THE “ $k = 1$ ” CASE OF A PROBLEM OF GREENE AND KLEITMAN FROM 1976: JOIN-IRREDUCIBLE ELEMENTS IN THE LATTICE OF SPERNER 1-FAMILIES

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ABSTRACT

Let $k \geq 1$. A *Sperner k -family* is a maximum-sized subset of a finite poset that contains no chain with $k + 1$ elements. In 1976, Greene and Kleitman defined a lattice-ordering on the set $S_k(P)$ of Sperner k -families of a finite poset P and posed the problem: “Characterize and interpret the join- and meet-irreducible elements of $S_k(P)$,” adding, “This has apparently not been done even for the case $k = 1$.”

In this article, the case $k = 1$ is done.

KEYWORDS

Sperner k -family, antichain, join-irreducible element, Dilworth’s Theorem, Hall’s Marriage Theorem, (partially) ordered set

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1. INTRODUCTION, DEFINITIONS, NOTATION, AND PREVIOUS RESULTS

For notation or definitions, see [1]. Let P be a finite poset. For $X \subseteq P$, let $\downarrow X = \{p \in P \mid p \leq x \text{ for some } x \in X\}$ and

$$\uparrow X = \{p \in P \mid p \geq x \text{ for some } x \in X\};$$

let $\uparrow X = (\uparrow X) \setminus X$ and write “ $\downarrow x$ ” if $X = \{x\}$. Let P^∂ denote the *dual* of P , the poset with underlying set P and ordering $p \leq q$ in P^∂ if and only if $p \geq q$ in P .

A subset D of P is a *down-set* or *order ideal* if for all $p \in P$ and $d \in D$, $p \leq d$ implies $p \in D$. Let $\mathcal{O}(P)$ be the poset of down-sets of P ordered by set-inclusion. It forms a distributive lattice under the operations of union and intersection. The set $J(L)$ of *join-irreducibles* of a lattice L consists of $j \in L \setminus 0_L$ such that for all $a, b \in L$, $j = a \vee b$ only if $j = a$ or $j = b$, where 0_L is the bottom element of L . Dually, we define the set $\mathcal{M}(L)$ of *meet-irreducible* elements and the top element 1_L .

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An *antichain* is a subset $A \subseteq P$ such that, for all $a, b \in A$, $a \leq b$ implies $a = b$. A *chain* is a subset C such that for all $c, d \in C$, $c \leq d$ or $d \leq c$. The *width* of P , $w(P)$, is the size of the largest antichain.

Dilworth's Theorem states that $w(P)$ is the smallest number of chains into which one can partition P [2].

One can order the set $\mathcal{A}(P)$ of antichains as a lattice; associate an antichain A with $\downarrow A \in \mathcal{O}(P)$. The inverse map takes a down-set D and associates to it its set $\text{Max } D$ of maximal elements. (For $D \subseteq P$, $\text{Min } D$ is its set of minimal elements.) This ordering is such that $A \leq B$ if and only if $\downarrow A \subseteq \downarrow B$ if and only if $A \subseteq \downarrow B$. If one restricts one's attention to the set of maximum-sized antichains, $S(P)$, one gets a sublattice of the distributive lattice $\mathcal{A}(P)$, where $A \vee B = \text{Max}(A \cup B)$ and $A \wedge B = \text{Min}(A \cup B)$ [3], [4]. For $k \geq 1$, Greene and Kleitman defined a *k-family* (a poset of height at most $k - 1$) to be a subset of P that contains no $(k + 1)$ -element chain [6]. Each *k-family* is a disjoint union of antichains $A_1, \dots, A_k$ such that $A_k \leq A_{k-1} \leq \dots \leq A_1$ ([6, Lemma 2.3]), and Greene and Kleitman defined a partial ordering on the set $\mathcal{A}_k(P)$ of *k-families* of P by: $A \leq B$ if and only if $A_i \leq B_i$ ($1 \leq i \leq k$), where $B_k \leq B_{k-1} \leq \dots \leq B_1$ are the "blocks" of B . See examples in the Appendix.

A *Sperner k-family* is a maximum-sized *k-family*, and the set of these, $S_k(P)$, is a distributive sublattice of $\mathcal{A}_k(P)$ ([6, Theorem 3.17]).

In 1976, Greene and Kleitman posed the following

PROBLEM. Characterize and interpret the join- and meet-irreducible elements of $S_k(P)$. (This has apparently not been done even for the case $k = 1$.) ([6, p. 55])

(The article [8] characterizes join-irreducible elements in the lattice of maximum-sized antichains of a poset derived from P , not P itself.)

We submit a solution to this problem for the case $k = 1$ and trust that Professors Greene and Kleitman approve (Corollary 2.2, Proposition 2.3, Proposition 2.4, and Corollary 2.6, taken together). We are not asserting that the lattice theory in this paper is as sophisticated as group cohomology – merely that a question was asked, which we believe we have answered. We use the word "believe" since Professors Greene and Kleitman asked a "characterization" question, and opinions may differ as to what constitutes a "good" characterization. We believe ours does, since, when specialized to height 1 or width 2 posets, we get reasonable characterizations, one of them reminiscent of a "Hall-type" condition [7].

The ideas in this paper owe their origin to correspondence with Professor Marcel Wild [10], but the author is solely responsible for the decision to publish this article. (The author is not certain Professor Wild would entirely support the author's decision!)

2. A CHARACTERIZATION OF $\mathcal{J}(S(P))$

If L is a finite lattice and $x \in L$, then $\downarrow x$ is a finite lattice and $\downarrow x \cap \mathcal{J}(L) = \mathcal{J}(\downarrow x)$, so in our study of $\mathcal{J}(S(P))$ for P a finite poset, we may assume $\text{Max } P \in S(P)$ – that is, $w(P) = |\text{Max } P|$.

PROPOSITION 2.1. Let P be a finite poset. Assume $w(P) = |\text{Max } P|$. Let $x \in \text{Max } P$. Let $Y = 0_{S(P \setminus \downarrow x)}$.

Then $w(P \setminus \downarrow x) = w(P) - 1$. Further, $\{x\} \cup Y \in \mathcal{J}(S(P))$ or $\{x\} \cup Y = 0_{S(P)}$. In the latter case, $\downarrow x \subseteq \downarrow Y \subseteq \downarrow[(\text{Max } P) \setminus \{x\}]$ and $x \in A$ for all $A \in S(P)$; moreover, the map $A \mapsto A \setminus \{x\}$ is an order-isomorphism from $S(P)$ onto $S(P \setminus \downarrow x)$.

Proof. Since $|(\text{Max } P) \setminus \{x\}| = w - 1$ where $w = w(P)$ and

$$(\text{Max } P) \setminus \{x\} \subseteq P \setminus \downarrow x,$$

$w(P \setminus \downarrow x) \geq w - 1$; but if $A \in \mathcal{A}(P \setminus \downarrow x)$, then $\{x\} \cup A \in \mathcal{A}(P)$, so $|A| \leq w - 1$ and hence $w(P \setminus \downarrow x) = w - 1$, $(\text{Max } P) \setminus \{x\} \in S(P \setminus \downarrow x)$, and if $A \in S(P \setminus \downarrow x)$, then $\{x\} \cup A \in S(P)$.

Thus $\{x\} \cup Y \in S(P)$. First assume $\{x\} \cup Y = 0_{S(P)}$. Then for all $A \in S(P)$, there exists $a \in A$ such that $x \leq a$ so $x = a$; also, for all $A \in S(P)$, $A \setminus \{x\} \subseteq P \setminus \downarrow x$ and for $A, B \in S(P)$, $A \subseteq \downarrow B$ if and only if $A \setminus \{x\} \subseteq \downarrow B$ if and only if $A \setminus \{x\} \subseteq \downarrow(B \setminus \{x\})$ since $x \in A \in \mathcal{A}(P)$. Given $C \in S(P \setminus \downarrow x)$, $C \cup \{x\} \in S(P)$, so the map $A \mapsto A \setminus \{x\}$ from $S(P)$ to $S(P \setminus \downarrow x)$ is onto. Finally, if $x' < x$, then $x' \notin Y$ and $\{x'\} \cup Y$ is not an antichain since otherwise $0_{S(P)} \leq \{x'\} \cup Y$ and $x \leq x'$ ($x \notin \downarrow Y$ because

$\{x\} \cup Y \in \mathcal{A}(P)$). Hence there exists $y \in Y$ such that $x' \leq y$ or $x' \geq y$, but the latter is impossible since $\{x\} \cup Y \in \mathcal{A}(P)$. As $Y = 0_{S(P \setminus \downarrow x)}$, $\downarrow Y \subseteq \downarrow(\text{Max } P) \setminus \{x\}$.

Now assume $\{x\} \cup Y \neq 0_{S(P)}$. Then there is a non-empty subset of $\mathcal{J}(S(P))$ whose supremum is $\{x\} \cup Y$; one of the members A must be such that $x \in A$. But $Y \subseteq \downarrow(A \setminus \{x\})$ since $A \setminus \{x\} \in S(P \setminus \downarrow x)$ (because $x \in A \in \mathcal{A}(P)$). We conclude that

$$A \setminus \{x\} = Y$$

and $\{x\} \cup Y = \{x\} \cup (A \setminus \{x\}) = A \in \mathcal{J}(S(P))$. $\square$

COROLLARY 2.2. Let P be a finite poset. Assume $|\text{Max } P| = w(P)$ and $S(P)$ is not trivial.

Then the following are equivalent:

- (1) $\text{Max}(P) \in \mathcal{J}(S(P))$.
- (2) There exists $x \in \text{Max } P$ such that $S(P \setminus \downarrow x)$ is trivial.
- (3) There exists $x \in \text{Max } P$ such that $S(P \setminus \downarrow x) = \{(\text{Max } P) \setminus \{x\}\}$.

Proof. Assume (1). For each $x \in \text{Max } P$, let $Y_x = 0_{S(P \setminus \downarrow x)}$. By Proposition 2.1, $\{x\} \cup Y_x \in S(P)$ and $\bigvee_{x \in \text{Max } P} (\{x\} \cup Y_x) = \text{Max } P$. (The set $\text{Max } P \neq \emptyset$ since $P \neq \emptyset$ because $|S(P)| \neq 1$.) By (1), there exists $x \in \text{Max } P$ such that $\text{Max } P = \{x\} \cup Y_x$. Hence $1_{S(P \setminus \downarrow x)} = (\text{Max } P) \setminus \{x\} = Y_x = 0_{S(P \setminus \downarrow x)}$. Therefore (2) holds.

Assume (2). Then (3) holds since $|S(P \setminus \downarrow x)| = 1$ and $(\text{Max } P) \setminus \{x\} \in S(P \setminus \downarrow x)$ by Proposition 2.1.

Assume (3). Using the notation of Proposition 1, $(\text{Max } P) \setminus \{x\} = 1_{S(P \setminus \downarrow x)} = 0_{S(P \setminus \downarrow x)} = Y$, so $\text{Max } P = \{x\} \cup Y \in \mathcal{J}(S(P))$ by Proposition 1, since $1_{S(P)} = \text{Max } P \neq 0_{S(P)}$ by hypothesis. $\square$

The book [5] gives a method for computing join-irreducibles in $S(P)$ on pages 239 and 240, but that method involves taking meets in the lattice, so in the author's opinion it would not count as a good "characterization" (not that the authors of that book were attempting to do this).

Given Corollary 2.2, it behooves us to understand when $|S(P)| = 1$ and how to find $0_{S(P)}$.

PROPOSITION 2.3. Let P be a finite poset. Let $A \in S(P)$. Then $|S(P)| = 1$ if and only if for all $a \in A$, $w(P \setminus \{a\}) = w(P) - 1$.

Proof. Since $|A \setminus \{a\}| = w(P) - 1$, $w(P \setminus \{a\}) \geq w(P) - 1$. If $w(P \setminus \{a\}) = w(P)$, then there is $B \in \mathcal{A}(P \setminus \{a\})$ with $|B| = w(P)$; but $B \neq A$ since $a \in A \setminus B$, so $|S(P)| \geq 2$, a contradiction. Hence $w(P \setminus \{a\}) = w(P) - 1$.

Conversely, if $B \in S(P) \setminus \{A\}$, then, since $|A| = |B|$, there is $a \in A \setminus B$ and thus $B \subseteq P \setminus \{a\}$ and $w(P \setminus \{a\}) = w(P)$. $\square$

Note that in the next proposition, we are viewing choice functions from the set of chains in a chain-partition of P as members of an ordered set. As Professor Wild has suggested, we could represent a choice function on a set of w chains as a w -tuple of elements of P , $(p_1, \dots, p_w)$, where $(p_1, \dots, p_w) \leq (q_1, \dots, q_w)$ if and only if for $1 \leq i \leq w$, $p_i \leq q_i$ in the poset P .

PROPOSITION 2.4. Let P be a finite poset and let C be a partition of P into $w(P)$ chains. Let f be a choice function for C . If $\text{im}(f)$, the image of f , is an antichain, let $\Phi(f) = f$. Otherwise, pick $p_1, p_2 \in \text{im } f$ such that $p_1 < p_2$ and let $\Phi(f)$ be the choice function of C such that $\text{im } \Phi(f) = ((\text{im } f) \setminus \{p_1\}) \cup \{p'_1\}$ where p'_1 is the smallest element in the chain of C containing p_1 such that $p'_1 \not\leq p_2$, or let $\Phi(f) = \text{STOP}$ if no such p'_1 exists.

Performing the following algorithm results in $0_{S(P)}$:

- (1) Let f_0 be the choice function that picks the bottom element of each chain in C .
- (2) Calculate $f_0, \Phi(f_0), \Phi^2(f_0), \dots$ until $\Phi^n(f_0) = \Phi^{n+1}(f_0)$.
- (3) The set $\text{im } \Phi^n(f_0)$ is $0_{S(P)}$.

Proof.

CLAIM 1. For every choice function f for C , $\Phi(f) = \text{STOP}$, $\Phi(f) = f$, or $f < \Phi(f)$ viewed as functions from C to P .

Proof of claim. Since $p'_1 \not\leq p_2$ but $p_1 \leq p_2$, then $p_1 < p'_1$. $\square$

CLAIM 2. For all $n \geq 0$, $\Phi^n(f_0) \neq \text{STOP}$ and if g_0 is the choice function for C such that $\text{im } g_0 = 0_{S(P)}$, then $\Phi^n(f_0) \leq g_0$.

Proof of claim. Each $A \in S(P)$ is the image of a choice function for C , so the claim holds for $n = 0$. Assume it holds for $n \geq 0$. If $\text{im } \Phi^n(f_0)$ is an antichain, the claim holds for $n + 1$. Otherwise, defining p_1 and p_2 as above and choosing $b_i \in 0_{S(P)}$ in the same chain of C as p_i so that $p_i \leq b_i$ ($i = 1, 2$), we see that if $b_1 \leq p_2$, then $b_1 \leq b_2$, so $b_1 = b_2$, contradicting the fact that b_1 and b_2 are in disjoint chains. Thus $b_1 \not\leq p_2$, so $\Phi^{n+1}(f_0) \neq \text{STOP}$ and $p'_1 \leq b_1$, so $\Phi^{n+1}(f_0) \leq g_0$. $\square$

By Claims 1 and 2, eventually $\Phi^n(f_0) = \Phi^{n+1}(f_0)$ – that is, $\text{im } \Phi^n(f_0) \in \mathcal{A}(P)$, and hence $\text{im } \Phi^n(f_0) \in S(P)$. Thus $0_{S(P)} \leq \text{im } \Phi^n(f_0)$, but by Claim 2 $\text{im } \Phi^n(f_0) \leq 0_{S(P)}$, so $\text{im } \Phi^n(f_0) = 0_{S(P)}$. $\square$

In Chapter XI, §7, Freese *et al.* [5] have an algorithm for finding a partition of a poset P into $w(P)$ chains and finding $0_{S(P)}$; it does more than our algorithm for doing the latter and is not the same as our algorithm.

The next two results show that we have also characterized $\mathcal{M}(S(P))$.

PROPOSITION 2.5. Let P be a finite poset. Then $S(P) = S(P^\partial)$ and for all $A, B \in S(P)$, $A \leq B$ in $S(P)$ if and only if $A \geq B$ in $S(P^\partial)$.

Proof. For $A, B \in S(P)$, $A \leq B$ in $S(P)$ if and only if $A \subseteq \downarrow B$ if and only if $\downarrow A \subseteq \downarrow B$, which implies $B \subseteq \uparrow A$ (since A is a maximal antichain and $B \in \mathcal{A}(P)$; or use [3], Lemma 2.3), i.e., $B \leq A$ in $S(P^\partial)$. $\square$

COROLLARY 2.6. Let P be a finite poset. Then $\mathcal{M}(S(P)) = \mathcal{J}(S(P^\partial))$. $\square$

As evidence that the results of this section provide a useful characterization of $\mathcal{J}(S(P))$, in the next two sections we use them to derive characterizations in the case of height 1 posets and width 2 posets.

3. HEIGHT 1 POSETS

In this section, let P be a finite 2-family. Let C be a partition of P into $w = w(P)$ chains.

Let $G_\leq = (P, E)$ be the directed graph with

$$E = \{(a, b) \in \text{Max } P \times \text{Min } P \mid b \leq a\} \cup \{(c, d) \in P \times P \mid \{c, d\} \in C \text{ and } c < d\}.$$

First we show that, when analyzing $S(P)$, we can assume that $|C| = 2$ for all $C \in \mathcal{C}$.

LEMMA 3.1. Assume $\{x\} \in C$ where $x \in \text{Max } P$. Then the map

$$A \mapsto A \setminus \downarrow x$$

is an order-isomorphism from $S(P)$ to $S(P \setminus \downarrow x)$ and

$$\{D \setminus \downarrow x \mid D \in C \setminus \{\{x\}\}\}$$

is a partition of $P \setminus \downarrow x$ into disjoint, non-empty chains. For each $D \in C \setminus \{\{x\}\}$, either $D \setminus \downarrow x = D$ or $D \setminus \downarrow x = \text{Max } D$. Moreover, $w - 1 = w(P \setminus \downarrow x)$.

Proof. Let $D \in C$, $D \neq \{x\}$. Assume for a contradiction that

$$D \setminus \downarrow x = \emptyset;$$

then $D \subseteq \downarrow x$, so $d < x$ for all $d \in D$. If $|D| > 1$, then there are distinct $d_1, d_2 \in D$ such that $d_1 < d_2 < x$, contradicting the fact that P has height 1. Hence $D = \{d\}$ for some $d \neq x$. Take $A \in S(P)$. Then $x, d \in A$ but $d < x$, a contradiction. Hence $D \setminus \downarrow x \neq \emptyset$. If $D \setminus \downarrow x \subsetneq D$, then $|D| = 2$ and $|D \setminus \downarrow x| = 1$, so if $D = \{d_1, d_2\}$ where $d_1 < d_2$, then either $d_1 \leq x$ or else $d_2 \leq x$, in which case $D \setminus \downarrow x = \emptyset$, which is false, so $D \setminus \downarrow x = \{d_2\} = \text{Max } D$. The chain partition of $P \setminus \downarrow x$ has $w - 1$ chains, so $w(P \setminus \downarrow x) \leq w - 1$.

Because there is an $A \in S(P)$, $|A| = w$, and hence $x \in A$, so the antichain $A \setminus \{x\} \subseteq P \setminus \downarrow x$ and $|A \setminus \{x\}| = w - 1$; therefore $w(P \setminus \downarrow x) = w - 1$ and $A \setminus \downarrow x \in S(P \setminus \downarrow x)$.

For $A, B \in S(P)$, $A \leq B$ in $S(P)$ if and only if $A \subseteq \downarrow B$ if and only if $A \setminus \{x\} \subseteq \downarrow B$ (since $x \in B$) if and only if $A \setminus \downarrow x \subseteq \downarrow B$ if and only if $A \setminus \downarrow x \subseteq \downarrow(B \setminus \{x\})$ (since $x \in A \in \mathcal{A}(P)$) if and only if $A \setminus \downarrow x \subseteq \downarrow(B \setminus \downarrow x)$.

Now let $A' \in S(P \setminus \downarrow x)$. Let A be $A' \cup \{x\}$. If $a' \in A'$ is comparable to x , then $a' < x$ since $x \in \text{Max } P$; thus $a' \notin P \setminus \downarrow x$. Hence $A \in S(P)$ and $A \setminus \{x\} = A'$. $\square$

PROPOSITION 3.2. For each $\{x\} \in \mathcal{C}$, remove the connected component of $G_{\leq}$ ($G_{\geq}$) containing x if $x \in \text{Max } P$ (resp., $x \in \text{Min } P$). There is a subfamily of $\mathcal{C}$ of cardinality $w(Q)$ partitioning the resulting subposet Q into two-element chains and the map $A \mapsto A \cap Q$ is an order-isomorphism from $S(P)$ onto $S(Q)$.

Proof. If $\{x\}, \{y\} \in \mathcal{C}$ and $x \neq y$, then y is not in the connected component of $G_{\leq}$ containing x – if $y \in \text{Min } P$, it is because there exists some $A \in S(P)$ and any such antichain must contain x and y and d_2 where $\{d_1, d_2\} \in \mathcal{C} \setminus \{\{x\}\}$ with $d_1 < d_2$. $\square$

The following corollary of Corollary 2.2 has a criterion resembling Hall's Marriage Theorem, so mathematicians ought to regard it as a solution to the problem of Greene and Kleitman in this special case.

COROLLARY 3.3. Assume $P \neq \emptyset$ is a finite height 1 poset that has a partition into $w(P)$ two-element chains. Then $\text{Max } P \in S(P)$, and $\text{Max } P \in \mathcal{J}(S(P))$ if and only if there exists $x \in \text{Max } P$ such that $|B| < |\uparrow_{\circ} B|$ for all non-empty $B \subseteq (\text{Min } P) \setminus \downarrow x$.

Proof. Since $\text{Max } P$ is the set of top elements in the chains of the partition and $\text{Min } P$ the set of bottom elements, $(\text{Max } P) \cap (\text{Min } P) = \emptyset$ and $|\text{Max } P| = w$. Since $P \neq \emptyset$, $\text{Max } P, \text{Min } P \in S(P)$ and are distinct.

Assume $\text{Max } P \in \mathcal{J}(S(P))$. Let $x \in \text{Max } P$ be such that

$$S(P \setminus \downarrow x) = \{(\text{Max } P) \setminus \{x\}\}$$

as per Corollary 2.2. Let $B \subseteq (\text{Min } P) \setminus \downarrow x$ be non-empty. From the chain partition, we get $|B| \leq |\uparrow_{\circ} B|$. If $|B| = |\uparrow_{\circ} B|$, then $[(\text{Max } P) \setminus (\{x\} \cup \uparrow_{\circ} B)] \cup B$ is an antichain of size $w - 1 = w(P \setminus \downarrow x)$ so belongs to $S(P \setminus \downarrow x)$, but is different from $(\text{Max } P) \setminus \{x\}$, a contradiction.

Now assume $x \in \text{Max } P$ and $|B| < |\uparrow_{\circ} B|$ for all non-empty $B \subseteq (\text{Min } P) \setminus \downarrow x$. Assume for a contradiction that

$$S(P \setminus \downarrow x) \neq \{(\text{Max } P) \setminus \{x\}\}.$$

Since $(\text{Max } P) \setminus \{x\} \in S(P \setminus \downarrow x)$, there is $A \neq (\text{Max } P) \setminus \{x\}$ in $S(P \setminus \downarrow x)$. Let $B = A \cap \text{Min } P$. Since $A \neq (\text{Max } P) \setminus \{x\}$ and $|A| = w - 1$, $\emptyset \neq B \subseteq P \setminus \downarrow x$. Thus $|B| < |\uparrow_{\circ} B|$, so

$$w - 1 = |A| \leq |B| + |[(\text{Max } P) \setminus \downarrow x] \setminus \uparrow_{\circ} B| = |B| + w - 1 - |\uparrow_{\circ} B| < w - 1,$$

a contradiction.

By Corollary 2.2, $\text{Max } P \in \mathcal{J}(S(P))$. $\square$

The following example and the “general result” to which we allude afterward come from Professor Marcel Wild [10].

EXAMPLE 3.4. Let P be the poset of Figure 1.

Consider the directed graph $H = (P, F)$ where

$$F = \{(i, i') \mid 1 \leq i \leq 10\} \cup \{(i', j) \mid 1 \leq i, j \leq 10 \text{ and } i' \geq j\}.$$

The strongly connected components (see [9]) are $a = \{1, 1', 2, 2'\}$, $b = \{3, 3', 4, 4', 5, 5'\}$, $c = \{6, 6', 7, 7'\}$, and $d = \{8, 8', 9, 9', 10, 10'\}$. The condensation of the graph is in Figure 2.

The set $S(P)$ is in one-to-one correspondence with $\mathcal{O}(Q)$. For example, $\{a, c\}$ corresponds to $X = \{1', 2', 3, 4, 5, 6', 7', 8, 9, 10\}$. To see if it is join-irreducible, we just consider Figure 3.

FIGURE 1.

FIGURE 2. Q

FIGURE 3.

Take any maximal element of this poset – to fix the ideas, $1'$. Then $\{6, 7\}$ is a non-empty set of minimal elements and $\uparrow_{\circ} \{6, 7\} = \{6', 7'\}$ so $|\uparrow_{\circ} \{6, 7\}| = |\{6', 7'\}|$. Hence $X \notin \mathcal{J}(S(P))$.

On the other hand, $\downarrow b$ in Q corresponds to $Y = \{1', 2', 3', 4', 5', 6', 7', 8, 9, 10\}$ in $S(P)$. To see if it is join-irreducible, consider the poset R of Figure 4. The element $3' \in \text{Max } R$ and $\text{Min}(R \setminus \downarrow 3') = \{1, 6, 7\}$. For all non-empty $B \subseteq \{1, 6, 7\}$, $\uparrow_{\circ} B$ has more elements: $\uparrow_{\circ} \{1\} = \{1', 2'\}$, $\uparrow_{\circ} \{6\} = \{4', 6', 7'\}$, $\uparrow_{\circ} \{7\} = \{5', 6', 7'\}$, $\uparrow_{\circ} \{1, 6\} = \{1', 2', 4', 6', 7'\}$, $\uparrow_{\circ} \{1, 7\} = \{1', 2', 5', 6', 7'\}$, $\uparrow_{\circ} \{6, 7\} = \{4', 5', 6', 7'\}$, $\uparrow_{\circ} \{1, 6, 7\} = \{1', 2', 4', 5', 6', 7'\}$. By Corollary 3.3, $Y \in \mathcal{J}(S(P))$.

FIGURE 4. R

One can guess the general result.

4. WIDTH 2 POSETS

COROLLARY 4.1. Let P be a finite width 2 poset. Let $\text{Max } P = \{a, b\}$ where $a \neq b$. Then $\text{Max } P \in \mathcal{J}(S(P))$ if and only if

- (i) $\downarrow a \subseteq \downarrow b$ and
- (ii) $w(P \setminus \{b\}) = 2$

or

- (i') $\downarrow b \subseteq \downarrow a$ and
- (ii') $w(P \setminus \{a\}) = 2$.

NOTE. Condition (i) is equivalent to: every lower cover of a is below b .

Proof. Assume $\text{Max } P \in \mathcal{J}(S(P))$. Then $|S(P)| \neq 1$. By Corollary 2.2, without loss of generality $S(P \setminus \downarrow b) = \{\{a\}\}$ – in particular,

$$w(P \setminus \downarrow b) = 1,$$

so $P \setminus \downarrow b$ is a non-empty chain and thus any singleton set is in $S(P \setminus \downarrow b)$ – hence $P \setminus \downarrow b = \{a\}$. Therefore $x < a$ implies $x \leq b$ and obviously $x \neq b$. Let $D \in S(P) \setminus \{\text{Max } P\}$. If $b \in D$, then there is $d \in D \setminus \{b\}$ such that $d \not\leq b$ and hence $\{d\} \in S(P \setminus \downarrow b)$ so that $d = a$ and therefore $D = \{a, b\} = \text{Max } P$, a contradiction. We deduce that $b \notin D$ and so $D \subseteq P \setminus \{b\}$ and $w(P \setminus \{b\}) = 2$.

Conversely, assume (i) and (ii). Then $|S(P)| \neq 1$ so $1_{S(P)} = \text{Max } P \neq 0_{S(P)}$. Obviously $a \in P \setminus \downarrow b$. If $x \in P \setminus \downarrow b$, then since $\text{Max } P = \{a, b\}$, we have $x \leq a$; but if $x < a$, then $x \leq b$, a contradiction. Thus $P \setminus \downarrow b = \{a\}$ and hence $S(P \setminus \downarrow b) = \{\{a\}\} = \{(\text{Max } P) \setminus \{b\}\}$. By Corollary 2.2, $\text{Max } P \in \mathcal{J}(S(P))$. $\square$

EXAMPLE 4.2. Consider the poset of Figure 5.

FIGURE 5. P

We see $S(P)$ in Figure 6: $S(P) = \{\{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}, \{c, f\}, \{e, f\}\}$.

FIGURE 6. $S(P)$

We see that $\downarrow b = \{d, e, f\} \not\subseteq \{c, e, f\} = \downarrow a$ and $\downarrow a \not\subseteq \downarrow b$, so $\{a, b\} \notin \mathcal{J}(S(P))$ by Corollary 4.1 (also evident from Figure 6).

EXAMPLE 4.3. Consider the poset R of Figure 2. Since $S(R)$ has three elements – $\{b, d\}$, $\{a, d\}$, and $\{a, c\}$ – and is a lattice, it must be the three-element chain, so $\{b, d\}$ and $\{a, d\}$ are join-irreducible. We also see $\{b, d\} \in \mathcal{J}(S(R))$ from Corollary 4.1, since $\downarrow d = \{c\} \subseteq \{a, c\} = \downarrow b$ and $R \setminus \{b\} = \{a, c, d\}$ contains the two-element antichain $\{a, c\}$. We also see it from Corollary 3.3, since $b \in \text{Max } R$ and $(\text{Min } R) \setminus \downarrow b = \emptyset$.

5. FUTURE WORK

Greene and Kleitman characterize $\mathcal{M}(\mathcal{A}_k(P))$ – essentially, each “block” A_i is the top of its respective portion except for exactly one, which is meet-irreducible ([6, Proposition 2.38]). The four-element Boolean lattice shows that the description of the covering relation in $S_2(P)$ can involve more than one block changing. (See the Appendix.)

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APPENDIX. EXAMPLES OF $\mathcal{A}_K(P)$ AND $S_K(P)$

FIGURE 7. U

FIGURE 8. $\mathcal{A}(U)$

FIGURE 9. $S(U)$

FIGURE 10. $\mathcal{A}_2(U)$

$$bcd/a \bullet$$

FIGURE 11. $S_2(U)$ FIGURE 12. V

$$\begin{array}{c} d/bc \bullet \\ \downarrow \\ bc/a \bullet \end{array}$$

FIGURE 13. $S_2(V)$

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