies $V.V \subseteq \langle S \rangle, \langle S \rangle.\langle S \rangle = \{0\}$, $(\langle S \rangle.V) \cup (V.\langle S \rangle) \subseteq \langle u_4 \rangle$ and $(\langle u_4 \rangle.V) \cup (V.\langle u_4 \rangle) = \{0\}$. This means that if an element of $\{u_i, u_j\}$ were to be in S , then $u_i u_j$ would be in $\langle u_4 \rangle$ and $(u_i u_j)u_k$ would be 0. However, if u_k were to be in S , $(u_i u_j)u_k$ would be in $S.S$ and again be 0. Thus, if an element of $\{u_i, u_j, u_k\}$ is in S , then $(u_i u_j)u_k = 0$. From this and the fact that $u_i(u_j u_k) = -(u_j u_k)u_i$ it follows that, if an element of $\{u_i, u_j, u_k\}$ were to be in S , then $u_i(u_j u_k) = 0$. Thus, if an element of $\{u_i, u_j, u_k\}$ is in S , then $(u_i u_j)u_k = 0 = u_i(u_j u_k)$.

CASE 2. Here we prove that if $\{u_i, u_j, u_k\}$ (assumed to be $\subseteq \{u_5, u_6, u_7\}$) is a two or one element subset of V , then $(u_i u_j)u_k = 0 = u_i(u_j u_k)$.

Since $u_i(u_j u_k) = -(u_j u_k)u_i$, this will follow if it is shown that for $|\{u_i, u_j, u_k\}| \leq 2$ we have $(u_i u_j)u_k = 0$. If $u_i = u_j$, then $(u_i u_j)u_k = 0$. Thus, it can be assumed either $u_i = u_k$ or $u_j = u_k$ and because $(u_i u_j)u_k = -(u_j u_i)u_k$ it is only required it be shown that $(u_i u_j)u_i = 0$. For $i = 5$ and j taken respectively as 5, 6 and 7, $u_i u_j$ becomes 0, u_1 and u_2 (see (10.5)). Because $\{u_1, u_2\}u_5 = \{0\}$ (see (10.1) and (10.2)), $(u_i u_j)u_i = 0$ when $i = 5$. For $i = 6$ and j taken respectively as 5, 6 and 7, $u_i u_j$ becomes $-u_1$, 0 and u_3 (see (10.5)). Because $\{u_1, u_3\}u_6 = \{0\}$ (see (10.1) and (10.3)), $(u_i u_j)u_i = 0$ when $i = 6$. For $i = 7$ and j taken respectively as 5, 6 and 7 $u_i u_j$ becomes $-u_2$, $-u_3$ and 0 (see (10.5)). Because $\{u_2, u_3\}u_7 = \{0\}$ (see (10.2) and (10.3)), $(u_i u_j)u_i = 0$ when $i = 7$. Thus, if $\{i, j, k\}$ has order ≤ 2 and is contained in $\{5, 6, 7\}$, then it must follow that $(u_i u_j)u_k = 0 = u_i(u_j u_k)$.

CASE 3. This is the final step in proving $(u_i u_j)u_k = -u_i(u_j u_k)$ ($\{i, j, k\} \subseteq \{1, \dots, 7\}$).

It will deal with $\{i, j, k\}$ coinciding with $\{5, 6, 7\}$. More work goes into this case than that of 1 or 2. There are six possibilities that must be checked out. These are $(u_5 u_6)u_7$ ((a) say), $(u_5 u_7)u_6$ ((b) say), $(u_6 u_5)u_7$ ((c) say), $(u_7 u_6)u_5$ (d), $(u_6 u_7)u_5$ (e) and $(u_7 u_5)u_6$ (f). For (a)

$$(u_5 u_6)u_7 = u_1 u_7 = u_4 \quad \text{while} \quad u_5(u_6 u_7) = u_5 u_3 = -u_4.$$

For (b)

$$(u_5 u_7)u_6 = u_2 u_6 = -u_4 \quad \text{while} \quad u_5(u_7 u_6) = u_5(-u_3) = u_4.$$

For (c)

$$(u_6 u_5)u_7 = (-u_1)u_7 = -u_4 \quad \text{while} \quad u_6(u_5 u_7) = u_6 u_2 = u_4.$$

For (d)

$$(u_7 u_6)u_5 = (-u_3)u_5 = -u_4 \quad \text{while} \quad u_7(u_6 u_5) = u_7(-u_1) = u_4.$$

For (e)

$$(u_6 u_7)u_5 = u_3 u_5 = u_4 \quad \text{while} \quad u_6(u_7 u_5) = u_6(-u_2) = -u_4.$$

For (f)

$$(u_7 u_5) u_6 = (-u_2) u_6 = u_4 \quad \text{while} \quad u_7 (u_5 u_6) = u_7 u_1 = -u_4.$$

The six possibilities (a) to (f) are dealt with. It has been shown that if $\{i, j, k\} = \{5, 6, 7\}$, then $(u_i u_j) u_k = -u_i (u_j u_k)$.

What cases 1 to 3 show is that $(u_i u_j) u_k = -u_i (u_j u_k)$ for all i, j and k of $\{1, \dots, 7\}$. It is now easy enough to prove that V (multiplication $e(\mu)$) is anti-associative. Indeed, if $v_1 = \sum_{i=1}^7 u_i a_i$, $v_2 = \sum_{j=1}^7 u_j b_j$ and $v_3 = \sum_{k=1}^7 u_k c_k$ (a_i, b_j and c_k integers), then the product $((v_1, v_2)e(\mu), v_3)e(\mu)$ is just

$$\sum_{k=1}^7 \left(\sum_{i=1}^7 \sum_{j=1}^7 (u_i u_j) a_i b_j \right) u_k c_k = \sum_{i=1}^7 \sum_{j=1}^7 \sum_{k=1}^7 (u_i u_j) u_k a_i b_j c_k,$$

where the right hand side is

$$\sum_{i=1}^7 \sum_{j=1}^7 \sum_{k=1}^7 -u_i (u_j u_k) a_i b_j c_k = - \sum_{i=1}^7 u_i \left(\sum_{j=1}^7 \sum_{k=1}^7 (u_j u_k) b_j c_k \right) a_i.$$

Since the right hand side of this equation is $-(v_1, (v_2, v_3)e(\mu))e(\mu)$ it follows that, when the multiplication in V is $e(\mu)$ ($(\{u_1, \dots, u_7\}, \mu)$ the given $\text{sac}(1, p-1, 1)$ -decomposition), V is anti-associative. However, the anti-commutative nature of V implies that for u and v in V , $(u, u)e(\mu) = (v, v)e(\mu) = 0$ so that

$$((u, u)e(\mu), v)e(\mu) + ((v, v)e(\mu), u)e(\mu) = 0$$

and V is terminal. Since $((u_5, u_6)e(\mu), u_7)e(\mu) = (u_5 u_6) u_7 \neq 0$, V is non-basic (see Theorem 7.3). Theorem 10.1 is proved. $\square$

COROLLARY 10.2. If V_i , $i = 1, 2$, are two non-basic terminal rings with additive groups (possibly the same) of the form $\mathcal{V}_p(7)$ (p an odd prime), then V_1 and V_2 are isomorphic.

Proof. By Theorem 10.1, there must exist respective $\text{sac}(1, p-1, 1)$ -decompositions $(\{u_1, \dots, u_7\}, \mu_1)$ and $(\{u'_1, \dots, u'_7\}, \mu_2)$ on $(V_1, +)$ and on $(V_2, +)$ as in Theorem 10.1. A group isomorphism τ of $(V_1, +)$ onto $(V_2, +)$ taking u_i to u'_i , $i = 1, \dots, 7$, is by Proposition 9.3, an isomorphism between V_1 and V_2 . Corollary 10.2 is proved. $\square$

By Corollary 10.2 there is, up to isomorphism, only one non-basic terminal ring on a $\mathcal{V}_p(7)$ (p an odd prime). Its rather exceptional nature makes it worthy of special notation. We denote it by $n\mathcal{B}(\mathcal{V}_p(7))$. Here the symbols $n\mathcal{B}$ are used to indicate it is non-basic.

Taken together theorems Theorem 8.3 and Theorem 10.1 provide real enough insight into finite low odd order terminal rings. More is provided in our final three sections.

11. CERTAIN BASIC TERMINAL RINGS

Section eight looked at terminal rings of order p^n (p an odd prime and n in $\{1, \dots, 6\}$). Although these were not classified, they were shown to be basic. At this stage the goal before us is to classify terminal rings of order p^n (p an odd prime and n in $\{1, 2, 3, 4\}$). The present section does not achieve this. It does however limit such terminal rings still further.

If V is an abelian group, then the null group on V is a basic terminal ring (here the U of Theorem 7.3 can be taken as $\{0\}$). These are the improper basic terminal rings. All others will be called proper. Quite a number of the terminal rings of order p^n (p an odd prime and $n \in \{1, 2, 3, 4\}$) are null groups but proper basic terminal rings can occur.

If k is an integer ≥ 1 then, as has previously been the case, the cyclic group of order k will be denoted by C_k . For the terminal rings under investigation minimal ideals turn out to be certain C_p subgroups (ie. subgroups isomorphic to a C_p). A property of some minimal ideals of a binary group was briefly touched on at the end of section six. The present need is to establish something definite in the case of the terminal rings we are investigating.

PROPOSITION 11.1. If V is a terminal ring of order p^n (p an odd prime and $n \geq 1$ an integer), then a subgroup A of $(V, +)$ is a minimal ideal if, and only if, it is a C_p subgroup of $(V, +)$ with the property that $AV = \{0\}$.

Proof. If A is a C_p subgroup of $(V, +)$ with the property that $AV = \{0\}$ then, as it is a minimal normal V -subgroup of V , it is a minimal ideal. Suppose A is a minimal ideal of V . V has ideals U_0, U_1, U_2 and U_3 with $\{0\} = U_0 \leq U_1 \leq U_2 \leq U_3 = V$ where $U_i V \subseteq U_{i-1}$ for $i = 1, 2$ and 3 (see Theorem 7.2). Take j in $\{0, 1, 2\}$ maximal such that $A \cap U_j = \{0\}$. $A \leq U_{j+1}$ and, as a consequence, $AV \subseteq U_j$. Thus AV is contained in A and U_j and must be $\{0\}$. As this means any subgroup of $(A, +)$ is a V -subgroup, the minimality of A forces it to be a C_p subgroup. Proposition 11.1 is proved. $\square$

Let V be as in Proposition 11.1. That result dealt with minimal ideals. The corollary of the next proposition looks at how certain maximal ideals can effect V 's structure.

PROPOSITION 11.2. Suppose V is an anti-commutative ring of order p^n (p an odd prime and $n \geq 1$ an integer) and A a subgroup of $(V, +)$ such that $AV = \{0\}$. If the additive group $(V, +)/A$ is cyclic, then V is a null group.

Proof. It is not difficult to show that the additive group of V is $A + \langle b \rangle$ where b can be taken as an element of $V \setminus A$. Let $a_i + bn_i$ ($i = 1, 2$, a_i in A and n_i integers ≥ 0) be elements of V . We have

$$(a_1 + bn_1)(a_2 + bn_2) = a_1 a_2 + a_1(bn_2) + (bn_1)a_2 + (bn_1)(bn_2) = 0 + 0 + 0 + (bb)n_1 n_2.$$

The anti-commutativity of V forces bb to be 0. Thus $(a_1 + bn_1)(a_2 + bn_2) = 0$ and V is a null group. Proposition 11.2 is proved. $\square$

COROLLARY 11.3. If V is a terminal ring of order p^n (p an odd prime and $n \geq 1$ an integer) then, when V has a maximal ideal A with $AV = \{0\}$, it must necessarily be a null group.

Proof. As V/A is a simple terminal ring of order p^k ($k \geq 1$ an integer) it is, by Proposition 11.1, of order p . Thus, the additive group of V/A is cyclic and the result follows from Proposition 11.2. $\square$

In what follows there will be some need to broaden considerations from very restricted terminal rings to anti-commutative rings with similar restriction. From now on (next two sections as well) V will be an anti-commutative ring of order p^n (p an odd prime and $n \in \{1, 2, 3, 4\}$). Although it is the case of V terminal that is of interest our study is facilitated by including anti-commutative rings. The results of the next two sections that do not assume V is terminal are Theorems 12.1, 12.3, 13.2, 13.5, 13.8 and 13.10.

PROPOSITION 11.4. If $n = 1$ or 2 and V is terminal, then V is a null group.

Proof. If $n = 1$, then V is a simple terminal ring and by Proposition 11.1, $V.V = \{0\}$. If $n = 2$, then a minimal ideal A of V is, by Proposition 11.1, maximal in V . Since, by Proposition 11.1, $AV = \{0\}$, it follows from Corollary 11.3 that V is null. Proposition 11.4 has been proved. $\square$

When $n = 3$ the possibility of proper basic V arises.

THEOREM 11.5. If $n = 3$ and V is terminal, then V is either a null group or properly basic. In the second case V has a unique minimal ideal A , the additive group of V/A is of the form $C_p \oplus C_p$, $VV \subseteq A$ and $AV = \{0\}$.

Proof. By Theorem 8.3 it follows that, if V is not null, then it is properly basic. Suppose the second case holds and there exist two distinct minimal ideals A_1 and A_2 of V . It follows from Proposition 11.1 that $A_1 \oplus A_2 (= B$ say) is maximal in V and $BV = \{0\}$. As this is contrary to Corollary 11.3, V has a unique minimal ideal A . Here Proposition 11.4 implies $VV \subseteq A$ (V/A has order p^2) and, by Proposition 11.1, $AV = \{0\}$. Also, in this situation, the order p^2 group $(V/A, +)$ cannot be cyclic (see Proposition 11.2) and must be of the form $C_p \oplus C_p$. The theorem is proved. $\square$

The case of $n = 4$ is considerably more complicated. Once again we have V is null or properly basic but now deeper analysis reveals more is going on.

THEOREM 11.6. If $n = 4$ and V is terminal, then V must be one of the following:

- (i) a null group,
- (ii) properly basic and of the form $A_1 \oplus A_2$ (A_i , $i = 1, 2$, ideals of V), where A_1 has order p and A_2 is properly basic of order p^3 ,
- (iii) properly basic with more than one minimal ideal and $(V, +)$ of the form $C_{p^2} \oplus C_{p^2}$ or $C_{p^2} \oplus C_p \oplus C_p$,
- (iv) properly basic with a unique minimal ideal A such that $VV \subseteq A$ and $AV = \{0\}$.

Proof. Let us start by supposing V is not a null group and has more than one minimal ideal. Since (ii) and (iii) are the only possibilities for non-null V with more than one minimal ideal, we need to show our V satisfies (ii) or (iii). Certainly V is properly basic (see Theorem 8.3). If B_1 and B_2 are two distinct minimal ideals of V , then $B_1 \oplus B_2 (= B \text{ say})$ is an ideal of V such that $BV = \{0\}$ (see Proposition 11.1). However, V/B is a terminal ring of order p^2 . By Proposition 11.4, it readily follows that $VV \subseteq B$. Another fact is that the additive group of V/B cannot be a C_{p^2} because this would imply V is null (see Proposition 11.2). Take a and b in $V \setminus B$ such that the subgroup $\langle a, b, ab \rangle$ of V generated by a , b and ab has the property that $\langle a, b, ab \rangle + B = V$. It is not difficult to see $\langle a, b, ab \rangle \langle a, b, ab \rangle \subseteq \langle a, b, ab \rangle$ and, because $\langle a, b, ab \rangle B = \{0\}$, $\langle a, b, ab \rangle$ is an ideal of V . Two cases arise. Either $\langle a, b, ab \rangle < V$ or $\langle a, b, ab \rangle = V$. It will be shown that in the first of these (ii) holds and in the second we have (iii).

In the first of the two cases it can be seen that, because B is of the form $C_p \oplus C_p$, it is necessarily of the form $B_3 \oplus (B \cap \langle a, b, ab \rangle)$ where B_3 is a subgroup of B . The fact that $\langle a, b, ab \rangle + B = V$ can only mean $\langle a, b, ab \rangle \oplus B_3 = V$. $B_3 \neq \{0\}$ since otherwise $\langle a, b, ab \rangle = V$. Also $B_3 \neq B$ as $B_3 = B$ would imply V is a direct sum of two null ideals of order p^2 (see Proposition 11.4) and is necessarily null. Thus, B_3 is a C_p subgroup of V with the property that $B_3 V = \{0\}$. It must be an ideal (see Proposition 11.1). Clearly the ideal $\langle a, b, ab \rangle$ (of order p^3) cannot be null and, by Theorem 11.5, is properly basic. Thus (ii) holds.

In the $\langle a, b, ab \rangle = V$ situation, V is additively generated by three elements and cannot be of the form $C_p \oplus C_p \oplus C_p \oplus C_p$. If it is ruled out that $(V, +)$ is of the form $C_{p^3} \oplus C_p$ (form C_{p^4}), then only the $C_{p^2} \oplus C_{p^2}$ and $C_{p^2} \oplus C_p \oplus C_p$ possibilities remain. $C_{p^3} \oplus C_p$ (C_{p^4}) can indeed be ruled out since such a group has exponent p^3 (exponent p^4) while V/B and B are $C_p \oplus C_p$'s (of exponent p) so that V has exponent p or p^2 . It has been shown that a non-null V with more than one minimal ideal satisfies (ii) or (iii).

If it is shown that a non-null V with a unique minimal ideal A satisfies (iv), then the theorem will be proved. This is because, by Theorem 8.3, V is basic. Certainly $AV = \{0\}$ (see Proposition 11.1) and it is only required that V/A be shown to be a null group. To prove this suppose the contrary. In this case there exists $c + A$ and $d + A$ (c and d in V) with $(c + A)(d + A) \neq A$. Here, $(c + A)(d + A)$ is a non-zero element of the unique minimal ideal of V/A (see Theorem 11.5). Let this ideal be A'/A where $A' > A$ is an ideal of V . As $(c + A)(d + A) = cd + A$, cd is in $A' \setminus A$. $(cd)P_0(V)$ is the ideal of V generated by cd and cannot be such that $A \cap ((cd)P_0(V)) = \{0\}$ (this would imply V had at least two minimal ideals). A consequence is that $A' = (cd)P_0(V)$. The next step in obtaining the required contradiction is to show that the subgroup $\langle c, d, cd \rangle$ of V is such that $\langle c, d, cd \rangle + A = V$.

To show this note that $cd + A$ must additively generate the unique ideal A'/A (a C_p subgroup of V/A) of V/A . Also, neither $c + A$ nor $d + A$ can be in A'/A (this would imply $(c + A)(d + A) = A$). Along with this the subgroup $\langle c + A + A'/A, d + A + A'/A \rangle$ (of order p or p^2) of $(V/A)/(A'/A)$ has order p^2 because otherwise $c + A + A'/A = dn + A + A'/A$ ($n \geq 1$ an integer) so that $c = dn + a'$ ($a' \in A'$). As this would imply cd were to be in A , $(V/A)/(A'/A)$ must coincide with $\langle c + A + A'/A, d + A + A'/A \rangle$. It has been shown $(V/A, +)$ is additively generated by $c + A$, $d + A$ and $cd + A$. Thus, $\langle c, d, cd \rangle + A = V$.

It is not difficult to see from the terminal nature of V that $\langle c, d, cd \rangle \langle c, d, cd \rangle \subseteq \langle c, d, cd \rangle$ and, as $\langle c, d, cd \rangle A = \{0\}$, $\langle c, d, cd \rangle$ is an ideal of V . A has order p so that in the case of $\langle c, d, cd \rangle < V$, $\langle c, d, cd \rangle \cap A = \{0\}$ and a minimal ideal of V distinct from A can be found in $\langle c, d, cd \rangle$. This is contrary to the uniqueness of A and the conclusion that $\langle c, d, cd \rangle = V$ has been reached.

The next step in obtaining a contradiction to the assumption that V/A is not a null group is to prove cd has order p^2 . This is easy enough to accomplish. The fact that V is basic and $= \langle c, d, cd \rangle$

implies $\langle cd \rangle V = \{0\}$ (see Theorem 7.3). This in turn implies $\langle cd \rangle$ is the ideal $(cd)P_0(V)$ of V generated by cd (ie. A'). A' was of order p^2 so that, as required, $|cd| = p^2$.

Assuming V/A is not a null group has lead to the conclusion that cd has order p^2 . Disproving this possibility will complete the proof. The V with elements of order p^2 are additively of the form $C_{p^2} \oplus C_{p^2}$, $C_{p^2} \oplus C_p \oplus C_p$, $C_{p^3} \oplus C_p$ and C_{p^4} . In the case of C_{p^4} , $(V, +)$ is a cyclic group $\langle u \rangle$ ($u \in V$). As $\langle u \rangle \langle u \rangle = \{0\}$, V is a null group contrary to the fact that cd is non-zero. In the case of $C_{p^2} \oplus C_p \oplus C_p$ and $C_{p^3} \oplus C_p$, $(V, +)$ is of the form $C_r \oplus D$ ($r = p^2$ or p^3 and D a abelian group of exponent p). A product of two elements of V is a sum of the product of two elements where at least one is from D . A product of two elements with one from D is 0 or of additive order p . Thus, the product of two elements is either 0 or of order p , contradicting cd being of order p^2 . When $(V, +)$ is of the form $\langle u_1 \rangle \oplus \langle u_2 \rangle$, (u_i , $i = 1, 2$, elements of V of order p^2) we have a product of two elements is of the form $(u_1 u_2)n$ (n an integer). In this case, if it is shown that $u_1 u_2$ is 0 or of order p , then the desired contradiction will be obtained. If $u_1 u_2$ did not have such an order, then $u_1 u_2 = u_1 n_1 + u_2 n_2$, where n_i are integers ≥ 1 with at least one of them being coprime to p . We may assume n_1 is coprime to p and consider $(u_1 u_2)u_2$. Here $(u_1 u_2)u_2 = (u_1 u_2)n_1$ (clearly non-zero) contrary to $(u_1 u_2)u_2 = -u_1(u_2 u_2)$ being 0. cd of order p^2 cannot exist and Theorem 11.6 is proved. $\square$

12. WHEN $N = 1, 2$ OR 3

It has been assumed that V is a anti-commutative ring of order p^n (p an odd prime and n in $\{1, 2, 3, 4\}$). This section will classify those V which are terminal and have $n = 1, 2$ or 3. For $n = 1$, there is up to isomorphism, one such V . It is the null group on a C_p (see Proposition 11.4). Also Proposition 11.4 tells us that for possible V when $n = 2$ there is, up to isomorphism, precisely two V . These are the null groups on C_{p^2} and $C_p \oplus C_p$. The $n = 3$ case is somewhat more complicated.

For $n = 3$ there is the null groups on our terminal V . Certainly these are determined by V 's additive structure. Thus, as $(V, +)$ can only be C_{p^3} , $C_{p^2} \oplus C_p$ or $C_p \oplus C_p \oplus C_p$, three different V arise. By Theorem 11.5, the non-null V are properly basic with a unique minimal ideal. For $(V, +) = C_{p^3}$, V has a unique minimal ideal (the C_p subgroup of V). However here, although V is basic it is not properly basic (ie. V is null). Thus in our classification we are seeking the terminal V with $(V, +)$ of the form $C_{p^2} \oplus C_p$ or $C_p \oplus C_p \oplus C_p$, which are properly basic (ie. non-null).

Things now return to V being anti-commutative.

THEOREM 12.1. Suppose $n = 3$ and $(V, +)$ is of the form $C_{p^2} \oplus C_p$. V is a terminal ring and non-null if, and only if, V has a $(\text{sac})_2$ -decomposition $(\{u, v\}, \mu)$ where $|u| = p^2$, $|v| = p$, $(u, u)\mu = (v, v)\mu = 0$, $(u, v)\mu = -(v, u)\mu = up$ and multiplication in V being $e(\mu)$.

Proof. First it is proved that, when V is terminal and non-null, $(\{u, v\}, \mu)$ exists. Here the nature of $(V, +)$ implies there exists u_1 and v_1 in V with $|u_1| = p^2$, $|v_1| = p$ and $(V, +) = \langle u_1 \rangle \oplus \langle v_1 \rangle$. It is readily seen that the ideal Vp of V must be $\langle u_1 p \rangle$ (a C_p subgroup of V). The anti-commutative nature of V ensures that a product of two elements is either 0 or of the form $(u_1 v_1)k$ ($k \geq 1$ an integer). As V is non-null and $V/\langle u_1 p \rangle$ is a terminal ring of order p^2 , we conclude (see Theorem 11.5) that $u_1 v_1 = (u_1 p)k_1$ where $k_1 \geq 1$ is an integer coprime to p . An integer $k_2 \geq 1$ with $k_2 k_1 \equiv 1 \pmod{p}$ which is coprime to p exists and we now take $u = u_1 k_1$ and $v = v_1 k_2$. It is certainly true that $V = \langle u \rangle \oplus \langle v \rangle$. The fact that $uv = up$ follows because $uv = (u_1 k_1)(v_1 k_2) = (u_1 v_1)(k_1 k_2) = (u_1 p)(k_1 k_1 k_2)$. Here, since $u_1 p$ has order p , the right hand side is just $(u_1 p)k_1 = (u_1 k_1)p = up$. It should now be clear that with μ defined by $(a, b)\mu = ab$ (a and b in $\{u, v\}$), $(\{u, v\}, \mu)$ is the required $(\text{sac})_2$ -decomposition (by Proposition 9.2 the multiplication in V is $e(\mu)$).

Proving the converse is easy enough. By Proposition 9.1, V under $e(\mu)$ is an anti-commutative ring and the product $(a_1 b_1)c_1$ (a_1, b_1 and c_1 in V) is of the form $[(uv)k_3]c_1 = [u(pk_3)]c_1$ ($k_3 \geq 1$ an integer). Because $c_1 = un_1 + vn_2$ (n_i , $i = 1, 2$, integers), it follows readily enough that $(a_1 b_1)c_1 = u(p^2 k_3 n_2) = 0$. Theorem 7.3 is applicable and V is terminal. Since $uv \neq 0$ it is non-null. Theorem 12.1 is proved. $\square$

COROLLARY 12.2. If V_i , $i = 1, 2$ are two non-null terminal rings with additive groups (possibly the same) of the form $C_{p^2} \oplus C_p$, then V_1 and V_2 are isomorphic.

Proof. By Theorem 12.1, there exist $(\text{sac})_2$ -decompositions $(\{u_i, v_i\}, \mu_i)$ on the $(V_i, +)$ as in the statement of Theorem 12.1. A group isomorphism τ of $(V_1, +)$ onto $(V_2, +)$ taking u_1 to u_2 and v_1 to v_2 is, by Proposition 9.3, an isomorphism between V_1 and V_2 . Corollary 12.2 is proved. $\square$

We have seen that up to isomorphism, there is a unique non-null terminal ring with an additive group of the form $C_{p^2} \oplus C_p$. This terminal ring will be denoted by $\mathcal{O}_p(2, 1)$. The notation here is essentially self explanatory except possibly for the $\mathcal{O}$. $\mathcal{O}$ is used because such a V has only one minimal ideal.

We are now ready to deal with the case where $(V, +)$ is a group of the form $C_p \oplus C_p \oplus C_p$.

THEOREM 12.3. Suppose $n = 3$ and $(V, +)$ is of the form $C_p \oplus C_p \oplus C_p$. V is a non-null terminal ring if, and only if, V has a $(\text{sac})_3$ -decomposition $(\{u, v, w\}, \mu)$ where $|u| = |v| = |w| = p$, $(u, u)\mu = (u, v)\mu = (v, u)\mu = (u, w)\mu = (w, u)\mu = 0$, $(v, w)\mu = -(w, v)\mu = u$ and multiplication in V being $e(\mu)$.

Proof. First it is proved that, when V is terminal and non-null, $(\{u, v, w\}, \mu)$ exists. Here the nature of V and $(V, +)$ implies there exists u_1, v_1 and w_1 in V with $|u_1| = |v_1| = |w_1| = p$, $(V, +) = \langle u_1 \rangle \oplus \langle v_1 \rangle \oplus \langle w_1 \rangle$ and $\langle u_1 \rangle$ taken as the unique minimal ideal of V (see Proposition 11.1). Since two element products of elements of V with one in $\langle u_1 \rangle$ are 0 it follows that $u_1 u_1 = u_1 v_1 = v_1 u_1 = u_1 w_1 = w_1 u_1 = 0$. Because V is non-null and $VV \subseteq \langle u_1 \rangle$ it must follow that $v_1 w_1 = u_1 k_1$ where k_1 is an integer ≥ 1 coprime to p . With $(\{u, v, w\}, \mu)$ defined by, $u = u_1 k_1$, $v = v_1$, $w = w_1$ and $(a, b)\mu = ab$ (a and b in $\{u, v, w\}$) it follows from the definition of sac -decomposition that $(\{u, v, w\}, \mu)$ is a $(\text{sac})_3$ -decomposition of V . Certainly $(\{u, v, w\}, \mu)$ is a $(\text{sac})_3$ -decomposition of the required type (by Proposition 9.2 the multiplication in V has to be $e(\mu)$).

Proving the converse is easy enough. By Proposition 9.1, V under $e(\mu)$ is an anti-commutative ring. It can be checked out that $\{u, v, w\} \cdot \{u, v, w\} \subseteq \{u, 0\}$ and $u \cdot \{u, v, w\} = \{0\}$. From this it readily follows that $VV \subseteq \langle u \rangle$ and $\langle u \rangle \cdot V = \{0\}$. Thus the product $(a_1 b_1) c_1$ (a_1, b_1 and c_1 in V) is contained in $\langle u \rangle \cdot V = \{0\}$ and $(a_1 b_1) c_1 = 0$. Theorem 7.3 is applicable and V is terminal. Since $vw \neq 0$ it is non-null. Theorem 12.3 is proved. $\square$

COROLLARY 12.4. If $V_i, i = 1, 2$ are two non-null terminal rings with additive groups (possibly the same) of the form $C_p \oplus C_p \oplus C_p$, then V_1 and V_2 are isomorphic.

Proof. By Theorem 12.3, there exist $(\text{sac})_3$ -decompositions $(\{u_i, v_i, w_i\}, \mu_i)$ on the $(V_i, +)$ as in the statement of Theorem 12.3. A group isomorphism τ of $(V_1, +)$ onto $(V_2, +)$ taking u_1 to u_2 , v_1 to v_2 and w_1 to w_2 is, by Proposition 9.3, an isomorphism between V_1 and V_2 . Corollary 12.4 is proved. $\square$

We have seen that up to isomorphism, there is a unique non-null terminal ring with an additive group of the form $C_p \oplus C_p \oplus C_p$. This terminal ring will be denoted by $\mathcal{O}_p(1, 1, 1)$. The notation here is rather self explanatory (see above explanation about the notation $\mathcal{O}_p(2, 1)$).

For given p with $n = 3$ the V that can arise have been classified. There are the null terminal rings (three in number) and the two non-null ones ($\mathcal{O}_p(2, 1)$ and $\mathcal{O}_p(1, 1, 1)$).

13. WHEN $N = 4$

In this section classification of the V which are terminal is completed. Dealing with $n = 4$ will finalize matters. Before doing this we introduce notation for the null groups ($n = 1, 2$ and 3 cases also) involved here. For $n = 1$, $(V, +)$ is a C_p . Here V is a null group and V is also written as $\mathcal{N}_p(1)$. For $n = 2$, $(V, +)$ is either C_{p^2} or $C_p \oplus C_p$. Again V is a null group. We respectively denote V by $\mathcal{N}_p(2)$ and $\mathcal{N}_p(1, 1)$. For $n = 3$, $(V, +)$ is either C_{p^3} , $C_{p^2} \oplus C_p$ or $C_p \oplus C_p \oplus C_p$. Only the $(V, +) = C_{p^3}$ case forces V to be a null group. We respectively denote the null groups involved by $\mathcal{N}_p(3)$, $\mathcal{N}_p(2, 1)$ and $\mathcal{N}_p(1, 1, 1)$. For $n = 4$, $(V, +)$ is either C_{p^4} , $C_{p^3} \oplus C_p$, $C_{p^2} \oplus C_{p^2}$, $C_{p^2} \oplus C_p \oplus C_p$ or $C_p \oplus C_p \oplus C_p \oplus C_p$. Only the $(V, +) = C_{p^4}$ case forces V to be a null group. We respectively denote the null groups involved by $\mathcal{N}_p(4)$, $\mathcal{N}_p(3, 1)$, $\mathcal{N}_p(2, 2)$, $\mathcal{N}_p(2, 1, 1)$ and $\mathcal{N}_p(1, 1, 1, 1)$.

Assume our anti-commutative V is terminal. The partial classification of the $n = 4$ situation addressed in Theorem 11.6 will supply background for what follows. (i) of Theorem 11.6 provides us with the five null groups just listed. As $\mathcal{O}_p(2, 1)$ and $\mathcal{O}_p(1, 1, 1)$ are the only non-null V of order

p^3 , (ii) of Theorem 11.6 provides us with $\mathcal{N}_p(1) \oplus \mathcal{O}_p(2, 1)$ and $\mathcal{N}_p(1) \oplus \mathcal{O}_p(1, 1, 1)$. They are distinct terminal rings as the first contains elements of order p^2 and the second does not. (iii) of Theorem 11.6 supplies us with the possibility of non-null V having more than one minimal ideal and additive group either $C_{p^2} \oplus C_{p^2}$ or $C_{p^2} \oplus C_p \oplus C_p$. Investigation into this situation begins with the $C_{p^2} \oplus C_{p^2}$ case. Any such V necessarily has more than one minimal ideal.

PROPOSITION 13.1. Suppose the anti-commutative ring V is terminal. If $(V, +)$ is of the form $\langle u \rangle \oplus \langle v \rangle$ where u and v in V have additive order p^2 , then the sum $\text{soc}(V)$ of all minimal ideals of V is the subgroup $\langle up \rangle \oplus \langle vp \rangle$ of $(V, +)$.

Proof. A minimal ideal of V is a C_p subgroup of V and $\langle up \rangle \oplus \langle vp \rangle$ contains all such subgroups. Thus $\text{soc}(V) \leq \langle up \rangle \oplus \langle vp \rangle$. If it is shown $(up)w = 0$ for all w in V , then $\langle up \rangle V = \{0\}$ and $\langle up \rangle$ is a minimal ideal of V . A similar argument will apply to $\langle vp \rangle$ so that $\langle up \rangle \oplus \langle vp \rangle$ will have been shown to be $\leq \text{soc}(V)$. In order to arrive at a contradiction assume $(up)w \neq 0$.

The assumption that $(up)w \neq 0$ implies $uw \neq 0$. Now $w = un_1 + vn_2$ ($n_i, i = 1, 2$, integers ≥ 1). Thus $uv \neq 0$. However uv is not of order p^2 . This is because if it were then uv would be of the form $uk_1 + vk_2$ where the $k_i, i = 1, 2$, are integers ≥ 1 with at least one of them (k_1 say) coprime to p . As this implies $(uv)v = (uv)k_1 \neq 0$ while $(uv)v = -u(vv) = 0$ we have a contradiction. Thus uv has order p and $(up)v = (uv)p = 0$. However $(up)w = (up)(un_1) + (up)(vn_2) = ((uv)p)n_2 = 0$. This proves Proposition 13.1. $\square$

We are now ready to determine those V with $(V, +)$ of the form $C_{p^2} \oplus C_{p^2}$ which are both terminal and properly basic (ie. by Theorem 11.6, non-null). Proposition 9.3 will supply the fact that, up to isomorphism, there is one such terminal V . The added information of Proposition 13.1 will be of use in labeling it.

THEOREM 13.2. Suppose $n = 4$ and $(V, +)$ is of the form $C_{p^2} \oplus C_{p^2}$. V is a terminal ring and non-null if, and only if, V has a $(\text{sac})_2$ -decomposition $(\{u, v\}, \mu)$ where $|u| = p^2, |v| = p^2, (u, u)\mu = (v, v)\mu = 0, (u, v)\mu = -(v, u)\mu = up$ and multiplication in V being $e(\mu)$.

Proof. First it is proved that, when V is terminal and non-null, $(\{u, v\}, \mu)$ exists. It may be assumed that $V = \langle u_1 \rangle \oplus \langle v_1 \rangle$ where u_1 and v_1 in V are such that $|u_1| = |v_1| = p^2$. If $w_i, i = 1, 2$, are in V , then the anti-commutative nature of V ensures $w_1 w_2 = (u_1 v_1)n$ where n is an integer ≥ 1 . Since V is non-null, it follows that $u_1 v_1 \neq 0$. It will be shown that $u_1 v_1$ has order p . Suppose it does not. Here $u_1 v_1 = u_1 n_1 + v_1 n_2$ where $n_i, i = 1, 2$, are integers ≥ 1 with at least one of them (n_1 say) coprime to p . As this implies $(u_1 v_1)v_1 = (u_1 n_1)v_1 = (u_1 v_1)n_1 \neq 0$ while $(u_1 v_1)v_1 = -u_1(v_1 v_1) = 0$, we conclude $u_1 v_1$ does indeed have order p . Three cases arise. These are $u_1 v_1 = (u_1 p)k_1, u_1 v_1 = (v_1 p)k_2$ or $u_1 v_1 = (u_1 p)m_1 + (v_1 p)m_2$ where, in each case, the integers k_1, k_2 and those of $\{m_1, m_2\}$ are ≥ 1 and coprime to p .

CASE 1. Here $u_1 v_1 = (u_1 p)k_1 = (u_1 k_1)p$.

An integer $k_3 \geq 1$ with $k_3 k_1 \equiv 1 \pmod{p}$ which is coprime to p exists and we now take $u = u_1 k_1$ and $v = v_1 k_3$. It is certainly true that $V = \langle u \rangle \oplus \langle v \rangle$. The fact that $uv = up$ follows because $uv = (u_1 k_1)(v_1 k_3) = (u_1 v_1)(k_1 k_3) = (u_1 p)(k_1 k_1 k_3)$. Here, since $u_1 p$ has order p , the right hand side is just $(u_1 p)k_1 = (u_1 k_1)p = up$. It should now be clear that with μ defined by $(a, b)\mu = ab$ (a and b in $\{u, v\}$), $(\{u, v\}, \mu)$ is the required $(\text{sac})_2$ -decomposition (by Proposition 9.2 the multiplication in V is $e(\mu)$).

CASE 2. Here $u_1 v_1 = (v_1 p)k_2 = (v_1 k_2)p$.

This follows in a straightforward manner from case 1. With v_1, u_1 and k_2 respectively taken as the u_1, v_1 and k_1 in the above case, the existence of the required $(\text{sac})_2$ -decomposition is established.

CASE 3. Here $u_1 v_1 = (u_1 p)m_1 + (v_1 p)m_2$.

An integer $m_3 \geq 1$ with $m_3 m_1 \equiv 1 \pmod{p}$ which is coprime to p exists and we now take $u = u_1 m_1 + v_1 m_2$ and $v = v_1 m_3$. Because m_1 and m_2 are coprime to p , u has order p^2 . Also the

subgroup $\langle u \rangle$ of V intersects $\langle v_1 \rangle (= \langle v \rangle)$ in $\{0\}$ (no non-zero element of $\langle u_1 m_1 + v_1 m_2 \rangle$ is in $\langle v_1 \rangle$). With the u and v so defined it follows that $V = \langle u \rangle \oplus \langle v \rangle$. The fact that $uv = up$ follows because

$$uv = (u_1 m_1 + v_1 m_2)(v_1 m_3) = (u_1 v_1)(m_1 m_3) = [(u_1 p)m_1 + (v_1 p)m_2](m_1 m_3).$$

Here, since $(u_1 p)m_1 + (v_1 p)m_2$ has order p , the right hand side is just $(u_1 p)m_1 + (v_1 p)m_2 = up$. It should now be clear that with μ defined by $(a, b)\mu = ab$ (a and b in $\{u, v\}$), $(\{u, v\}, \mu)$ is the required $(\text{sac})_2$ -decomposition (by Proposition 9.2 the multiplication in V is $e(\mu)$).

Proving the converse is easy enough. By Proposition 9.1, V under $e(\mu)$ is an anti-commutative ring and the product $(a_1 b_1)c_1$ (a_1, b_1 and c_1 in V) is of the form $[(uv)k_4]c_1 = [u(pk_4)]c_1$ ($k_4 \geq 1$ an integer). Because $c_1 = uk_5 + vk_6$ ($k_i, i = 5, 6$, integers), it follows readily enough that $(a_1 b_1)c_1 = u(p^2 k_4 k_6) = 0$. Theorem 7.3 is applicable and V is terminal. Since $uv \neq 0$ it is non-null. Theorem 13.2 is proved. $\square$

COROLLARY 13.3. If $V_i, i = 1, 2$ are two non-null terminal rings with additive groups (possibly the same) of the form $C_{p^2} \oplus C_{p^2}$, then V_1 and V_2 are isomorphic.

Proof. By Theorem 13.2, there exist $(\text{sac})_2$ -decompositions $(\{u_i, v_i\}, \mu_i)$ on the $(V_i, +)$ as in the statement of Theorem 13.2. A group isomorphism τ of $(V_1, +)$ onto $(V_2, +)$ taking u_1 to u_2 and v_1 to v_2 is, by Proposition 9.3, an isomorphism between V_1 and V_2 . Corollary 13.3 is proved. $\square$

We have seen that up to isomorphism, there is a unique non-null terminal ring with an additive group of the form $C_{p^2} \oplus C_{p^2}$. This terminal ring will be denoted by $\mathcal{T}_p(2, 2)$. The notation here is essentially self explanatory except possibly for the $\mathcal{T}$. $\mathcal{T}$ is used because such a V has at least two minimal ideals (see Proposition 13.1). This terminal ring is distinct from others that have already arisen. The only possibility of any overlap is with the $\mathcal{N}_p(1) \oplus \mathcal{O}_p(2, 1)$ and $\mathcal{N}_p(1) \oplus \mathcal{O}_p(1, 1, 1)$. However, in both these cases the additive group has a C_p subgroup direct summand. No such subgroup can exist for $C_{p^2} \oplus C_{p^2}$ so $\mathcal{T}_p(2, 2)$ is something new.

At this stage things move toward finding an interesting terminal ring with additive group $C_{p^2} \oplus C_p \oplus C_p$. It will turn out to have more than one minimal ideal but not be of type (i) or (ii) of Theorem 11.6. A proposition will be of help in obtaining it.

PROPOSITION 13.4. In the terminal ring $\mathcal{N}_p(1) \oplus \mathcal{O}_p(2, 1)$ an element v of order p^2 is such that $\langle v \rangle$ is an ideal.

Proof. Clearly, $v = v_1 + v_2$ where v_1 in $\mathcal{N}_p(1)$ is 0 or of order p and v_2 is in $\mathcal{O}_p(2, 1)$ and of order p^2 . The additive group of $\mathcal{O}_p(2, 1)$ is of the form $\langle v_2 \rangle \oplus \langle h \rangle$ where h in $\mathcal{O}_p(2, 1)$ has order p . As $\langle vp \rangle = \langle v_2 p \rangle = [\mathcal{N}_p(1) \oplus \mathcal{O}_p(2, 1)]p$, $\langle v_2 p \rangle$ is an ideal of $\mathcal{O}_p(2, 1)$. Here, $\mathcal{O}_p(2, 1)/\langle v_2 p \rangle$ is a null group (see Proposition 11.4) and so therefore is $\mathcal{N}_p(1) \oplus [\mathcal{O}_p(2, 1)/\langle v_2 p \rangle]$. Thus $\langle v \rangle/\langle v_2 p \rangle$ is an ideal of $\mathcal{N}_p(1) \oplus [\mathcal{O}_p(2, 1)/\langle v_2 p \rangle]$ so that $\langle v \rangle$ must be an ideal of $\mathcal{N}_p(1) \oplus \mathcal{O}_p(2, 1)$. Proposition 13.4 is proved. $\square$

THEOREM 13.5. Suppose V has additive group $C_{p^2} \oplus C_p \oplus C_p$. V is a terminal ring not of type (i), (ii) or (iv) of Theorem 11.6 if, and only if, V has a $(\text{sac})_3$ -decomposition $(\{u, v, w\}, \mu)$, with $|u| = p^2$, $|v| = |w| = p$,

$$(u, u)\mu = (v, v)\mu = (w, w)\mu = (u, v)\mu = (v, u)\mu = (v, w)\mu = (w, v)\mu = 0,$$

$(u, w)\mu = -(w, u)\mu = v$ and $e(\mu)$ the multiplication in V .

Proof. First it is shown that if V is a terminal ring not of type (i), (ii) or (iv) of Theorem 11.6, then $(\{u, v, w\}, \mu)$ exists.

The group $(V, +)$ has elements of order p^2 and we take u as one of them. It is not difficult to see from the nature of this group that it can be written as $\langle u \rangle \oplus D$ where D is a subgroup of the form $C_p \oplus C_p$. As $Vp = \langle up \rangle$ is an ideal of V of order p it is a minimal ideal. Having found u the next step is to find v and w . We start with finding w .

Let H be the sum of all C_p subgroups of V and $\text{soc}(V)$ the sum of all ideals. From the $C_{p^2} \oplus C_p \oplus C_p$ nature of $(V, +)$ it follows without any real difficulty that $|H| = p^3$. By Proposition 11.1, $\text{soc}(V) \leq H$ and $\text{soc}(V) \cdot V = \{0\}$. If $\text{soc}(V)$ were to be H , then it would be maximal in V and Corollary 11.3

would imply V is null (ie. (i) of Theorem 11.6 holds). Thus $\text{soc}(V) < H$ and $|\text{soc}(V)| = p$ or p^2 . If $|\text{soc}(V)| = p$, then V would have unique minimal ideal, contrary to the fact that (i) and (iv) of Theorem 11.6 do not hold. Thus $|\text{soc}(V)| = p^2$. At this stage we take w in $H \setminus \text{soc}(V)$ and, after noting $|w| = p$, proceed to find v .

From the fact that $|V/\text{soc}(V)| = p^2$ it follows, by Proposition 11.4, that $V/\text{soc}(V)$ is a null group. Thus $(u + \text{soc}(V))(w + \text{soc}(V)) = \text{soc}(V)$ and uw is in $\text{soc}(V)$. This element of V is of some interest. It cannot be 0 as w is such that $Hw = [\text{soc}(V) \oplus \langle w \rangle]w = 0$ (see Proposition 11.1) and if uw were to be 0 then, because $H + \langle u \rangle = V$ we would have $V\langle w \rangle = \{0\}$. Since this would imply the C_p subgroup $\langle w \rangle$ of V is a minimal ideal (see Proposition 11.1) contrary to w being in $H \setminus \text{soc}(V)$ we see $uw \neq 0$. Also, as uw is in $\text{soc}(V)$ it has order p . The crucial thing here is that v can be taken as uw . Furthermore, as v is in $\text{soc}(V)$, $\langle v \rangle V = 0$, $uu = vv = ww = vu = uv = vw = wv = 0$ and $uw = -wu = v$. It should be clear from this that if it were to be shown that $V = \langle u \rangle \oplus \langle v \rangle \oplus \langle w \rangle$, then the $(\text{soc})_3$ -decomposition $(\{u, v, w\}, \mu)$ given by $(a, b)\mu = ab$ (a and b in $\{u, v, w\}$) would be as in the statement of the theorem (here Proposition 9.2 will ensure $e(\mu)$ is the multiplication of V). Thus the theorem will be proved in the forward direction by showing $(V, +) = \langle u \rangle \oplus \langle v \rangle \oplus \langle w \rangle$.

The required direct sum nature of $(V, +)$ is now established by first proving $(\langle u \rangle + \langle v \rangle) \cap \langle w \rangle = \{0\}$ and then showing $\langle u \rangle \cap \langle v \rangle = \{0\}$. $\langle w \rangle$ is a group of order p so that if $(\langle u \rangle + \langle v \rangle) \cap \langle w \rangle \neq \{0\}$, then $\langle w \rangle \leq \langle u \rangle + \langle v \rangle$. The subgroup of $\langle u \rangle + \langle v \rangle$ of elements of order p can, without any real difficulty, be shown to be $\langle up \rangle + \langle v \rangle$. Thus w is in $\langle up \rangle + \langle v \rangle$. The manner in which v was defined ensures $\langle v \rangle \leq \text{soc}(V)$. Also, $\langle up \rangle (= Vp)$ is a minimal ideal of V . This is a contradiction as here w (an element of $H \setminus \text{soc}(V)$) has been shown to be in $\text{soc}(V)$. We conclude that $(\langle u \rangle + \langle v \rangle) \cap \langle w \rangle = \{0\}$.

To show $\langle u \rangle \cap \langle v \rangle = \{0\}$ suppose otherwise. As $\langle v \rangle$ is a C_p subgroup of V this must mean v is in $\langle u \rangle$. However, $v = uw$ so that $\langle u, w \rangle \langle u, w \rangle \subseteq \langle u, w \rangle$. It is not difficult to see that an element h of $\text{soc}(V)$ not in $\langle up \rangle$ is such that the sum $\langle h \rangle + \langle u, w \rangle$ is direct. From above $\langle u, w \rangle = \langle u \rangle \oplus \langle w \rangle$ and has order p^3 . Thus $V = \langle u, w \rangle \oplus \langle h \rangle$. The fact that $\langle h \rangle \leq \text{soc}(V)$ means it is an ideal so that, because $\langle u, w \rangle \langle h \rangle = \{0\}$ and $\langle u, w \rangle \langle u, w \rangle \subseteq \langle u, w \rangle$, it can be concluded that $\langle u, w \rangle$ is an ideal. $\langle u, w \rangle$ (of order p^3) is properly basic since $uw \neq 0$. As this contradicts (ii) of Theorem 11.6 not holding, it follows that $\langle u \rangle \cap \langle v \rangle = \{0\}$. Theorem 13.5 has been proved in the forward direction and the only need is to now prove the converse.

Certainly V is an anti-commutative ring under $e(\mu)$. From the definition of $(\{u, v, w\}, \mu)$ we see $\{u, v, w\}\{u, v, w\} \subseteq \{v, 0\}$ and $v\{u, v, w\} = \{0\}$. This implies $VV \subseteq \langle v \rangle$ and $\langle v \rangle V = \{0\}$ so that $(a_1 b_1) c_1$ (a_1, b_1 and c_1 in V) is contained in $\langle v \rangle V$ and must be 0. By Theorem 7.3, V is terminal. Since $uw \neq 0$, (i) of Theorem 11.6 does not hold. Since $uw = v$ (not an element of $\langle u \rangle$) $\langle u \rangle$ is not an ideal and, by Proposition 13.4, (ii) of Theorem 11.6 does not hold (previous material implies the only terminal V satisfying (ii) of Theorem 11.6 are $\mathcal{N}_p(1) \oplus \mathcal{O}_p(2, 1)$ and $\mathcal{N}_p(1) \oplus \mathcal{O}_p(1, 1, 1)$ where the latter case only has elements of order p). Finally, the definition of $(\{u, v, w\}, \mu)$ implies $up\{u, v, w\} = v\{u, v, w\} = \{0\}$ so that $\langle up \rangle V = \langle v \rangle V = \{0\}$ and $\text{soc}(V) \geq \langle up \rangle \oplus \langle v \rangle$ which must mean V has more than one minimal ideal. Thus (iv) of Theorem 11.6 does not hold and Theorem 13.5 is completely proved. $\square$

With straightforward adjustment the proofs of some previous corollaries (eg. Corollary 12.2) supply the verification that:

COROLLARY 13.6. Suppose $V_i, i = 1, 2$, are terminal rings with additive group of the form $C_{p^2} \oplus C_p \oplus C_p$. If the V_i do not satisfy (i), (ii) or (iv) of Theorem 11.6, then they are isomorphic.

The unique terminal ring of Corollary 13.6 will be denoted by $\mathcal{T}_p(2, 1, 1)$ (see the definition of $\mathcal{T}_p(2, 2)$).

Terminal rings of type (i), (ii) and (iii) of Theorem 11.6 have been dealt with. We now look at those covered by (iv). The possible additive groups are $C_{p^4}, C_{p^3} \oplus C_p, C_{p^2} \oplus C_{p^2}, C_{p^2} \oplus C_p \oplus C_p$ and $C_p \oplus C_p \oplus C_p \oplus C_p$. Not all these possibilities can occur.

PROPOSITION 13.7. If $n = 4$, then V cannot be a terminal ring of type (iv) of Theorem 11.6 when $(V, +)$ has the form C_{p^4} or $C_{p^2} \oplus C_{p^2}$.

Proof. It is easily seen that when $(V, +) = C_{p^4}$, V must be null. The $(V, +) = C_{p^2} \oplus C_{p^2}$ case is excluded by Proposition 13.1. Proposition 13.7 is proved. $\square$

The study of terminal V with $n = 4$, $(V, +)$ of the form $C_{p^3} \oplus C_p$, $C_{p^2} \oplus C_p \oplus C_p$ or $C_p \oplus C_p \oplus C_p \oplus C_p$ and of type (iv) of Theorem 11.6 is now addressed. Things start with $C_{p^3} \oplus C_p$.

THEOREM 13.8. Suppose $n = 4$ and $(V, +)$ is of the form $C_{p^3} \oplus C_p$. V is a terminal ring and satisfies (iv) of Theorem 11.6 if, and only if, V has a $(\text{sac})_2$ -decomposition $(\{u, v\}, \mu)$ where $|u| = p^3$, $|v| = p$, $(u, u)\mu = (v, v)\mu = 0$, $(u, v)\mu = -(v, u)\mu = up^2$ and multiplication in V being $e(\mu)$.

Proof. Theorem 13.8 is established in a somewhat similar manner to that used for Theorem 12.1. For the sake of clarity full treatment is given.

First it is proved that, when V is terminal and satisfies (iv) of Theorem 11.6, $(\{u, v\}, \mu)$ exists. Here the nature of $(V, +)$ implies there exists u_1 and v_1 in V with $|u_1| = p^3$, $|v_1| = p$ and $(V, +) = \langle u_1 \rangle \oplus \langle v_1 \rangle$. It is readily seen that the ideal Vp^2 of V must be $\langle u_1 p^2 \rangle$ (a C_p subgroup of V). Thus Vp^2 must be the A of (iv) of Theorem 11.6. The anti-commutative nature of V ensures that a product of two elements of V is either 0 or of the form $(u_1 v_1)k$ ($k \geq 1$ an integer). Also, because (iv) of Theorem 11.6 holds, the non-zero case occurs. Again, since (iv) of Theorem 11.6 holds, $V/\langle u_1 p^2 \rangle$ is a null ring and we conclude that $u_1 v_1 = (u_1 p^2)k_1$ where $k_1 \geq 1$ is an integer coprime to p . An integer $k_2 \geq 1$ with $k_2 k_1 \equiv 1 \pmod{p}$ which is coprime to p exists and we now take $u = u_1 k_1$ and $v = v_1 k_2$. It is certainly true that $V = \langle u \rangle \oplus \langle v \rangle$. The fact that $uv = up^2$ follows because $uv = (u_1 k_1)(v_1 k_2) = (u_1 v_1)(k_1 k_2) = (u_1 p^2)(k_1 k_1 k_2)$. Here, since $u_1 p^2$ has order p , the right hand side is just $(u_1 p^2)k_1 = (u_1 k_1)p^2 = up^2$. It should now be clear that with μ defined by $(a, b)\mu = ab$ (a and b in $\{u, v\}$), $(\{u, v\}, \mu)$ is the required $(\text{sac})_2$ -decomposition (by Proposition 9.2 the multiplication in V is $e(\mu)$).

Proving the converse is easy enough. Certainly V under $e(\mu)$ is an anti-commutative ring and the product $(a_1 b_1)c_1$ (a_1, b_1 and c_1 in V) is of the form $[(uv)k_3]c_1 = [u(p^2 k_3)]c_1$ ($k_3 \geq 1$ an integer). Because $c_1 = un_1 + vn_2$ ($n_i, i = 1, 2$, integers), it follows readily enough that $(a_1 b_1)c_1 = u(p^4 k_3 n_2) = 0$. Theorem 7.3 is applicable and V is terminal. If A_1 were a minimal ideal of V distinct from the minimal ideal $\langle up^2 \rangle$, then $\langle up^2 \rangle \oplus A_1$ would have to be the subgroup of $(V, +)$ consisting of all elements of order p . It is readily seen that if this were to be the case v (of order p) would have to be such that $Vv = 0$. Since $uv \neq 0$, (iv) of Theorem 11.6 must hold. Theorem 13.8 is proved. $\square$

As is easily seen:

COROLLARY 13.9. If $V_i, i = 1, 2$ are two terminal rings as in (iv) of Theorem 11.6 with additive groups of the form $C_{p^3} \oplus C_p$, then V_1 and V_2 are isomorphic.

The natural notation $\mathcal{O}_p(3, 1)$ is used for the terminal ring of Corollary 13.9 (see explanation for the notation $\mathcal{O}_p(2, 1)$).

The examination of terminal V with $n = 4$ and V satisfying (iv) of Theorem 11.6 continues. All that remains to be considered are the cases of $(V, +)$ being $C_{p^2} \oplus C_p \oplus C_p$ or $C_p \oplus C_p \oplus C_p \oplus C_p$. A definition of help in the $(V, +) = C_{p^2} \oplus C_p \oplus C_p$ situation follows.

If W is an abelian group of the form $C_{p^2} \oplus C_p \oplus C_p$, then a $(2, 1, 1)$ -triple of W will be a triple (a, b, c) where a, b and c are in W , $|a| = p^2$, $|b| = |c| = p$ and $W = \langle a \rangle \oplus \langle b \rangle \oplus \langle c \rangle$.

THEOREM 13.10. Suppose V has additive group $C_{p^2} \oplus C_p \oplus C_p$. V is a terminal ring satisfying (iv) of Theorem 11.6 if, and only if, V has a $(\text{sac})_3$ -decomposition $(\{u, v, w\}, \mu)$, with $|u| = p^2$, $|v| = |w| = p$,

$$(u, u)\mu = (v, v)\mu = (w, w)\mu = (u, v)\mu = (v, u)\mu = (u, w)\mu = (w, u)\mu = 0,$$

$(v, w)\mu = -(w, v)\mu = up$ and $e(\mu)$ the multiplication in V .

Proof. First it is shown that if V is a terminal ring satisfying (iv) of Theorem 11.6, then $(\{u, v, w\}, \mu)$ exists. A step toward this is to show for any $(2, 1, 1)$ -triple (a, b, c) of $(V, +)$, $bc \neq 0$. Vp ($= \langle ap \rangle$) is an ideal of V of order p and therefore the unique minimal ideal. By (iv) of Theorem 11.6 we have $VV \subseteq \langle ap \rangle$ and the map taking d in $\langle b, c \rangle$ to ad is a group homomorphism of $\langle b, c \rangle$ to $\langle ap \rangle$. Because $\langle b, c \rangle$ has order p^2 and $\langle ap \rangle$ order p , it follows that there exists a $g \neq 0$ in $\langle b, c \rangle$ such that $ag = 0$.

This allows us to conclude that $bc \neq 0$ since if $bc = 0$, then $gb = gc = 0$ and $g\{a, b, c\} = \{0\}$. It is clear enough from this that, $\langle g \rangle (\leq \langle b, c \rangle)$ is a minimal ideal distinct from $\langle ap \rangle$, forcing bc to be $\neq 0$. This argument in fact gives more. It supplies an element $g \neq 0$ of $\langle b, c \rangle$ with $ag = 0$.

Using what has just been established, it is shown that there is a $(2, 1, 1)$ -triple (a_1, b_1, c_1) of $(V, +)$ with $a_1c_1 = 0$ and $b_1c_1 \neq 0$. Let (a, b, c) and g be as above. It is not hard to see that if g is in $\langle b \rangle$, then (a_1, b_1, c_1) can be taken as (a, c, b) . When g is in $\langle c \rangle$, (a_1, b_1, c_1) can be taken as (a, b, c) . Also, it follows without any real difficulty, that when $g \notin \langle b \rangle \cup \langle c \rangle$, we may take (a_1, b_1, c_1) as (a, b, g) . (a_1, b_1, c_1) has been found (in any $(2, 1, 1)$ -triple (a', b', c') on $(V, +)$, $b'c' \neq 0$). It will be used to establish the existence of a yet more specialized $(2, 1, 1)$ -triple of $(V, +)$.

From the $(2, 1, 1)$ -triple (a_1, b_1, c_1) of $(V, +)$ another (a_2, b_2, c_2) is now constructed. Like (a_1, b_1, c_1) it will have the property $a_2c_2 = 0$ (the property $b_2c_2 \neq 0$ must always hold) but a_2b_2 will also be 0. Clearly if $a_1b_1 = 0$, (a_2, b_2, c_2) can be taken as (a_1, b_1, c_1) . If $a_1b_1 \neq 0$ then, as $VV \subseteq \langle a_1p \rangle$, $a_1b_1 = a_1pr_1$ ($r_1 \geq 1$ an integer coprime to p). Now $c_1b_1 \neq 0$ so that $c_1b_1 = a_1pr_2$ ($r_2 \geq 1$ an integer coprime to p). There exists an integer $r_3 \geq 1$ such that $r_2r_3 \equiv r_1 \pmod{p}$ and, with (a_2, b_2, c_2) taken as $(a_1 - c_1r_3, b_1, c_1)$, it will be shown (a_2, b_2, c_2) is the required $(2, 1, 1)$ -triple. Here c_1r_3 has order p and $a_2p = (a_1 - c_1r_3)p = a_1p$ so that $\langle a_2 \rangle$ has order p^2 and $\langle a_2p \rangle (= \langle a_1p \rangle)$ is the unique non-zero proper subgroup of $\langle a_2 \rangle$. From this it is easy enough to see that $\langle b_2, c_2 \rangle$ (of exponent p and $= \langle b_1, c_1 \rangle$) intersects $\langle a_2 \rangle$ in $\{0\}$. Therefore, order considerations ensure $(V, +) = \langle a_2 \rangle \oplus \langle b_2 \rangle \oplus \langle c_2 \rangle$ and (a_2, b_2, c_2) is indeed a $(2, 1, 1)$ -triple of $(V, +)$. Also, $a_2c_2 = (a_1 - c_1r_3)c_1 = a_1c_1 = 0$ while

$$a_2b_2 = (a_1 - c_1r_3)b_1 = a_1b_1 - c_1b_1r_3 = a_1pr_1 - a_1pr_2r_3 = a_1pr_1 - a_1pr_1 = 0.$$

The $(2, 1, 1)$ -triple (a_2, b_2, c_2) of $(V, +)$ has been constructed. It now turns out to be a relatively easy matter to make use of it to find the $(\text{sac})_2$ -decomposition $(\{u, v, w\}, \mu)$ of V .

The V we are dealing with is such that $Vp = \langle a_2p \rangle$ and $\langle a_2p \rangle$ is the unique minimal ideal of V . As $VV \subseteq \langle a_2p \rangle$, it follows that $b_2c_2 = a_2pr$ ($r \geq 1$ an integer coprime to p). Take $r' \geq 1$ as an integer with the property that $rr' \equiv 1 \pmod{p}$. We now take (u, v, w) as (a_2, b_2, c_2r') . It is easy enough to see (u, v, w) is a $(2, 1, 1)$ -triple of $(V, +)$ with the property that $uu = vv = ww = uv = vu = uw = wu = 0$ and $vw = b_2c_2r' = a_2prr' = a_2p = up$ so that $wv = -up$. Thus with $(\{u, v, w\}, \mu)$ defined by $(x, y)\mu = xy$ (x and y in $\{u, v, w\}$) we have the required $(\text{sac})_3$ -decomposition (by Proposition 9.2, $e(\mu)$ is the multiplication of V). Theorem 13.10 has been proved in the forward direction.

Certainly V is an anti-commutative ring when the multiplication is taken as $e(\mu)$. To prove the converse of Theorem 13.10 note that $\{u, v, w\}\{u, v, w\} \subseteq \{u, 0\}$ and $u\{u, v, w\} = \{0\}$. This means $VV \subseteq \langle u \rangle$ and $\langle u \rangle V = \{0\}$. As this implies $(h_1h_2)h_3 = 0$ ($h_i, i = 1, 2, 3$, in V), V is terminal (see Theorem 7.3). It is properly basic since $vw \neq 0$. Furthermore, if B were a minimal ideal of V distinct from $\langle up \rangle$, then $(\langle up \rangle \oplus B) \cap \langle v, w \rangle$ would contain an element $z \neq 0$. Here $\langle v, w \rangle$ is either $\langle z, v \rangle$ or $\langle z, w \rangle$ so that (see above) $zv \neq 0$ or $zw \neq 0$. However, since $(\langle up \rangle \oplus B).V = \{0\}$, no such B can exist. Thus (iv) of Theorem 11.6 holds and Theorem 13.10 has been proved. $\square$

As is easily seen:

COROLLARY 13.11. If $V_i, i = 1, 2$ are two terminal rings as in (iv) of Theorem 11.6 with additive groups of the form $C_{p^2} \oplus C_p \oplus C_p$, then V_1 and V_2 are isomorphic.

The natural notation $\mathcal{O}_p(2, 1, 1)$ is used for the terminal ring of Corollary 13.11 (see explanation for the notation $\mathcal{O}_p(2, 1)$).

The examination of terminal V with $n = 4$ and V satisfying (iv) is almost complete. In the case of $(V, +) = C_p \oplus C_p \oplus C_p$ and V with a unique minimal ideal, $\mathcal{O}_p(1, 1, 1)$ was the terminal ring we obtained. On the surface it may look as though something similar will take place for $(V, +) = C_p \oplus C_p \oplus C_p \oplus C_p$ (ie. there is an $\mathcal{O}_p(1, 1, 1, 1)$). There is however no such terminal ring. A definition will be of help in proving this.

Suppose V is a terminal ring with $(V, +)$ of the form $C_p \oplus C_p \oplus C_p \oplus C_p$. A quadruple (a, b, c, d) (a, b, c and d in V) will be called a $(1, 1, 1, 1)$ -quadruple of V if $|a| = |b| = |c| = |d| = p$, $(V, +) = \langle a \rangle \oplus \langle b \rangle \oplus \langle c \rangle \oplus \langle d \rangle$ and $\langle a \rangle$ is a minimal ideal of V .

THEOREM 13.12. There is no terminal V with a unique minimal ideal and $(V, +)$ of the form $C_p \oplus C_p \oplus C_p$.

Proof. Suppose the theorem is false and our V has a unique minimal ideal with $(V, +)$ being of the form $C_p \oplus C_p \oplus C_p$. It is not difficult to see that the null group on C_{p^4} is the only terminal ring with a unique minimal ideal satisfying (i), (ii) or (iii) of Theorem 11.6. Thus (iv) of Theorem 11.6 holds and if A is the unique minimal ideal of V , then $VV \subseteq A$. A has order p so that with a in A^* , $\langle a \rangle = A$. The nature of $(V, +)$ ensures there is a $(1, 1, 1, 1)$ -quadruple (a, b, c, d) of V . The first step toward obtaining a contradiction is to use (a, b, c, d) to provide a $(1, 1, 1, 1)$ -quadruple (a_1, b_1, c_1, d_1) of V with $b_1 c_1 = 0$. As $VV \subseteq \langle a \rangle$, there exists a group homomorphism mapping h of $\langle c, d \rangle$ to bh of $\langle a \rangle$. Now $|\langle c, d \rangle| = p^2$ and $|\langle a \rangle| = p$ so that it is possible to find non-zero g in $\langle c, d \rangle$ with $bg = 0$. If g is in $\langle c \rangle$ we take (a_1, b_1, c_1, d_1) as (a, b, c, d) . When g is in $\langle d \rangle$, (a_1, b_1, c_1, d_1) is taken as (a, b, d, c) . It is not hard to see that for g not in $\langle c \rangle \cup \langle d \rangle$, (a_1, b_1, c_1, d_1) can be taken as (a, b, g, d) .

Assuming the existence of our V there cannot exist a $(1, 1, 1, 1)$ -quadruple (a_2, b_2, c_2, d_2) of V with $b_2 c_2 = d_2 c_2 = 0$. This is because in this case $\{a_2, c_2\}c_2 = \{0\}$, $\{a_2, b_2, c_2, d_2\}c_2 = \{0\}$, $V \cdot \langle c_2 \rangle = \{0\}$ and the subgroup $\langle c_2 \rangle$ of order p (disjoint from $\langle a_2 \rangle$) is a minimal ideal (see Proposition 11.1). (a_1, b_1, c_1, d_1) will be used to obtain just such a $(1, 1, 1, 1)$ -quadruple (a_2, b_2, c_2, d_2) of V .

In constructing (a_2, b_2, c_2, d_2) things start by noting that if $d_1 c_1 = 0$, then (a_2, b_2, c_2, d_2) can be taken as (a_1, b_1, c_1, d_1) while if $d_1 b_1$ were to be 0 then because $c_1 b_1 = -b_1 c_1 = 0$, (a_2, b_2, c_2, d_2) can be taken as (a_1, c_1, b_1, d_1) . Thus, it can be assumed that $d_1 c_1 \neq 0$ and $d_1 b_1 \neq 0$. Here we have $d_1 c_1 = a_1 r_1$ and $d_1 b_1 = a_1 r_2$ where $r_i, i = 1, 2$, are integers ≥ 1 and coprime to p . Let r_3 be an integer ≥ 1 such that $r_2 r_3 \equiv r_1 \pmod{p}$. Now take (a_2, b_2, c_2, d_2) as $(a_1, b_1, c_1 - b_1 r_3, d_1)$. It is not hard to see (a_2, b_2, c_2, d_2) is again a $(1, 1, 1, 1)$ -quadruple of V . It is certainly true that $b_2 c_2 = b_1(c_1 - b_1 r_3) = 0$ and that

$$d_2 c_2 = d_1(c_1 - b_1 r_3) = d_1 c_1 - d_1 b_1 r_3 = a_1 r_1 - a_1 r_2 r_3 = a_1 r_1 - a_1 r_1 = 0.$$

(a_2, b_2, c_2, d_2) has been constructed and Theorem 13.12 is proved. $\square$

The categories (i), (ii) (iii) and (iv) of Theorem 11.6 cover terminal V with $n = 4$. The only overlap between them occurs with (ii) and (iii). Those of (ii) or (iii) are $\mathcal{N}_p(1) \oplus \mathcal{O}_p(2, 1)$, $\mathcal{N}_p(1) \oplus \mathcal{O}_p(1, 1, 1)$, $\mathcal{T}_p(2, 2)$ and $\mathcal{T}_p(2, 1, 1)$ (see Corollary 13.3, Proposition 13.1 and Corollary 13.6). Null V with $n = 4$ have been discussed and terminal V satisfying (iv) of Theorem 11.6 are $\mathcal{O}_p(3, 1)$ and $\mathcal{O}_p(2, 1, 1)$ (see Corollary 13.9, 13.11 and Theorem 13.12). For given p the classification of terminal V with $n = 4$ is as follows:

THEOREM 13.13. For $n = 4$ there are, up to isomorphism, eleven different terminal V . They are, $\mathcal{N}_p(4)$, $\mathcal{N}_p(3, 1)$, $\mathcal{N}_p(2, 2)$, $\mathcal{N}_p(2, 1, 1)$, $\mathcal{N}_p(1, 1, 1, 1)$, $\mathcal{N}_p(1) \oplus \mathcal{O}_p(2, 1)$, $\mathcal{N}_p(1) \oplus \mathcal{O}_p(1, 1, 1)$, $\mathcal{T}_p(2, 2)$, $\mathcal{T}_p(2, 1, 1)$, $\mathcal{O}_p(3, 1)$ and $\mathcal{O}_p(2, 1, 1)$.

Theorem 13.13 is the final result of this paper. Some of the pertinent aspects of terminal rings have been covered. As has been seen these non-associative rings arise from studying the nearring of zero fixing polynomial maps on a binary group. Binary groups are much more general structures. A start on investigating them has been made by me in the 137 page manuscript [6]. Regarding all this it is most certainly appropriate to acknowledge support I have received. The Department of Algebra at Johannes Kepler Universität (Linz, Austria) graciously hosted me and Gary Peterson for a five week period starting from the end of April 2019. This visit was partially supported by the Austrian Science Fund (FWF) P29931. While there I gave talks relating to structures covered here. Indeed, this paper is a result of the encouragement Erhard Aichinger supplied. I am also very grateful to Gary Peterson for reading through a draft copy and helping with referencing. The support received has been very much an integral part of what has preceded.

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