ARITHMETICAL FUNCTIONS COMMUTABLE WITH SUMS OF SQUARES II.

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(Dedicated to Professor Ha Huy Khoai on occasion of his 75th birthday)

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ABSTRACT

We give all functions $f, h : \mathbb{N} \rightarrow \mathbb{C}$ which satisfy the relation

$$f(a^2 + b^2 + c^2 + k) = h(a) + h(b) + h(c) + K$$

for every $a, b, c \in \mathbb{N}$, where $k \geq 0$ is an integer and K is a complex number. If n cannot be written as $a^2 + b^2 + c^2 + k$ for suitable $a, b, c \in \mathbb{N}$, then $f(n)$ is not determined. This is more complicated if we assume that f and h are multiplicative functions.

KEYWORDS

Arithmetical function, function equation, Dirichlet character, multiplicative function

MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 11K65; Secondary 11N37, 11N64

1. INTRODUCTION

Let, as usual, $\mathcal{P}$, $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$ be the set of primes, non negative integers, positive integers, integers and complex numbers, respectively. Let $\mathcal{M}$ be the set of all complex valued multiplicative functions.

We define S and $\mathcal{A}$ as follows:

$$S := \{4^s(8t+7) \mid s, t \in \mathbb{N}_0\}$$

and

$$\mathcal{A} := \{r \in \mathbb{N} \mid r^2 = k^2 + h^2 + t^2 \text{ and } r^2 = u^2 + v^2 \text{ for some } k, h, t, u, v \in \mathbb{N}\}.$$

It is clear to check that

$$\{7, 15, 23, 28, 31, 39, 47, 55, 60, 63, 71, 79, 87, 92, 95, \dots\} \subseteq S$$

and

$$\{13, 15, 17, 25, 26, 29, 30, 34, 35, 37, 39, 41, 45, 50, \dots\} \subseteq \mathcal{A}.$$

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For some $k \in \mathbb{N}$ let $\bar{k} \equiv k \pmod{4}$ such that $\bar{k} \in \{0, 1, 2, 3\}$. Let $\mathbb{N}_k = \{n \in \mathbb{N} \mid (n, k) = 1\}$ and let $\mathcal{M}_k^*$ be the set of all completely multiplicative functions on the set $\mathbb{N}_k$.

Recently, in [8] and [9] we gave all arithmetical functions $f, h : \mathbb{N} \rightarrow \mathbb{C}$, which satisfy the relations

$$f(a^2 + b^2 + c^2 + d^2 + k) = h(a) + h(b) + h(c) + h(d) + K \quad (1.1)$$

for every $a, b, c, d \in \mathbb{N}$, where $k \in \mathbb{N}_0$ and $K \in \mathbb{C}$. We proved that (1.1) holds only if there are numbers $A, B \in \mathbb{C}$ such that

$$h(m) = Am^2 + B$$

and

$$f(a^2 + b^2 + c^2 + d^2 + k) = A(a^2 + b^2 + c^2 + d^2) + 4B + K$$

hold for every $n, m \in \mathbb{N}$.

Poo-Sung Park [6] proved that if $f \in \mathcal{M}$ and $k \in \mathbb{N}$, $k \geq 3$ satisfy one of following two conditions

$$f(x_1^2 + \dots + x_k^2) = f(x_1)^2 + \dots + f(x_k)^2$$

or

$$f(x_1^2 + \dots + x_k^2) = f(x_1^2) + \dots + f(x_k^2)$$

for all positive integers $x_1, \dots, x_k$, then f is the identity function.

B.M.M.Khanh [4] determined the solution of the equation

$$f(n^2 + m^2 + k) = f(n)^2 + f(m)^2 + K \quad (\forall n, m \in \mathbb{N}),$$

where $k \in \mathbb{N}_0$ and $K \in \mathbb{C}$. In [3] and [5] she gave all functions $f : \mathbb{N} \rightarrow \mathbb{C}$ which satisfy the equation

$$f(n^2 + Dm^2 + k) = f(n)^2 + Df(m)^2 + k \text{ for every } n, m \in \mathbb{N}.$$

In this paper we improve the above results for two functions. We prove the following theorems.

THEOREM 1.1. Let $k \in \mathbb{N}_0$ and $K \in \mathbb{C}$. Assume that the arithmetical functions $f, h : \mathbb{N} \rightarrow \mathbb{C}$ satisfy the equation

$$f(a^2 + b^2 + c^2 + k) = h(a) + h(b) + h(c) + K$$

for every $a, b, c \in \mathbb{N}$. Then there are numbers $A, B, C \in \mathbb{C}$ such that

$$h(m) = Am^2 + B\chi_2(m) + C$$

and

$$f(a^2 + b^2 + c^2 + k) = A(a^2 + b^2 + c^2) + B(\chi_2(a) + \chi_2(b) + \chi_2(c)) + D_3,$$

hold for every $m, a, b, c \in \mathbb{N}$, where $D_3 = 3C + K$ and $\chi_2(r)$ is the Dirichlet character $\pmod{2}$, that is

$$\chi_2(r) = \begin{cases} 1 & \text{if } (r, 2) = 1 \\ 0 & \text{if } 2 \mid r. \end{cases}$$

Furthermore

$$f(r^2n + k) = Ar^2n + \bar{n}B\chi_2(r) + D_3$$

holds for every $n \in \mathbb{N}$, $n \notin S$ and for every $r \in \mathcal{A}$.

In the following let

$$\mathbb{M} := \{a^2 + b^2 + c^2 \mid a, b, c \in \mathbb{N}\}$$

and for each $k \in \mathbb{N}_0$ let

$$H_k = \begin{cases} \{1, \dots, e+2\} & \text{if } k = 2^e k_1, (k_1, 2) = 1, 2 \mid e \text{ and } k_1 \equiv 1 \pmod{8}, \\ \mathbb{N} \setminus \{e+2\} & \text{if } k = 2^e k_1, (k_1, 2) = 1, 2 \mid e \text{ and } k_1 \equiv 5 \pmod{8}, \\ \mathbb{N} & \text{in any other cases.} \end{cases}$$

THEOREM 1.2. Let $k \in \mathbb{N}_0$ and $K \in \mathbb{C}$. Assume that $F, H \in \mathcal{M}$ satisfy the relation

$$F(a^2 + b^2 + c^2 + k) = H(a) + H(b) + H(c) + K$$

for every $a, b, c \in \mathbb{N}$.

Then one of the following assertions holds:

- (1) $H(m) = 1$ and $F(\eta + k) = 0$ ($\forall m \in \mathbb{N}, \forall \eta \in \mathbb{M}$) if $K = -3$,
- (2) $H(m) = 1$ and $F(2n + 1) = 1, F(2^\alpha) = K + 3$ ($\forall m, n \in \mathbb{N}, \forall \alpha \in \mathcal{H}_k$) if $K \neq -3$,
- (3) $H(m) = \chi_2(m)$ and $F(2n + 1) = 1$ ($\forall m, n \in \mathbb{N}$) if $(\bar{k}, K) \neq (3, -1)$,
- (4) $H(m) = \chi_2(m)$ and $F(2n + 1) = (-1)^n, F(2) = (-1)^{\frac{k+1}{4}} 2, F(2^\alpha) = 0$ ($\forall m, n \in \mathbb{N}, \forall \alpha \in \mathcal{H}_k, \alpha \geq 2$) if $(\bar{k}, K) = (3, -1)$,
- (5) $H(m) = m^2, F(2n + 1) = 2n + 1$ and $F(2^\alpha) = 2^\alpha$ ($\forall m, n \in \mathbb{N}, \forall \alpha \in \mathcal{H}_k$).

We derive the following corollaries from Theorem 1.1.

COROLLARY 1.3. Let $\ell \in \mathbb{N}$ and $\ell \geq 3$. Assume that the arithmetical function $F : \mathbb{N} \rightarrow \mathbb{C}$ satisfies the relation

$$F(n_1^2 + \dots + n_\ell^2) = F(n_1)^2 + \dots + F(n_\ell)^2$$

for every $n_1, \dots, n_\ell \in \mathbb{N}$. Then one of the following assertions holds:

- (i) $F(n) = 0$ for every $n \in \mathbb{N}$,
- (ii) $F(n) = \frac{\varepsilon(n)}{\ell}$ for every $n \in \mathbb{N}$,
- (iii) $F(n) = \varepsilon(n)n$ for every $n \in \mathbb{N}$, where $\varepsilon(n) \in \{-1, 1\}$ and $\varepsilon(n_1^2 + \dots + n_\ell^2) = 1$ for every $n_1, \dots, n_\ell \in \mathbb{N}$.

COROLLARY 1.4. Let $\ell \in \mathbb{N}$ and $\ell \geq 4$. Assume that the arithmetical function $F : \mathbb{N} \rightarrow \mathbb{C}$ satisfies the relation

$$F(n_1^2 + \dots + n_\ell^2) = F(n_1^2) + \dots + F(n_\ell^2)$$

holds for every $n_1, \dots, n_\ell \in \mathbb{N}$. Then there is a complex A such that

$$F(m^2) = Am^2 \text{ for every } m \in \mathbb{N}$$

and

$$F(n_1^2 + \dots + n_\ell^2) = A(n_1^2 + \dots + n_\ell^2)$$

hold for every $m, n_1, \dots, n_\ell \in \mathbb{N}$. Consequently, $F(n) = An$ for every n which can be written as $n = n_1^2 + \dots + n_\ell^2$ for $n_1, \dots, n_\ell \in \mathbb{N}$.

2. PROOF OF THEOREM 1.1

Assume that the arithmetical functions $f, h : \mathbb{N} \rightarrow \mathbb{C}$ satisfy the relation

$$f(a^2 + b^2 + c^2 + k) = h(a) + h(b) + h(c) + K \quad (2.1)$$

for every $a, b, c \in \mathbb{N}$.

We shall prove that there are numbers $A, B, C \in \mathbb{C}$ such that

$$h(m) = Am^2 + B\chi_2(m) + C \quad (2.2)$$

and

$$f(a^2 + b^2 + c^2 + k) = A(a^2 + b^2 + c^2) + B(\chi_2(a) + \chi_2(b) + \chi_2(c)) + D_3, \quad (2.3)$$

hold for every $m, a, b, c \in \mathbb{N}$, where $D_3 = 3C + K$ and $\chi_2(r)$ is the principal character (mod 2).

We define numbers A, B, C as follows:

$$A = \frac{h(3) - h(1)}{8}, B = \frac{3h(3) - 8h(2) + 5h(1)}{8} \text{ and } C = \frac{-h(3) + 2h(2) + h(1)}{2}.$$

It is obvious from the definitions of A, B and C that (2.2) is true for $m \in \{1, 2, 3\}$. Indeed, we have

$$A \cdot 1^2 + B\chi_2(1) + C = A + B + C = \frac{h(3) - h(1)}{8} + \frac{3h(3) - 8h(2) + 5h(1)}{8} + \frac{-h(3) + 2h(2) + h(1)}{2} = h(1),$$

$$A \cdot 2^2 + B\chi_2(2) + C = 4A + C = 4 \frac{h(3) - h(1)}{8} + \frac{-h(3) + 2h(2) + h(1)}{2} = h(2)$$

and

$$A \cdot 3^2 + B\chi_2(3) + C = 9A + B + C = 9 \frac{h(3) - h(1)}{8} + \frac{3h(3) - 8h(2) + 5h(1)}{8} + \frac{-h(3) + 2h(2) + h(1)}{2} = h(3).$$

Since $5^2 + 1^2 + 1^2 = 3^2 + 3^2 + 3^2$, we infer from (2.1) that

$$h(5) + 2h(1) + K = f(5^2 + 1^2 + 1^2 + k) = f(3^2 + 3^2 + 3^2 + k) = 3h(3) + K$$

and so

$$h(5) = 3h(3) - 2h(1).$$

Now we infer from the fact $5^2 + 2^2 + 2^2 = 4^2 + 4^2 + 1^2$ that $h(5) + 2h(2) + K = 2h(4) + h(1) + K$, which implies

$$h(4) = \frac{h(5) + 2h(2) - h(1)}{2} = \frac{(3h(3) - 2h(1)) + 2h(2) - h(1)}{2} = \frac{3h(3) + 2h(2) - 3h(1)}{2}.$$

On the other hand, we have

$$A \cdot 4^2 + B\chi_2(4) + C = 16 \frac{h(3) - h(1)}{8} + \frac{-h(3) + 2h(2) + h(1)}{2} = \frac{3h(3) + 2h(2) - 3h(1)}{2},$$

which proves that $h(4) = A \cdot 4^2 + B\chi_2(4) + C$, and so (2.2) is true for $m = 4$.

Assume that (2.2) is true for every $m < M$, where $M \geq 5$. Since $M - 4 \in \mathbb{N}$ and $M - 2 \in \mathbb{N}$, we infer from the next relation

$$M^2 + (M - 4)^2 + 1^2 = (M - 2)^2 + (M - 2)^2 + 3^2,$$

that

$$h(M) + h(M - 4) + h(1) = 2h(M - 2) + h(3).$$

Thus, from our assumptions, we have

$$\begin{aligned} h(M) &= 2h(M - 2) - h(M - 4) + h(3) - h(1) \\ &= 2(A(M - 2)^2 + B\chi_2(M - 2) + C) - (A(M - 4)^2 + B\chi_2(M - 4) + C) + h(3) - h(1) \\ &= AM^2 + B\chi_2(M) + C, \end{aligned}$$

which proves that (2.2) holds for M , and so (2.2) holds for every $m \in \mathbb{N}$.

Finally, from (2.1) and (2.2), we have

$$\begin{aligned} f(a^2 + b^2 + c^2 + k) &= h(a) + h(b) + h(c) + K \\ &= (Aa^2 + B\chi_2(a) + C) + (Ab^2 + B\chi_2(b) + C) + (Ac^2 + B\chi_2(c) + C) + K \\ &= A(a^2 + b^2 + c^2) + B(\chi_2(a) + \chi_2(b) + \chi_2(c)) + 3C + K. \end{aligned}$$

The proof of (2.3) is finished with $D_3 = 3C + K$.

Now we prove that

$$f(r^2n + k) = Ar^2n + \bar{n}B\chi_2(r) + D_3 \quad (2.4)$$

holds for every $n \in \mathbb{N}$, $n \notin S$ and for every $r \in \mathcal{A}$, where $\bar{n} \in \{0, 1, 2, 3\}$ such that $n \equiv \bar{n} \pmod{4}$.

In order to prove (2.4), we need the following result concerning the sum of three squares.

LEMMA 2.1 (Legendre's three-square theorem). A number $n \in \mathbb{N}$ can be represented as the sum of three squares of integers if and only if $n \notin S$.

Let $n \in \mathbb{N}$, $n \notin S$, $r \in \mathcal{A}$, $r^2 = k^2 + h^2 + t^2$ and $r^2 = u^2 + v^2$ for some $k, h, t, u, v \in \mathbb{N}$. Since $n \notin S$, Lemma 2.1 implies that $n = x^2 + y^2 + z^2$ with $x, y, z \in \mathbb{Z}$. Then there are three options:

- (a) $n = x^2 + y^2 + z^2$ with $x, y, z \in \mathbb{N}$,
- (b) $n = x^2 + y^2$ with $x, y \in \mathbb{N}$,
- (c) $n = x^2$ with $x \in \mathbb{N}$.

In the case (a), we have $r^2n = (rx)^2 + (ry)^2 + (rz)^2$. It is clear to check that

$$\chi_2(rx) + \chi_2(ry) + \chi_2(rz) = \bar{n}\chi_2(r). \quad (2.5)$$

Indeed, since $n = x^2 + y^2 + z^2$, we have

$$(\bar{n}, \bar{x}, \bar{y}, \bar{z}) \in \{(0, 0, 0, 0), (1, 1, 0, 0), (2, 1, 1, 0), (3, 1, 1, 1)\},$$

which proves (2.5). Therefore (2.3) and (2.5) imply

$$\begin{aligned} f(r^2n + k) &= f((rx)^2 + (ry)^2 + (rz)^2 + k) \\ &= A((rx)^2 + (ry)^2 + (rz)^2) + B(\chi_2(rx) + \chi_2(ry) + \chi_2(rz)) + D_3 \\ &= Ar^2n + \bar{n}B\chi_2(r) + D_3, \end{aligned}$$

which proves (2.4) for the case (a).

In the case (b), we have $r^2n = (rx)^2 + (ry)^2 = (rx)^2 + (uy)^2 + (vy)^2$ and

$$\chi_2(rx) + \chi_2(uy) + \chi_2(vy) = \bar{n}\chi_2(r).$$

This with (2.3) implies (2.4).

In the case (c), we have

$$r^2n = (k^2 + h^2 + t^2)x^2 = (kx)^2 + (hx)^2 + (tx)^2$$

and

$$\chi_2(kx) + \chi_2(hx) + \chi_2(tx) = \bar{n}\chi_2(r)$$

which by using (2.3) proves (2.4). Therefore, the proof of (2.4) is finished.

Theorem 1.1 is proved.

3. LEMMAS FOR THE PROOF OF THEOREM 1.2

LEMMA 3.1 (P. D. T. A. Elliott, [1] Lemma 19.3). Assume that $F \in \mathcal{M}$ satisfies the relation

$$F(an + b) = d$$

for every $n \in \mathbb{N}$, where $a, b \in \mathbb{N}$ and $d \in \mathbb{C} \setminus \{0\}$. Then $F(n) = \chi_a(n)$ for every $n \in \mathbb{N}_a$, where $\chi_a(n)$ is a Dirichlet real character (mod a).

Proof. This lemma is identical to Lemma 19.3 in [1]. □

LEMMA 3.2. Assume that $F \in \mathcal{M}$ satisfies the equation

$$F(an + b) = cn + d \tag{3.1}$$

for every $n \in \mathbb{N}$, where $a, b \in \mathbb{N}$ and $c, d \in \mathbb{C}$ are given. Then one of the following assertions holds:

- (a) $c = d = 0$ and $F(an + b) = 0$ for every $n \in \mathbb{N}$,
- (b) $c = 0$, $F(b) = d \neq 0$ and $F(n) = \chi_a(n)$ for every $n \in \mathbb{N}_a$,
- (c) $c \neq 0$, $ad = bc$, $F(b) = d \neq 0$ and $F(n) = \chi_a(n)n$ for every $n \in \mathbb{N}_a$,

where $\chi_a(n)$ is a Dirichlet real character (mod a).

Proof. Let $M \in \mathbb{N}$, $M \equiv 1 \pmod{a}$ be an arbitrary. Let

$$I_M := \{n \in \mathbb{N} \mid (M, an + b) = 1\}$$

It follows from the fact $M \equiv 1 \pmod{a}$ that $|I_M| = \infty$. We infer from the facts $F \in \mathcal{M}$ and (3.1) that

$$\begin{aligned} cF(M)n + dF(M) &= F(M)(cn + d) = F(M)F(an + b) = \\ &= F\left[a\left(Mn + \frac{M-1}{a}b\right) + b\right] = c\left(Mn + \frac{M-1}{a}b\right) + d = \\ &= cMn + bc\frac{M-1}{a} + d \end{aligned}$$

holds for every $n \in I_M$, consequently

$$\begin{cases} cF(M) = cM \\ dF(M) = bc\frac{M-1}{a} + d. \end{cases}$$

The first equation shows that either $c = 0$ or $F(M) = M$. If $c = 0$, then the second equation gives $dF(M) = d$, consequently either $d = 0$ or $F(M) = 1$. If $(c, d) = (0, 0)$, then $F(an + b) = 0$ for every $n \in \mathbb{N}$, and so (a) is proved.

If $c = 0$, $d \neq 0$, then the second equation shows that $F(M) = 1$ holds for every $M \in \mathbb{N}$, $M \equiv 1 \pmod{a}$. Lemma 3.1 implies that $F(n) = \chi_a(n)$ for every $n \in \mathbb{N}_a$. Since $F \in \mathcal{M}_a^*$ and $F(n) = \chi_a(n)$ for every $n \in \mathbb{N}_a$, we infer from (3.1) that

$$F(b) = F(b)F(at + 1) = F(abt + b) = d,$$

which proves (b) of Lemma 3.2.

Now assume that $c \neq 0$ and $F(M) = M$ for every $M \in \mathbb{N}, M \equiv 1 \pmod{a}$. Let $g(n) := \frac{F(n)}{n}$ for every $n \in \mathbb{N}$. Then $g \in \mathcal{M}$ and $g(M) = 1$ for every $M \in \mathbb{N}, M \equiv 1 \pmod{a}$, Lemma 3.1 implies that $g(n) = \chi_a(n)$ for every $n \in \mathbb{N}_a$. Therefore

$$F(n) = \chi_a(n)n \text{ for every } n \in \mathbb{N}_a.$$

Now we prove that

$$ad = bc \text{ and } F(b) = d.$$

Indeed, since $F \in \mathcal{M}_a^*$ and $F(n) = \chi_a(n)n$ for every $n \in \mathbb{N}_a$, we infer from (3.1) that

$$F(b)(at + 1) = F(b)F(at + 1) = F(abt + b) = bct + d,$$

which implies that

$$aF(b) = bc, F(b) = d \text{ and } ad = bc.$$

Since $c \neq 0, b \in \mathbb{N}$, we have $d \neq 0$, and so the above relation proves the assertion (c) of Lemma 3.2.

Lemma 3.2 is proved. $\square$

LEMMA 3.3. Let $a, b \in \mathbb{N}, D = (a, b)$. If $F \in \mathcal{M}$ such that $F(n) = \chi_a(n)$ for every $n \in \mathbb{N}_a$ and $F(n) = \chi_b(n)$ for every $n \in \mathbb{N}_b$, then $F(n) = \chi_D(n)$ for every $n \in \mathbb{N}_D$.

Proof. Since F is a completely multiplicative function on $\mathbb{N}_a$ and $\mathbb{N}_b$, we have F is completely multiplicative function on $\mathbb{N}_D$. We prove that F is periodic $\pmod{D}$.

Since $F(n) = \chi_a(n)$ for every $n \in \mathbb{N}_a$ and $F(n) = \chi_b(n)$ for every $n \in \mathbb{N}_b$, we have F is periodic $\pmod{a}$ and $\pmod{b}$. Let $u, v \in \mathbb{N}$ such that $D = (a, b) = au - bv$. Then

$$F(n + D) = F(n + D + bv) = F(n + au) = F(n) \text{ for every } n \in \mathbb{N}_D.$$

Therefore, we have $F(N) = \chi_D(n)$. Lemma 3.3 is proved. $\square$

LEMMA 3.4. Let $k \in \mathbb{N}_0$ and

$$\mathcal{H}_k := \{\alpha \in \mathbb{N} \mid 2^\alpha \parallel a^2 + b^2 + c^2 + k \text{ for some } a, b, c \in \mathbb{N}\}.$$

Then

$$\mathcal{H}_k = \begin{cases} \{1, \dots, e + 2\} & \text{if } k = 2^e k_1, (k_1, 2) = 1, 2 \mid e \text{ and } k_1 \equiv 1 \pmod{8}, \\ \mathbb{N} \setminus \{e + 2\} & \text{if } k = 2^e k_1, (k_1, 2) = 1, 2 \mid e \text{ and } k_1 \equiv 5 \pmod{8}, \\ \mathbb{N} & \text{in any other cases.} \end{cases}$$

Proof. First we consider the case $k = 0$. Since

$$2^{2j} \parallel 2^{2j} + 2^{2j+2} + 2^{2j+2} \text{ and } 2^{2j+1} \parallel 2^{2j} + 2^{2j} + 2^{2j+2},$$

consequently $2j \in \mathcal{H}_0$ and $2j + 1 \in \mathcal{H}_0$ for every $j \in \mathbb{N}$. This shows that $\mathcal{H}_0 = \mathbb{N}$.

Let $k > 0, k = 2^e k_1, (k_1, 2) = 1$ and

$$\mathbb{M}_0 := \{a^2 + b^2 + c^2 \mid a, b, c \in \mathbb{N}_0\} \text{ and } \mathbb{M} := \{a^2 + b^2 + c^2 \mid a, b, c \in \mathbb{N}\}.$$

It is known that $n \in \mathbb{M}_0$ if and only if $n \notin S$ and if $\eta = a^2 + b^2 + c^2 \in \mathbb{M}_0$ with $a, b, c \in \mathbb{N}_0$, then

$$\eta_t = (a + 2^t)^2 + (b + 2^t)^2 + (c + 2^t)^2 \in \mathbb{M} \text{ and } \eta_t - \eta \equiv 0 \pmod{2^{t+1}}$$

hold for every $t \in \mathbb{N}$. Therefore, if $\alpha \in \mathbb{N}$ such that $2^\alpha \parallel \eta + k$ for some $\eta \in \mathbb{M}_0$, then $2^\alpha \parallel \eta_t + k$ for every $t \in \mathbb{N}, t \geq \alpha$. This means that

$$\text{If } 2^\alpha \parallel \eta + k \text{ for some } \eta \in \mathbb{M}_0, \text{ then } \alpha \in \mathcal{H}_k.$$

Since $k = 2^e k_1, (k_1, 2) = 1$, we consider the equation

$$\eta + 2^e k_1 = 2^\alpha r, (r, 2) = 1, \text{ where } \eta \in \mathbb{M}_0, \alpha, r \in \mathbb{N} \text{ are unknown.} \quad (3.2)$$

CASE 1. $2 \mid e$

CASE 1a. $2 \mid e, \alpha \geq e$.

If (3.2) is true, then $2^e \mid \eta$, and so

$$\theta = \frac{\eta}{e^e} \in \mathbb{M}_0 \iff \eta \in \mathbb{M}_0,$$

thus (3.2) is true if and only if

$$\theta + k_1 = 2^{\alpha-e}r, (r, 2) = 1, \theta \in \mathbb{M}_0, \alpha, r \in \mathbb{N}. \quad (3.3)$$

◦ If $\alpha \geq e + 3$, then

$$\theta \notin \mathbb{M}_0 \iff \theta = 2^{\alpha-e}r - k_1 \in S \iff k_1 \equiv 1 \pmod{8},$$

and so no exist solution $\theta \in \mathbb{M}_0$ if $\alpha \geq e + 3$ and $k_1 \equiv 1 \pmod{8}$ and in any other case (3.3) has a solution.

◦ If $\alpha = e + 2$, then

$$\theta \notin \mathbb{M}_0 \iff \theta = 2^2r - k_1 \in S \iff k_1 \equiv 5 \pmod{8}.$$

Therefore there is no solution $\theta \in \mathbb{M}_0$ if $\alpha = e + 2$ and $k_1 \equiv 5 \pmod{8}$ and in any other case (3.3) has a solution.

◦ If $\alpha = e + 1$, then $\eta = \overline{k_1}2^e \in \mathbb{M}_0$ is a solution of (3.2).

◦ If $\alpha = e$, then $\eta = 2^e + 2^e \in \mathbb{M}_0$ is a solution of (3.2).

CASE 1b. $2 \mid e, \alpha < e$.

Then $\eta = 2^e \in \mathbb{M}_0$ is a solution of (3.2).

CASE 2. $2 \nmid e$.

First we prove that $\{1, \dots, e\} \subset \mathcal{H}_k$. It is clear that $2 \parallel \eta_1 + 2^e k_1$ with η_1 is defined by

$$\eta_1 = \begin{cases} 2 = 1^2 + 1^2 \in \mathbb{M}_0 & \text{if } e > 1 \\ 4 = 2^2 \in \mathbb{M}_0 & \text{if } e = 1. \end{cases}$$

Let

$$\eta_2 = \begin{cases} 2^2 \in \mathbb{M}_0 & \text{if } e > 2 \\ 2^4 \in \mathbb{M}_0 & \text{if } e = 2 \\ (k_1 - 1)^2 + 1^2 + 1^2 \in \mathbb{M}_0 & \text{if } e = 1. \end{cases}$$

Then $2^2 \parallel \eta_2 + 2^e k_1$, because in the case $e = 1$, we have

$$\eta_2 + 2^e k_1 = (k_1 - 1)^2 + 1^2 + 1^2 + 2k_1 = 4 \left(\left(\frac{k_1 - 1}{2} \right)^2 + \frac{k_1 - 1}{2} + 1 \right).$$

Thus we have proved that $1 \in \mathcal{H}_k$ and $2 \in \mathcal{H}_k$.

Now for each $2j < e$ and $2\ell + 1 < e$, we have $2^{2j} \in \mathbb{M}_0, 2^{2\ell} + 2^{2\ell} \in \mathbb{M}_0$ and

$$2^{2j} \parallel 2^{2j} + 2^e k_1, 2^{2\ell+1} \parallel 2^{2\ell} + 2^{2\ell} + 2^e k_1 \text{ and } 2^e \parallel 2^{e+1} + 2^{e+1} + 2^{e+1} + 2^e k_1.$$

Thus, we proved that $\{1, \dots, e\} \subseteq \mathcal{H}_k$.

Finally, for every $\alpha \geq e + 1$, we infer from $2 \nmid e$ that $2^\alpha r - 2^e k_1 = 2^e(2^{\alpha-e}r - k_1) \notin S$, i.e. $\eta = 2^\alpha r - 2^e k_1 \in \mathbb{M}_0$. This shows that $\alpha \in \mathcal{H}_k$ for every $\alpha \geq e + 1$, which with the fact $\{1, \dots, e\} \subseteq \mathcal{H}_k$ shows that $\mathcal{H}_k = \mathbb{N}$.

The proof of Lemma 3.4 is finished. $\square$

4. PROOF OF THEOREM 1.2

Assume that $k \in \mathbb{N}_0, K \in \mathbb{C}$ and $F, H \in \mathcal{M}$ satisfy the relation (2.1) for every $a, b, c \in \mathbb{N}$. Then Theorem 1.1 shows that there are numbers $A, B, C, D_3 = 3C + K \in \mathbb{C}$ such that (2.2) and (2.3) are satisfied for every $m, a, b, c \in \mathbb{N}$. Furthermore (2.2)–(2.4) are true, i.e.

$$H(m) = Am^2 + B\chi_2(m) + C, \quad (4.1)$$

$$F(a^2 + b^2 + c^2 + k) = A(a^2 + b^2 + c^2) + B(\chi_2(a) + \chi_2(b) + \chi_2(c)) + D_3, \quad (4.2)$$

and

$$F(r^2n + k) = Ar^2n + \bar{n}B\chi_2(r) + D_3 \quad (4.3)$$

hold for every $m, n \in \mathbb{N}$, $n \notin S$ and $r \in \mathcal{A}$, where $\chi_2(r)$ is the principal character (mod 2) and $D_3 = 3C + K$.

We shall deduce from (4.2) and the fact $H \in \mathcal{M}$ that only the following three cases can occur:

$$\begin{cases} (1) & (A, B, C) = (0, 0, 1) \text{ and } H(m) = 1 & \text{for every } m \in \mathbb{N}, \\ (2) & (A, B, C) = (0, 1, 0) \text{ and } H(m) = \chi_2(m) & \text{for every } m \in \mathbb{N}, \\ (3) & (A, B, C) = (1, 0, 0) \text{ and } H(m) = m^2 & \text{for every } m \in \mathbb{N}. \end{cases} \quad (4.4)$$

Indeed, since $H \in \mathcal{M}$, we have

$$\begin{aligned} 0 &= H(nm) - H(n)H(m) = \\ &= An^2m^2 + B\chi_2(nm) + C - (An^2 + B\chi_2(n) + C)(Am^2 + B\chi_2(m) + C) = \\ &= (A - A^2)n^2m^2 - An^2(B\chi_2(m) + C) - Am^2(B\chi_2(n) + C) + B\chi_2(nm) + C - (B\chi_2(n) + C)(B\chi_2(m) + C) \end{aligned}$$

for every $n, m \in \mathbb{N}$, $(n, m) = 1$. This implies that

$$\begin{cases} A = A^2, B = B^2 \text{ and } C = C^2 \\ AB = AC = BC = 0 \end{cases}$$

Since $H(1) = A + B + C = 1$, these relations imply that

$$(A, B, C) \in \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\},$$

which proves (4.4).

In the proof of Theorem 1.2 we distinguish the cases 1, 2, 3 in (4.4).

(1) Assume that $(A, B, C) = (0, 0, 1)$ and $H(m) = 1$ for every $m \in \mathbb{N}$. Then we infer from (4.2) and (4.3) that

$$F(\eta + k) = D_3$$

hold for every $\eta \in \mathbb{M}$, where $D_3 = 3C + K = 3 + K$.

Since $4n + 1 \notin S$ and $4n + 2 \notin S$ for every $n \in \mathbb{N}$, we infer from (4.3) that

$$F(4r^2n + r^2 + k) = D_3 \text{ and } F(4r^2n + 2r^2 + k) = D_3 \quad (4.5)$$

is satisfied for every $n \in \mathbb{N}$ and $r \in \mathcal{A}$.

◦ If $D_3 = 0$, $K = -3$, then $H(m) = C = 1$ and $F(\eta + k) = 0$ for every $m \in \mathbb{N}$, $\eta \in \mathbb{M}$, which proves (1) of Theorem 1.2.

◦ If $D_3 \neq 0$, then it follows from (4.5) that

$$\begin{cases} F(2)F\left(2r^2n + r^2 + \frac{k}{2}\right) = D_3 & \text{if } \bar{k} = 0, (r, 2) = 1, \\ F(2)F\left(2r^2n + \frac{r^2+k}{2}\right) = D_3 & \text{if } \bar{k} = 1, (r, 2) = 1, \\ F(2^{\alpha_r})F\left(2r^2n + \frac{2r^2+k}{2^{\alpha_r}}\right) = D_3 & \text{if } \bar{k} = 2, (r, 2) = 1, \\ F(2^{\beta_r})F\left(2r^2n + \frac{r^2+k}{2^{\beta_r}}\right) = D_3 & \text{if } \bar{k} = 3, (r, 2) = 1, \end{cases}$$

where $2^{\alpha_r} \parallel 2r^2 + k$, $2^{\beta_r} \parallel r^2 + k$, $(\alpha_2 \geq 2, \beta_r \geq 2)$. These with Lemma 3.1 imply that

$$F(n) = \chi_{2r^2}(n) \text{ for every } n \in \mathbb{N}_{2r^2}, r \in \mathcal{A}, (r, 2) = 1,$$

which with $r = 13$ and $r = 15$ shows that

$$F(n) = \chi_{4 \cdot 13^2}(n) \text{ and } F(n) = \chi_{4 \cdot 15^2}(n).$$

Therefore Lemma 3.3 implies that

$$F(n) = \chi_{(2 \cdot 13^2, 2 \cdot 15^2)}(n) = \chi_2(n) = 1 \text{ for every } n \in \mathbb{N}_2,$$

which implies that $F(2n + 1) = 1$ for every $m \in \mathbb{N}$. Thus, we infer from Lemma 3.4 and the fact $F(\eta + k) = D_3$ that $F(2^\alpha) = D_3$ for every $\alpha \in \mathcal{H}_k$. The assertion (2) of Theorem 1.2 is proved.

(2) Assume now that $(A, B, C) = (0, 1, 0)$ and $H(m) = \chi_2(m)$ for every $m \in \mathbb{N}$. Then $D_3 = 3C + K = K$ and

$$F(a^2 + b^2 + c^2 + k) = \chi_2(a) + \chi_2(b) + \chi_2(c) + K \quad (4.6)$$

hold for every $a, b, c \in \mathbb{N}$, furthermore

$$F(4r^2n + r^2 + k) = \chi_2(r) + K \text{ and } F(4r^2n + 2r^2 + k) = 2\chi_2(r) + K \quad (4.7)$$

is satisfied for every $n \in \mathbb{N}$ and $r \in \mathcal{A}$. If r is odd, then at least one of $K + \chi_2(r) = K + 1$, or $K + 2\chi_2(r) = K + 2$ is not zero. Let $K_1 \in \{K + 1, K + 2\}$ such that $K_1 \neq 0$. Thus, it follows from (4.7) and Lemma 3.1 that

$$F(n) = \chi_{4r^2}(n) \text{ for every } n \in \mathbb{N}_{4r^2} \text{ and } r \in \mathcal{A}, (r, 2) = 1.$$

Since this relation holds for every $r \in \mathcal{A}, (r, 2) = 1$, then an application of Lemma 3.3 with $r = 13$ and $r = 15$, we have

$$F(n) = \chi_{(4 \cdot 13^2, 4 \cdot 15^2)}(n) = \chi_4(n) \text{ for every } n \in \mathbb{N}_4. \quad (4.8)$$

On the other hand, it follows from (4.7) that

$$\begin{cases} F(2)F\left(2r^2n + r^2 + \frac{k}{2}\right) = 2 + K & \text{if } \bar{k} = 0, (r, 2) = 1, \\ F(2)F\left(2r^2n + \frac{r^2+k}{2}\right) = 1 + K & \text{if } \bar{k} = 1, (r, 2) = 1, \\ F(2^{\alpha_r})F\left(2r^2n + \frac{2r^2+k}{2^{\alpha_r}}\right) = 2 + K & \text{if } \bar{k} = 2, (r, 2) = 1, \\ F(2^{\beta_r})F\left(2r^2n + \frac{r^2+k}{2^{\beta_r}}\right) = 1 + K & \text{if } \bar{k} = 3, (r, 2) = 1, \end{cases}$$

where $2^{\alpha_r} \parallel 2r^2 + k$, $2^{\beta_r} \parallel r^2 + k$, $(\alpha_r \geq 2, \beta_r \geq 2)$.

If

$$(\bar{k}, K) \notin \{(0, -2), (1, -1), (2, -2), (3, -1)\}, \quad (4.9)$$

then the above relations with Lemma 3.1 imply that

$$F(n) = \chi_{2r^2}(n) \text{ for every } n \in \mathbb{N}_{2r^2} \text{ and } r \in \mathcal{A}, (r, 2) = 1.$$

An application of Lemma 3.3 with $r = 13$ and $r = 15$ shows that

$$F(n) = \chi_{(2 \cdot 13^2, 2 \cdot 15^2)}(n) = \chi_2(n) = 1 \text{ for every } n \in \mathbb{N}_2,$$

consequently

$$F(2n + 1) = 1 \text{ for every } n \in \mathbb{N}.$$

The assertion (3) of Theorem 1.2 is proved for the cases in (4.9).

Now we consider the following cases: $(\bar{k}, K) \in \{(0, -2), (1, -1), (2, -2), (3, -1)\}$.

It is clear to check that $8n+3 \notin S$ for every $n \in \mathbb{N}_0$ and so Lemma 2.1 shows that $8n+3 = a^2 + b^2 + c^2$, $a, b, c \in \mathbb{Z}$. Since $a^2 + b^2 + c^2 \equiv 3 \pmod{8}$, we have $a, b, c \in \mathbb{N}$ and $a \equiv b \equiv c \equiv 1 \pmod{2}$. Thus, we infer from (4.6) that

$$F(8n + 3 + k) = \chi_2(a) + \chi_2(b) + \chi_2(c) + K = 3 + K \neq 0 \text{ if } K \in \{-1, -2\} \quad (4.10)$$

for every $n \in \mathbb{N}$.

If $\bar{k} = 1$, then $3 + k = 2^\beta t$, $(t, 2) = 1$, $\beta \geq 2$ and by applying (4.10) with $n = 2^{\beta-2}t$, we have

$$F(2^\beta)F(2t + t) = 3 + K \neq 0 \text{ for every } t \in \mathbb{N}.$$

This with Lemma 3.1 imply that

$$F(n) = \chi_2(n) = 1 \text{ for every } n \in \mathbb{N}_2,$$

which proves (3) of Theorem 1.2.

If $\bar{k} = 0$, then (4.8) and (4.10) with the facts $K \in \{-1, -2\}$ implies that

$$\chi_4(3) = F(8n + 3 + k) = 3 + K \in \{2, 1\},$$

which implies that $\chi_4(3) = 1$. Thus, this relation with (4.8) proves that $F(n) = \chi_2(n) = 1$ for every $n \in \mathbb{N}_2$.

Now we consider the case $(\bar{k}, K) = (2, -2)$, $k = 4t + 2$, $t \in \mathbb{N}_0$. Since $2r \in \mathcal{A}$ for every $r \in \mathcal{A}$, therefore from (4.7) we obtain that

$$\begin{aligned} F(2)F(8r^2n + 2(r^2 + t) + 1) &= F(4(2r)^2n + (2r)^2 + k) \\ &= \chi_2(2r) + K = -2 \text{ for every } n \in \mathbb{N}, r \in \mathcal{A}. \end{aligned} \quad (4.11)$$

We can find $r_1, r_2 \in \{13, 26\} \subset \mathcal{A}$ such that $r_1^2 + t$ is odd and $r_2^2 + t$ is even. Consequently, (4.8) and (4.11) show that

$$F(2)F(8r_1^2n + 2(r_1^2 + t) + 1) = F(2)\chi_4(3) = -2$$

and

$$F(2)F(8r_2^2n + 2(r_2^2 + t) + 1) = F(2)\chi_4(1) = F(2) = -2.$$

These imply that $\chi_4(3) = 1$, and so (4.8) implies that $F(n) = \chi_2(n) = 1$ for every $n \in \mathbb{N}_2$. The assertion (3) of Theorem 1.2 is proved.

◦ Finally, we consider the case $(\bar{k}, K) = (3, -1)$. Let $k = 4q + 3$, $q \in \mathbb{N}_0$ and $A, B, C \in \mathbb{N}$. Then, by setting $a = 2A$, $b = 2B$ and $c = 2C$ into (4.6), we infer from (4.8) that

$$\begin{aligned} \chi_4(3) &= \chi_4((2A)^2 + (2B)^2 + (2C)^2 + 4q + 3) = \\ &= F((2A)^2 + (2B)^2 + (2C)^2 + 4q + 3) = \\ &= \chi_2(2A) + \chi_2(2B) + \chi_2(2C) - 1 = -1, \end{aligned}$$

which shows that

$$F(2n + 1) = (-1)^n \text{ for every } n \in \mathbb{N}. \quad (4.12)$$

Now we prove that

$$F(2) = (-1)^{q+1} \cdot 2 = (-1)^{\frac{k+1}{4}} \cdot 2 \text{ and } F(2^\alpha) = 0 \text{ for every } \alpha \geq 2. \quad (4.13)$$

By setting $a = 2A + 1$, $b = 2B + 1$ and $c = 2C + 1$ into (4.6), we infer from (4.12) that

$$\begin{aligned} F(2)(-1)^{q+1} &= F(2)F(4t + 2q + 3) = F(8t + 4q + 6) = \\ &= F((2A + 1)^2 + (2B + 1)^2 + (2C + 1)^2 + 4q + 3) = \\ &= \chi_2(2A + 1) + \chi_2(2B + 1) + \chi_2(2C + 1) - 1 = 2, \end{aligned}$$

where $t = \frac{A(A+1)}{2} + \frac{B(B+1)}{2} + \frac{C(C+1)}{2}$. The last relation proves that $F(2) = (-1)^{q+1} \cdot 2$.

It can prove that if $\bar{k} = 3$, then for each $\alpha \in \mathbb{N}$, $\alpha \geq 2$ there are $a, b, c \in \mathbb{N}$ such that $2 \mid a$, $2 \mid b$, $(c, 2) = 1$ and $2^\alpha \parallel a^2 + b^2 + c^2 + k$. Therefore, (4.6) implies that

$$F(2^\alpha)F\left(\frac{a^2 + b^2 + c^2 + k}{2^\alpha}\right) = F(a^2 + b^2 + c^2 + k) = \chi_2(a) + \chi_2(b) + \chi_2(c) - 1 = 0,$$

which completes the proof of (4.13). The proof of (4) is finished.

(3) Assume now that $(A, B, C) = (1, 0, 0)$ and $H(m) = m^2$ for every $m \in \mathbb{N}$. Then $D_3 = 3C + K = K$ and

$$F(\eta + k) = \eta + K$$

hold for every $\eta \in \mathbb{M}$, furthermore

$$F(4r^2n + r^2 + k) = 4r^2n + r^2 + K \text{ and } F(4r^2n + 2r^2 + k) = 4r^2n + 2r^2 + K \quad (4.14)$$

is satisfied for every $n \in \mathbb{N}$ and $r \in \mathcal{A}$. By using the assertion (c) of Lemma 3.2, we infer from (4.14) that

$$k = K \text{ and } F(n) = \chi_{4r^2}(n)n \text{ for every } n \in \mathbb{N}_{4r^2} \text{ and } r \in \mathcal{A}.$$

On the other hand, it follows from (4.14) that

$$\begin{cases} F(2)F\left(2r^2n + r^2 + \frac{k}{2}\right) = 4r^2n + 2r^2 + k & \text{if } \bar{k} = 0, (r, 2) = 1, \\ F(2)F\left(2r^2n + \frac{r^2+k}{2}\right) = 4r^2n + r^2 + k & \text{if } \bar{k} = 1, (r, 2) = 1, \\ F(2^{\alpha_r})F\left(2r^2n + \frac{2r^2+k}{2^{\alpha_r}}\right) = 4r^2n + 2r^2 + k & \text{if } \bar{k} = 2, (r, 2) = 1, \\ F(2^{\beta_r})F\left(2r^2n + \frac{r^2+k}{2^{\beta_r}}\right) = 4r^2n + r^2 + k & \text{if } \bar{k} = 3, (r, 2) = 1, \end{cases}$$

where $2^{\alpha r} \parallel 2r^2 + k$, $2^{\beta r} \parallel r^2 + k$, ($\alpha_2 \geq 2$, $\beta_r \geq 2$). These imply that

$$F(n) = \chi_{2r^2}(n)n \text{ for every } n \in \mathbb{N}_{2r^2} \text{ and } r \in \mathcal{A}.$$

We apply this relation for $r = 13$ and $r = 15$, Lemma 3.3 gives

$$F(n) = \chi_{(2 \cdot 13^2, 2 \cdot 15^2)}(n)n = \chi_2(n)n \text{ for every } n \in \mathbb{N}_2,$$

which implies that

$$F(2n+1) = 2n+1 \text{ for every } n \in \mathbb{N}.$$

Finally, we infer from Lemma 3.4 that for each $\alpha \in \mathcal{H}_k$ there is a $\eta \in \mathbb{M}$ such that $\eta + k = 2^\alpha r$, ($r, 2$) = 1. Thus, we have

$$2^\alpha r = \eta + k = F(\eta + k) = F(2^\alpha)F(r) = F(2^\alpha)r,$$

and so $F(2^\alpha) = 2^\alpha$.

The assertion (5) and Theorem 1.2 is proved. $\square$

5. PROOFS OF COROLLARIES

Proof of Corollary 1.3. Assume that $\ell \in \mathbb{N}$, $\ell \geq 3$ and the arithmetical functions $F: \mathbb{N} \rightarrow \mathbb{C}$ satisfy the relation

$$F(n_1^2 + \dots + n_\ell^2) = F(n_1)^2 + \dots + F(n_\ell)^2$$

holds for every $n_1, \dots, n_\ell \in \mathbb{N}$.

From Theorem 1.1, there are numbers $A, B, C \in \mathbb{C}$ such that

$$F(m)^2 = Am^2 + B\chi_2(m) + C$$

and

$$F(n_1^2 + \dots + n_\ell^2) = A(n_1^2 + \dots + n_\ell^2) + B(\chi_2(n_1) + \dots + \chi_2(n_\ell)) + \ell C,$$

hold for every $m, n_1, \dots, n_\ell \in \mathbb{N}$. These imply that

$$\begin{aligned} & \left(A(n_1^2 + \dots + n_\ell^2) + B(\chi_2(n_1) + \dots + \chi_2(n_\ell)) + \ell C \right)^2 = \\ & = A(n_1^2 + \dots + n_\ell^2)^2 + B\chi_2(n_1^2 + \dots + n_\ell^2) + C \end{aligned}$$

holds for every $n_1, \dots, n_\ell \in \mathbb{N}$. This equation implies the following relations

$$\begin{cases} A^2 = A \\ A(B(\chi_2(n_1) + \dots + \chi_2(n_\ell)) + \ell C) = 0 \\ (B(\chi_2(n_1) + \dots + \chi_2(n_\ell)) + \ell C)^2 = B\chi_2(n_1^2 + \dots + n_\ell^2) + C \end{cases}$$

hold for every $n_1, \dots, n_\ell \in \mathbb{N}$. The solutions of this system are:

$$(A, B, C) \in \left\{ (0, 0, 0), \left(0, 0, \frac{1}{\ell^2} \right), (1, 0, 0) \right\}.$$

If $(A, B, C) = (0, 0, 0)$, then $F(n) = 0$ for every $n \in \mathbb{N}$. If $(A, B, C) = \left(0, 0, \frac{1}{\ell^2} \right)$, then $F(n)^2 = \frac{1}{\ell^2}$, $F(n) = \frac{\varepsilon(n)}{\ell}$, where $\varepsilon(n) \in \{-1, 1\}$ and $\varepsilon(n_1^2 + \dots + n_\ell^2) = 1$. $(A, B, C) = (1, 0, 0)$, then $F(n)^2 = n^2$, $F(n) = \varepsilon(n)n$.

Thus, the proof of Corollary 1.3 is complete. $\square$

Proof of Corollary 1.4. Assume that $\ell \in \mathbb{N}$, $\ell \geq 4$ and the arithmetical functions $F: \mathbb{N} \rightarrow \mathbb{C}$ satisfy the relation

$$F(n_1^2 + \dots + n_\ell^2) = F(n_1)^2 + \dots + F(n_\ell)^2$$

holds for every $n_1, \dots, n_\ell \in \mathbb{N}$. Similarly as the proof of Corollary 1.3, there are numbers $A, B, C \in \mathbb{C}$ such that

$$F(m^2) = Am^2 + B\chi_2(m) + C \quad (5.1)$$

holds for every $m \in \mathbb{N}$, and so

$$F(n_1^2 + \dots + n_\ell^2) = A(n_1^2 + \dots + n_\ell^2) + B(\chi_2(n_1) + \dots + \chi_2(n_\ell)) + \ell C. \quad (5.2)$$

Since

$$(2\ell - 3)^2 = (2\ell - 4)^2 + \underbrace{2^2 + \dots + 2^2}_{\ell-2} + 1^2,$$

we infer from (5.1) and (5.2) that

$$F((2\ell - 3)^2) = A(2\ell - 3)^2 + B\chi_2(2\ell - 3) + C$$

and

$$F((2\ell - 3)^2) = A(2\ell - 3)^2 + B\chi_2(1) + \ell C.$$

These imply that $\ell C = C$, consequently $C = 0$. Finally, since $\ell \geq 3$, we have

$$\begin{cases} \left(\frac{\ell}{2}\right)^2 = \left(\frac{\ell-2}{2}\right)^2 + \underbrace{1^2 + \dots + 1^2}_{\ell-1} & \text{if } 2 \mid \ell \\ \left(\frac{\ell+3}{2}\right)^2 = \left(\frac{\ell+1}{2}\right)^2 + \underbrace{1^2 + \dots + 1^2}_{\ell-2} + 2^2 & \text{if } 2 \nmid \ell, \end{cases}$$

which with (5.1) and (5.2) imply

$$\begin{cases} A\left(\frac{\ell}{2}\right)^2 + B\chi_2\left(\frac{\ell}{2}\right) = A\left(\frac{\ell}{2}\right)^2 + B\left(\chi_2\left(\frac{\ell-2}{2}\right) + \ell - 1\right) & \text{if } 2 \mid \ell \\ A\left(\frac{\ell+3}{2}\right)^2 + B\chi_2\left(\frac{\ell+3}{2}\right) = A\left(\frac{\ell+3}{2}\right)^2 + B\left(\chi_2\left(\frac{\ell+1}{2}\right) + \ell - 2\right) & \text{if } 2 \nmid \ell. \end{cases}$$

One can check easily that these relations with $\ell \geq 4$ imply that $B = 0$, consequently $F(m^2) = m^2$ for every $m \in \mathbb{N}$. Hence, from (5.2), we have

$$F(n_1^2 + \dots + n_\ell^2) = A(n_1^2 + \dots + n_\ell^2)$$

for every $n_1, \dots, n_\ell \in \mathbb{N}$. The proof of Corollary 1.4 is finished. $\square$

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