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AKADÉMIAI KIADÓ

SOME NEW INEQUALITIES INVOLVING THE GENERALIZED HARDY OPERATOR

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ABSTRACT

In this paper we derive new inequalities involving the generalized Hardy operator. The obtained results generalized known inequalities involving the Hardy operator. We also get new inequalities involving the classical Hardy–Hilbert inequality.

KEYWORDS

Hardy's inequality, Hardy-Hilbert's inequality, Hardy's operator, kernel

MATHEMATICS SUBJECT CLASSIFICATION (2020)

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1. INTRODUCTION

First, we recall some well-known integral inequalities. If $p > 1$, then Hardy's integral inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)y\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx \tag{1.1}$$

holds for all non-negative functions $f \in L^p(\mathbb{R}_+)$, where $\mathbb{R}_+ = (0,\infty)$, see [2]. Hardy proved in 1928 the following generalization of inequality (1.1)

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^\alpha dx \leq \left(\frac{p}{p-1-\alpha}\right)^p \int_0^\infty f^p(x)x^\alpha\,dx \tag{1.2}$$

where $p \geq 1$ and $\alpha < p-1$. Inequality (1.2) is not a genuine generalization of inequality (1.1) since both are equivalent to the Hardy inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(y)\,dy\right)^p \frac{dx}{x} \leq \int_0^\infty g^p(x)\,\frac{dx}{x} \tag{1.3}$$

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If we do substitution $f(x) = g(x^{\frac{p-1}{p}})x^{-\frac{1}{p}}$ and $f(x) = g(x^{\frac{p-\alpha p-1}{p}})x^{-\frac{\alpha+1}{p}}$ For finite integrals we have that for $0 < d \leq \infty$, $p \geq 1$ or $p < 0$ the inequality

$$\int_0^d \left(\frac{1}{x}\int_0^x g(y)\,dy\right)^p \frac{dx}{x} \leq 1 \cdot \int_0^d g^p(x)\left(1-\frac{x}{d}\right)\frac{dx}{x} \tag{1.4}$$

holds. With the same substitutions inequality (1.4) is equivalent to the inequality

$$\int_0^{d_0} \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^\alpha\,dx \leq \left(\frac{p}{p-1-\alpha}\right)^p \int_0^{d_0} f^p(x)x^\alpha \left(1-\left(\frac{x}{d_0}\right)^{\frac{p-1-\alpha}{p}}\right)dx \tag{1.5}$$

where $d_0 = d^{\frac{p}{p-1-\alpha}}$, $0 < d_0 \leq \infty$, $p \geq 1$, $\alpha < p-1$ or $p < 0, \alpha > p-1$.

Another important inequality, closely related to (1.1), is Hardy–Hilbert's inequality,

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}\,dx\right)^p dy \leq \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^p \int_0^\infty f^p(t)\,dt, \tag{1.6}$$

which holds for $p > 1$ and non-negative functions $f \in L^p(\mathbb{R}_+)$. Moreover, we mention Pólya–Knopp's inequality,

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(t)\,dt\right)dx < e\int_0^\infty f(x)\,dx, \tag{1.7}$$

for positive functions $f \in L^1(\mathbb{R}_+)$. Since (1.7) can be obtained from (1.1) by rewriting it with the function f replaced with $f^{\frac{1}{p}}$ and then by letting $p \longrightarrow \infty$, Pólya–Knopp's inequality may be considered as a limiting case of Hardy's inequality. Observe that the constants $\left(\frac{p}{p-1}\right)^p$, $\left(\frac{p}{p-1-\alpha}\right)^p$, $\left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^p$, and e, respectively appearing on the right-hand sides of (1.1), (1.2), (1.6), and (1.7), are the best possible, that is, none of them can be replaced with any smaller constant.

Further information and remarks concerning generalizations, and applications of inequalities (1.1) – (1.7) can be found e.g. in the monographs [2, 5, 7, 6, 8, 9, 11].

We define the Jensen functional connected to the inequality (1.4)

$$J_p(f(x)) = \int_0^d f^p(x)\left(1-\frac{x}{d}\right)\frac{dx}{x} - \int_0^d \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p \frac{dx}{x} \tag{1.8}$$

as a measure of the so called "Jensen gap".

In particular it was pointed out by S. Kaijser et al. in [3] that both (1.1) and (1.7) are just special cases of a much more general Hardy–Knopp's type inequality,

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x} \leq \int_0^\infty \Phi(f(x))\frac{dx}{x}, \tag{1.9}$$

where Φ is a convex function on $\mathbb{R}_+$ and $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a locally integrable positive function. Note that (1.9) follows by a standard application of Jensen's inequality and Fubini's theorem.

We also note that Hardy's inequality (1.1) shows that the Hardy operator H, defined by

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt,$$

maps L^p into itself with operator norm $p/(p-1)$. The operator can be generalized by adding a kernel

$$A_k f(x) = \frac{1}{K(x)}\int_0^\infty f(t)k(x,t)\,dt, \tag{1.10}$$

where

$$K(x) = \int_0^\infty k(x,t)\,dt < \infty.$$

Here $k(x,y)$ is a general measurable and non-negative function.

Further generalization include measure spaces $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$, so A_k from (1.10) can be generalized as follows:

$$A_k f(x) := \frac{1}{K(x)} \int_{\Omega_2} k(x,y) f(y)\, d\mu_1(y), \tag{1.11}$$

where $f : \Omega_2 \longrightarrow \mathbb{R}$ is a measurable function, $k : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R}$ is a measurable and non-negative kernel and

$$K(x) := \int_{\Omega_2} k(x,y)\, d\mu_1(y) < \infty, \quad x \in \Omega_1. \tag{1.12}$$

Finally, K. Krulić et al. [4] and [5] unified the above results by studying the measure spaces $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$, and the general integral operator A_k defined by

$$A_k f(x) = \frac{1}{K(x)} \int_{\Omega_2} k(x,y) f(y)\, d\mu_1(y), \quad x \in \Omega_1, \tag{1.13}$$

where $f : \Omega_2 \longrightarrow \mathbb{R}$ is a measurable function, $k : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R}$ is measurable and non-negative, and

$$K(x) = \int_{\Omega_2} k(x,y)\, d\mu_1(y) > 0, \quad x \in \Omega_1. \tag{1.14}$$

Just by using Jensen's inequality and Fubini's theorem, they elegantly proved the weighted inequality

$$\int_{\Omega_1} u(x) \Phi(A_k f(x))\, d\mu_1(x) \le \int_{\Omega_2} v(y) \Phi(f(y))\, d\mu_1(y), \tag{1.15}$$

where $u : \Omega_1 \longrightarrow \mathbb{R}$ is a non-negative measurable function, $x \mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$, v is defined on Ω_2 by

$$v(y) = \int_{\Omega_1} u(x) \frac{k(x,y)}{K(x)}\, d\mu_1(x), \tag{1.16}$$

Φ is a convex function on an interval $I \subseteq \mathbb{R}$, and $f : \Omega_2 \longrightarrow \mathbb{R}$ is such that $f(y) \in I$, for all $y \in \Omega_2$.

2. THE MAIN RESULTS

We define the generalized Jensen functional

$$J_\Phi(f(x)) = \frac{1}{K(x)} \int_{\Omega_2} \Phi(f(y)) k(x,y) d\mu_1(y) - \Phi\left(\frac{1}{K(x)} \int_{\Omega_2} k(x,y) f(y) d\mu_1(y)\right) \tag{2.1}$$

By the Jensen inequality for convex function Φ $J_\Phi(f) \ge 0$. For concave functions $J_\Phi(f) \le 0$.

In this paper we consider the generalized Jensen functional and prove some more general inequalities. Finally applying it to some important particular measure spaces, kernels, and weights, we derive a series of new refined Hardy–type inequalities.

LEMMA 2.1. Le Φ be a differentiable convex function, $k(x,y)$ is a measurable and non-negative kernel, $K(x)$ is defined by (1.14) and A_k is defined by (1.13). Then the inequalities

$$\begin{aligned} 0 &\le A_k\Phi(f(x)) - \Phi(A_k(f(x)) \\ &\le \frac{1}{K(x)} \int_{\Omega_2} \Phi'(f(y)) f(y) k(x,y) d\mu_1(y) - A_k\Phi'(f(x)) \cdot A_k f(x) \end{aligned} \tag{2.2}$$

Proof. The first inequality is just Jensen's inequality. For a differentiable convex function it holds

$$\Phi(a) - \Phi(b) \ge \phi'(b)(a-b) \tag{2.3}$$

Inequality (2.3) holds in reversed direction when Φ is concave function. Now let

$$a = \frac{1}{K(x)} \int_{\Omega_2} f(y) k(x,y) d\mu_1(y) \quad \text{and} \quad b = f(y)$$

We get

$$\Phi\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)d\mu_1(y)\right) - \Phi(f(y)) \geq \Phi'(f(y))\cdot\left(\frac{1}{K(x)}\int_{\Omega_2} f(y)k(x,y)d\mu_1(y) - f(y)\right) \tag{2.4}$$

Multiplying by $-k(x,y)$ and integrating over Ω_2 we get

$$\int_{\Omega_2} k(x,y)\Phi(f(y))d\mu_1(y) - \int_{\Omega_2} k(x,y)\Phi\left(\frac{1}{K(x)}\int_{\Omega_2} f(y)k(x,y)d\mu_1(y)\right)d\mu_1(y) \leq \int_{\Omega_2} k(x,y)\Phi'(f(y))f(y)d\mu_1(y) - \int_{\Omega_2} k(x,y)\Phi'(f(y))\left(\frac{1}{K(x)}\int_{\Omega_2} f(y)k(x,y)d\mu_1(y)\right)d\mu_1(y) \tag{2.5}$$

Since

$$K(x) = \int_{\Omega_2} k(x,y)\, d\mu_1(y) > 0$$

and

$$A_k f(x) = \frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)\, d\mu_1(y)$$

and dividing the above inequalities with $K(x)$ we get inequalities (2.2). □

REMARK 2.2. The reverse of Jensen's inequality in Lemma 2.1 is known in the general settings of positive linear functionals (see [1]) and Lemma 2.1 can be derived from Theorem 3.1 from [1].

LEMMA 2.3. Let $A_k = A_k(f(x))$, $0 < x < \infty$ be the generalized Hardy operator.

- If $p \geq 1$ or $p < 0$, then
$$A_k(f(x))^p \leq A_k(f^p(x)) \tag{2.6}$$
- If $0 < p \leq 1$, then (2.6) holds in reversed direction.

By using this lemma we conclude that for $p \geq 1$ or $p < 0$

$$A_k(f^p(x)) - (A_k(f(x))^p \geq 0.$$

If we apply convex function x^p, $p \leq 1$ or $p < 0$ we get the following result.

THEOREM 2.4. Let $A_k f$ be the generalized Hardy operator. Then we have the following two-sided estimates:

a) If $p \geq 1$ or $p < 0$, then
$$0 \leq A_k f^p(x) - (A_k f(x))^p \leq pA_k f^p(x) - pA_k f^{p-1}(x)A_k f(x) \tag{2.7}$$

b) If $0 < p < 1$ then the both inequalities in (2.7) hold in reversed direction.

We continue with the following Corollary.

COROLLARY 2.5. Let $A_k f$ be the generalized Hardy operator. If $p \geq 2$ the the following refinement of (2.6) holds

$$(A_k f(x))^p \leq \frac{1}{p-1}[pA_k f^{p-1}(x)A_k f(x) - (A_k f(x))^p] \leq A_k f^p(x) \tag{2.8}$$

Proof. First we prove the first inequality. We start with (2.7).

$$\begin{gathered} A_k f^p(x) - (A_k f(x))^p \leq pA_k f^p(x) - pA_k f^{p-1}(x)A_k f(x) \\ A_k f^p(x) - pA_k f^p(x) \leq (A_k f(x))^p - pA_k f^{p-1}(x)A_k f(x) \\ (1-p)A_k f^p(x) \leq (A_k f(x))^p - pA_k f^{p-1}(x)A_k f(x) \end{gathered} \tag{2.9}$$

Dividing by $p-1$ we get the first inequality in (2.8). Next we apply inequality (2.6) for $p-1 \geq 1$ and get that

$$A_k\big(f(x)\big)^{p-1} \leq A_k\big(f^{p-1}(x)\big)$$

If we multiply both sides by $pA_kf(x)$ and substract $(A_kf(x))^p$ from both sides we get the second inequality. □

We give examples of Corollary 2.5.

EXAMPLE 2.6. Let $\Omega_2 = (0, \infty)$ and let $k(x, y)$ be defined as $k(x, y) = 1$ for $0 \leq y \leq x$ and $k(x, y) = 0$ for $y > x$. Let $d\mu_1(y) = dy$ and $u(x) = \frac{1}{x}$. Then we calculate $K(x)$. With upper substitutions $K(x) = x$ and $A_kf(x) = Hf(x)$ is the Hardy operator. Then inequality (2.8) becomes

$$(H(f(x)))^p \leq \frac{1}{p-1}[pH(f^{p-1}(x))H(f(x)) - (H(f(x)))^p] \leq H(f^p(x)). \tag{2.10}$$

Result (2.10) is given in Corollary 2.6 in [10].

As a consequence of Example 2.6 we can get the inequalities given in [10]. There the authors obtain refinements of the fundamental Hardy inequality 1.4.

Now we give the new refinement of the Hardy-Hilbert inequality (1.6)

THEOREM 2.7. Let f be a non-negative function on $(0, \infty)$ and $p \geq 2$. Then

$$\begin{aligned}\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}\,dx\right)^p dy &\leq \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^{p-1} \frac{1}{p-1}\cdot \\ &\left[p\int_0^\infty x^{\frac{2}{p}-1}\int_0^\infty \frac{f^{p-1}(y)y^{1-\frac{2}{p}}}{x+y}dy\int_0^\infty \frac{f(y)}{x+y}\,dy\,dx - \frac{\sin\frac{\pi}{p}}{\pi}\int_0^\infty\left(\int_0^\infty \frac{f(y)}{x+y}\,dy\right)^p\right] \\ &\leq \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^p \int_0^\infty f^p(y)\,dy,\end{aligned} \tag{2.11}$$

Proof. This result is a direct consequence of Corollary 2.5. Let $\Omega_2 = (0, \infty)$, $k(x, y)$ be defined as

$$k(x,y) = \frac{\left(\frac{y}{x}\right)^{-\frac{1}{p}}}{x+y}, \quad p > 1$$

Let $u(x) = \frac{1}{x}$, $v(y) = \frac{1}{y}$ and $d\mu_1(y) = dy$. Then we get

$$K(x) = \frac{\pi}{\sin\frac{\pi}{p}} \quad \text{and} \quad A_kf(x) = \frac{\sin\frac{\pi}{p}}{\pi}x^{\frac{1}{p}}\int_0^\infty \frac{y^{-\frac{1}{p}}}{x+y}f(y)dy$$

If we substitute the above parameters in Corollary 2.5 we get

$$\begin{aligned}&\left(\frac{\sin\frac{\pi}{p}}{\pi}x^{\frac{1}{p}}\int_0^\infty \frac{y^{-\frac{1}{p}}}{x+y}f(y)dy\right)^p \\ &\leq \frac{\sin\frac{\pi}{p}}{\pi}\frac{1}{p-1}\cdot\left[px^{\frac{2}{p}}\int_0^\infty \frac{y^{-\frac{1}{p}}}{x+y}f^{p-1}(y)dy\int_0^\infty \frac{y^{-\frac{1}{p}}}{x+y}f(y)dy - \left(\frac{\sin\frac{\pi}{p}}{\pi}\right)^{p-1} x\int_0^\infty \frac{y^{-\frac{1}{p}}}{x+y}f(y)dy\right] \\ &\leq \frac{\sin\frac{\pi}{p}}{\pi}x^{\frac{1}{p}}\int_0^\infty \frac{y^{-\frac{1}{p}}}{x+y}f^p(y)dy\end{aligned} \tag{2.12}$$

If we replace $f(t)t^{-\frac{1}{p}}$ with $f(t)$ and multiply by $\frac{dx}{x}$, integrate over the interval $(0, \infty)$ we get the required result. □

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