THE METRICS $G = ag^S + bg^H + cg^V$ ON THE TANGENT BUNDLE OF A WEYL MANIFOLD

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ABSTRACT

Let $(M, [g])$ be a Weyl manifold and TM be its tangent bundle equipped with Riemannian g -natural metrics which are linear combinations of Sasaki, horizontal and vertical lifts of the base metric with constant coefficients. The aim of this paper is to construct a Weyl structure on TM and to show that TM cannot be Einstein-Weyl even if (M, g) is flat.

KEYWORDS

Riemannian metric, Weyl structure, tangent bundle

MATHEMATICS SUBJECT CLASSIFICATION (2020)

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1. INTRODUCTION

A Weyl manifold is a conformal manifold equipped with an affine connection preserving the conformal structure, called a Weyl connection. It is said to be Einstein-Weyl if and only if the symmetric part of Ricci tensor is proportional to a Riemannian metric in the conformal class (see [6],[7],[9]). On the other hand, the Einstein-Weyl manifolds play an important role in physics as the pure Einstein theory is too strong as a system model for various physical questions. On the contrary, Einstein-Weyl manifolds appear naturally as the background of the static Yang-Mills-Higgs theory (for a physical survey of Weyl geometry see [10]).

In tangent bundles, there are classic metrics such as Sasaki and Cheeger-Gromoll as well as more general g -natural metrics (see [1],[3],[8]). A general metric is defined by Abbassi and Sarik in [2] using three classical lifts, namely Sasaki, horizontal and vertical lifts of the base metric.

In their very recent paper [4], Bejan and Gul constructed a Weyl structure on the tangent bundle and find conditions under which the tangent bundle is an Einstein-Weyl manifold. As a generalization, in [5], Bejan *et al.* obtained some conditions such that the Weyl structure on the tangent bundle to be a gradient Weyl-Ricci soliton. In both studies, the tangent bundle is considered with the Sasaki metric.

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In this paper, we consider the tangent bundle endowed with the metric in [2] and introduce a Weyl structure on it. We will prove that the tangent bundle cannot be an Einstein-Weyl manifold with respect to this structure.

Unless otherwise stated, throughout the paper, the Einstein summation convention is used and all geometric objects are assumed as smooth.

2. PRELIMINARIES

We review some background material related to Weyl geometry from [4]. Let M be an m -dimensional manifold endowed with a conformal class of Riemannian metrics $[g]$. A torsion-free connection D is said to be a Weyl connection if it preserves the conformal class $[g]$. For a Riemannian metric $g \in [g]$, there exists a 1-form ω determined by D as $Dg = -2\omega \otimes g$. If ∇ is the Levi-Civita connection of g , then D is given as follows:

$$D_X Y = \nabla_X Y + \omega(Y)X + \omega(X)Y - g(X, Y)\xi, \quad \forall X, Y \in \Gamma(TM), \quad (2.1)$$

where ξ is the dual vector field of ω with respect to g . Conversely, when ω is given, if we use the equation (2.1) to define D , then D is a Weyl connection.

We also have $g(\xi, \xi) = \|\xi\|^2 = \omega(\xi)$ and the relation (2.1) is invariant under the Weyl transformation $e \rightarrow e^{2f}g$, $\omega' = \omega - df$. The pair (g, ω) is called a Weyl structure on M .

Denote by $R_g = [\nabla, \nabla] - \nabla_{[\cdot]} \nabla_{[\cdot]}$ and $R_{[g]} = [D, D] - D_{[\cdot]} D_{[\cdot]}$ the curvature tensors of the Levi-Civita connection ∇ and the Weyl connection D , respectively. Then the relation between R_g and $R_{[g]}$ is given by

$$\begin{aligned} R_{[g]}(X, Y)Z &= R_g(X, Y)Z + d\omega(X, Y)Z - ((\nabla_Y \omega)(Z))X + ((\nabla_X \omega)(Z))Y \\ &\quad + \omega(Y)\omega(Z)X - g(Y, Z)\nabla_X \xi - g(Y, Z)\omega(\xi)X \\ &\quad + g(Y, Z)\omega(X)\xi - \omega(X)\omega(Z)Y + g(X, Z)\nabla_Y \xi \\ &\quad + g(X, Z)\omega(\xi)Y - g(X, Z)\omega(Y)\xi, \end{aligned} \quad (2.2)$$

$\forall X, Y, Z \in \Gamma(TM)$.

From the above equation, the relation between the Ricci tensor field $\text{Ric}_{[g]}$ of the Weyl connection D and the Ricci tensor field Ric_g of the Levi-Civita connection ∇ is given by

$$\begin{aligned} \text{Ric}_{[g]}(X, Y) &= \text{Ric}_g(X, Y) + d\omega(X, Y) + (\delta\omega - (m-2)\|\xi\|^2)g(X, Y) \\ &\quad - (m-2)(\nabla_X \omega)Y + (m-2)\omega(X)\omega(Y), \end{aligned}$$

$\forall X, Y \in \Gamma(TM)$, where the co-differential $\delta\omega$ of ω is defined by $\delta\omega = -\text{tr}_g\{(U, V) \rightarrow (\nabla_U \omega)V\}$.

The symmetric part $\text{Ric}_{[g]}^{\text{sym}}$ of $\text{Ric}_{[g]}$ is given by following formula:

$$\begin{aligned} \text{Ric}_{[g]}^{\text{sym}}(X, Y) &= \text{Ric}_g(X, Y) + (\delta\omega - (m-2)\|\xi\|^2)g(X, Y) \\ &\quad - \frac{1}{2}(m-2)[(\nabla_X \omega)Y + (\nabla_Y \omega)X] + (m-2)\omega(X)\omega(Y), \end{aligned} \quad (2.3)$$

$\forall X, Y \in \Gamma(TM)$.

3. TANGENT BUNDLE

Let M denotes an m -dimensional manifold. Its tangent bundle is denoted by TM and $\pi : TM \rightarrow M$ is the natural projection mapping. It is known that TM is a $2m$ -dimensional differentiable manifold. If $(U, x^j); j = 1, \dots, m$ is a local chart on M , then the local chart $(\pi^{-1}(U), x^j, u^j)$ is induced on TM , where $x^j = x^j \circ \pi$ and u^j are the vector space coordinates of $v \in \pi^{-1}(U) \subset TM$ with respect to the natural local frame $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m})$ i.e. $v = u^j (\frac{\partial}{\partial x^j})_{\pi(v)}$.

If $X = X^i \frac{\partial}{\partial x^i}$ is the local expression of a vector field X in U , then the vertical lift X^V and the horizontal lift X^H of X are given, with respect to the induced coordinates, by

$$X^V = X^i \frac{\partial}{\partial u^i}, \quad X^H = X^i \frac{\partial}{\partial x^i} - X^j \Gamma_{jk}^i u^k \frac{\partial}{\partial u^i},$$

where Γ_{ij}^h are the Christoffel symbols of a torsion-free affine connection ∇ .

If f is a function on M , then the vertical lift f^V of f is defined by $f^V = f \circ \pi$. The horizontal lift f^H of f is given by $f^H = 0$.

Let ω be a 1-form on M . Then the horizontal lift ω^H of ω is given by $\omega^H(X^H) = 0$, $\omega^H(X^V) = (\omega(X))^V$. The vertical lift ω^V of ω is given by $\omega^V(X^V) = 0$, $\omega^V(X^H) = (\omega(X))^V$. For further details on tangent bundle we refer to [11].

The Riemannian metric $G = ag^S + bg^H + cg^V$ on the tangent bundle TM over the Riemannian manifold (M, g) is defined by the following three equations:

$$\begin{aligned} G(X^H, Y^H) &= (a+c)(g(X, Y))^V, \\ G(X^V, Y^H) &= G(X^H, Y^V) = b(g(X, Y))^V, \\ G(X^V, Y^V) &= a(g(X, Y))^V, \end{aligned} \quad (3.1)$$

$\forall X, Y \in \Gamma(TM)$, where a, b, c are constants which satisfy $a > 0$ and $a(a+c) - b^2 > 0$ ([2]).

For the Levi-Civita connection $\bar{\nabla}$ of the metrics G , we have the following proposition.

PROPOSITION 3.1 ([2]). Let (M, g) be a Riemannian manifold, ∇ its Levi-Civita connection and R_g its curvature tensor. Then the corresponding Levi-Civita connection $\bar{\nabla}$ of the metric $G = ag^S + bg^H + cg^V$ is characterized by the following equalities

$$\begin{aligned} \bar{\nabla}_{X^H} Y^H &= (\nabla_X Y)^H + \frac{ab}{2\alpha} [R_g(u, X) + R_g(u, Y)X]^H \\ &\quad + \frac{b^2}{\alpha} [R_g(X, u)Y - \frac{a(a+c)}{2\alpha} R_g(X, Y)u]^V, \\ \bar{\nabla}_{X^H} Y^V &= -\frac{a^2}{2\alpha} (R_g(Y, u)X)^H + (\nabla_X Y)^V + \frac{ab}{2\alpha} (R_g(Y, u)X)^V, \\ \bar{\nabla}_{X^V} Y^H &= -\frac{a^2}{2\alpha} (R_g(X, u)Y)^H + \frac{ab}{2\alpha} (R_g(X, u)Y)^V, \\ \bar{\nabla}_{X^V} Y^V &= 0, \end{aligned} \quad (3.2)$$

$\forall X, Y \in \Gamma(TM)$, where $\alpha = a(a+c) - b^2$ and the inequalities $a > 0$ and $a(a+c) - b^2 > 0$ hold.

For the Ricci tensor $\bar{\text{Ric}}_G$ of the metric G , we write the proposition below.

PROPOSITION 3.2 ([2]). Let (M, g) be a Riemannian manifold, ∇ its Levi-Civita connection, R_g its curvature tensor and Ric_g its Ricci tensor. Then the corresponding Ricci tensor $\bar{\text{Ric}}_G$ of the metric $G = ag^S + bg^H + cg^V$ is characterized by the following equalities

$$\begin{aligned} \bar{\text{Ric}}_G(X^H, Y^H) &= \left(1 - \frac{b^2}{\alpha}\right) \text{Ric}_g(X, Y) \\ &\quad + \frac{ab}{2\alpha} [2(\nabla_u \text{Ric}_g)(X, Y) - (\nabla_Y \text{Ric}_g)(u, X) - (\nabla_X \text{Ric}_g)(u, Y)] \\ &\quad - \frac{a^2}{2\alpha} \sum_{i=1}^m g(R_g(u, E_i)X, R_g(u, E_i)Y) \\ &\quad + \left(\frac{a^2 b^2}{4\alpha^2} - \frac{a^2 b^4}{4\alpha^3}\right) \sum_{i=1}^m g(R_g(u, X)E_i, R_g(u, Y)E_i) \\ &\quad + \frac{a^2 b^4}{4\alpha^3} \sum_{i=1}^m [g(R_g(u, E_i)X, R_g(u, Y)E_i) \\ &\quad + g(R_g(u, E_i)Y, R_g(u, X)E_i)], \end{aligned} \quad (3.3)$$

$$\begin{aligned}\overline{\text{Ric}}_G(X^H, Y^V) &= -\frac{ab}{2\alpha} \text{Ric}_g(X, Y) + \frac{a^2}{2\alpha} [(\nabla_u \text{Ric}_g)(X, Y) - (\nabla_Y \text{Ric}_g)(u, X)] \\ &\quad + \frac{a^3 b}{4\alpha^2} \sum_{i=1}^m g(R_g(u, X)E_i, R_g(u, Y)E_i), \\ \overline{\text{Ric}}_G(X^V, Y^V) &= \frac{a^4}{4\alpha^2} \sum_{i=1}^m g(R_g(u, X)E_i, R_g(u, Y)E_i),\end{aligned}\quad (3.4)$$

$\forall X, Y \in \Gamma(TM)$, where $\{E_1, \dots, E_m\}$ is an orthonormal basis on M .

4. A WEYL STRUCTURE ON TM

In this section, we construct a Weyl structure on (TM, G) using the vertical lift of a 1-form on M . We have the following proposition from the definition of the metric G in (3.1).

PROPOSITION 4.1. Let (M, g) be a Riemannian manifold and TM its tangent bundle with the metric $G = ag^S + bg^H + cg^V$. Any conformal change $g \rightarrow e^{2f}g$ on M corresponds the change of the metric $G \rightarrow (e^{2f})^V G$ on TM .

Now we can express the proposition below.

PROPOSITION 4.2. Let (M, g) be a Riemannian manifold and TM its tangent bundle with the metric $G = ag^S + bg^H + cg^V$. If the pair (g, ω) is a Weyl structure on M , then the pair (G, ω^V) is a Weyl structure on TM and its Weyl connection is given by

$$\begin{aligned}\overline{D}_{X^H} Y^H &= (D_X Y)^H + \left[\frac{ab}{2\alpha} (R_g(u, X) + R_g(u, Y)X) - (a + c - 1)g(X, Y)\xi \right]^H \\ &\quad + \frac{b^2}{\alpha} \left[R_g(X, u)Y - \frac{a(a + c)}{2\alpha} R_g(X, Y)u \right]^V, \\ \overline{D}_{X^H} Y^V &= -\frac{a^2}{2\alpha} [(R_g(Y, u)X) + bg(X, Y)\xi]^H \\ &\quad + \left[D_X Y - \omega(Y)X + g(X, Y)\xi + \frac{ab}{2\alpha} R_g(Y, u)X \right]^V, \\ \overline{D}_{X^V} Y^H &= \left[-\frac{a^2}{2\alpha} R_g(X, u)Y - bg(X, Y)\xi \right]^H + \left[\frac{ab}{2\alpha} R_g(X, u)Y + \omega(Y)X \right]^V, \\ \overline{D}_{X^V} Y^V &= -ag(X, Y)\xi^h,\end{aligned}\quad (4.1)$$

where D is the Weyl connection on M , R_g is the curvature tensor field of g and ξ is the dual vector field of ω with respect to g .

Proof. Using the relations (3.2) in (2.1) and direct computations give the result. Here, we will prove only the first statement.

$$\begin{aligned}\overline{D}_{X^H} Y^H &= \overline{\nabla}_{X^H} Y^H + \omega^V(Y^H)X^H + \omega^V(X^H)Y^H - G(X^H, Y^H)\xi \\ &= (\nabla_X Y)^H + \frac{ab}{2\alpha} [R_g(u, X) + R_g(u, Y)X]^H + \frac{b^2}{\alpha} \left[R_g(X, u)Y - \frac{a(a + c)}{2\alpha} R_g(X, Y)u \right]^V \\ &\quad + (\omega(Y)X)^H + (\omega(X)Y)^H - (a + c)(g(X, Y)\xi)^H \\ &= [\nabla_X Y + \omega(Y)X + \omega(X)Y - g(X, Y)\xi]^H \\ &\quad + \left[\frac{ab}{2\alpha} (R_g(u, X) + R_g(u, Y)X) - (a + c - 1)g(X, Y)\xi \right]^H \\ &\quad + \frac{b^2}{\alpha} \left[R_g(X, u)Y - \frac{a(a + c)}{2\alpha} R_g(X, Y)u \right]^V\end{aligned}$$

$$\begin{aligned}
&= (D_X Y)^H + \left[\frac{ab}{2\alpha} (R_g(u, X) + R_g(u, Y)X) - (a + c - 1)g(X, Y)\xi \right]^H \\
&\quad + \frac{b^2}{\alpha} [R_g(X, u)Y - \frac{a(a+c)}{2\alpha} R_g(X, Y)u]^V. \quad \square
\end{aligned}$$

LEMMA 4.3. Let M be an m -dimensional manifold ($m > 2$) endowed with the Weyl structure (g, ω) and TM its tangent bundle endowed with the Weyl structure (G, ω^V) , where $G = ag^S + bg^H + cg^V$. The symmetric part $\overline{\text{Ric}}_{[G]}^{sym}$ of the Ricci tensor field of the Weyl structure (G, ω^V) satisfies the following relations

$$\begin{aligned}
\overline{\text{Ric}}_{[G]}^{sym}(X^H, Y^H) &= \overline{\text{Ric}}_G(X^H, Y^H) + (a + c)(\delta\omega - 2(m - 1)(a + c))\|\xi\|^2 g(X, Y) \\
&\quad - (m - 1)[(\nabla_X \omega)Y + (\nabla_Y \omega)X] + 2(m - 1)\omega(X)\omega(Y), \quad (4.2)
\end{aligned}$$

$$\begin{aligned}
\overline{\text{Ric}}_{[G]}^{sym}(X^V, Y^H) &= \overline{\text{Ric}}_G(X^V, Y^H) + b(\delta\omega - 2(m - 1)(a + c))\|\xi\|^2 g(X, Y) \\
&\quad - \frac{a^2}{2\alpha}(m - 1)[\omega(R(X, u)Y) + \omega(R(Y, u)X)], \quad (4.3)
\end{aligned}$$

$$\overline{\text{Ric}}_{[G]}^{sym}(X^V, Y^V) = \overline{\text{Ric}}_G(X^V, Y^V) + a(\delta\omega - 2(m - 1)(a + c))\|\xi\|^2 g(X, Y), \quad (4.4)$$

where ∇ is the Levi-Civita connection on M and $\overline{\text{Ric}}_G$ is the Ricci tensor field of G .

Proof. We use the formulas (2.3) and (3.1). Since TM is a $2m$ -dimensional manifold, we have

$$\begin{aligned}
\overline{\text{Ric}}_{[G]}^{sym}(X^H, Y^H) &= \overline{\text{Ric}}_G(X^H, Y^H) \\
&\quad + (\delta(\omega^V) - 2(m - 1)G(\xi^H, \xi^H))G(X^H, Y^H) \\
&\quad - (m - 1)[(\overline{\nabla}_{X^H} \omega^V)Y^H + (\overline{\nabla}_{Y^H} \omega^V)X^H] \\
&\quad + 2(m - 1)\omega^V(X^H)\omega^V(Y^H) \\
&= \overline{\text{Ric}}_G(X^H, Y^H) \\
&\quad + [(a + c)(\delta\omega - 2(m - 1)(a + c))\|\xi\|^2 g(X, Y)]^V \\
&\quad - (m - 1)[((\nabla_X \omega)Y)^V + ((\nabla_Y \omega)X)^V] \\
&\quad + 2(m - 1)[\omega(X)\omega(Y)]^V \\
&= \overline{\text{Ric}}_G(X^H, Y^H) \\
&\quad + (a + c)(\delta\omega - 2(m - 1)(a + c))\|\xi\|^2 g(X, Y) \\
&\quad - (m - 1)[(\nabla_X \omega)Y + (\nabla_Y \omega)X] \\
&\quad + 2(m - 1)\omega(X)\omega(Y).
\end{aligned}$$

By the similar way, we get (4.3) and (4.4). $\square$

The main result of the paper is given in the following theorem.

THEOREM 4.4. Let M be an m -dimensional manifold ($m > 2$) endowed with the Weyl structure (g, ω) and TM its tangent bundle endowed with the Weyl structure (G, ω^V) , where $G = ag^S + bg^H + cg^V$. In this case, TM cannot be an Einstein-Weyl manifold.

Proof. It is known that TM is an Einstein-Weyl manifold if and only if there exists a function $\tilde{f}$ such that $\overline{\text{Ric}}_{[G]}^{sym} = \tilde{f}G(\tilde{X}, \tilde{Y})$ for all vector fields $\tilde{X}, \tilde{Y}$ on TM. To find such a function, we should get rid of the terms except for including $g(X, Y)$ in (2.3). For this, assume that $(\nabla_X \omega)Y + (\nabla_Y \omega)X = 2\omega(X)\omega(Y)$. Then (2.3) becomes

$$\text{Ric}_{[g]}^{sym}(X, Y) = \text{Ric}_g(X, Y) + (\delta\omega - (m - 2)\|\xi\|^2)g(X, Y), \quad (4.5)$$

$\forall X, Y \in \Gamma(TM)$. Moreover, if we suppose M is flat, i.e., $R_g = 0$, then the formulas (4.2), (4.3) and (4.4) reduce to

$$\begin{aligned}\overline{\text{Ric}}_{[G]}^{\text{sym}}(X^H, Y^H) &= (a + c)(\delta\omega - 2(m - 1)(a + c))\|\xi\|^2 g(X, Y), \\ \overline{\text{Ric}}_{[G]}^{\text{sym}}(X^V, Y^H) &= b(\delta\omega - 2(m - 1)(a + c))\|\xi\|^2 g(X, Y), \\ \overline{\text{Ric}}_{[G]}^{\text{sym}}(X^V, Y^V) &= a(\delta\omega - 2(m - 1)(a + c))\|\xi\|^2 g(X, Y).\end{aligned}$$

Since assuming $a + c = b = a$ contradicts the definition of the metric G , we cannot find a function $\tilde{f}$ which satisfies $\overline{\text{Ric}}_{[G]}^{\text{sym}} = \tilde{f}G(\tilde{X}, \tilde{Y})$. So, TM cannot be an Einstein-Weyl manifold when it is endowed with the Weyl structure (G, ω^V) . $\square$

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