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A MARCINKIEWICZ TYPE INTERPOLATION THEOREM FOR ORLICZ SPACES AND ITS APPLICATION

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ABSTRACT

In this paper, we show a Marcinkiewicz type interpolation theorem for Orlicz spaces. As an application, we obtain an existence result for a parabolic equation in divergence form.

KEYWORDS

Orlicz space, interpolation, parabolic equation, divergence form

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1. INTRODUCTION

Marcinkiewicz interpolation is one of the important and fundamental results in the theory of interpolation. The classical interpolation theorem for L^p spaces says that if the quasi-linear operator T is of weak (p, p) and of weak (q, q), where $p < q$, then for every $s \in (p, q)$, T is of strong (s, s).

The first interpolation theorem concerning Orlicz spaces as intermediate is due to Orlicz [1]. He proved that any separable Orlicz space $L^\varphi(a, b)$ is an interpolation space between $L^1(a, b)$ and $L^\infty(a, b)$, i.e., if T is any linear bounded operator from $L^1(a, b)$ into itself and bounded from $L^\infty(a, b)$ into itself, then T is bounded from $L^\varphi(a, b)$ into itself. For some generalizations of Orlicz's nonlinear interpolation theorem, we refer to [2].

After that, the problem is described as: if T is any bounded linear operator from L^{φ_i} into L^{ψ_i}, $i = 0, 1$, then under what conditions on φ and ψ is it true that T is also bounded from L^φ into L^ψ?

The assumption is

$$\varphi^{-1} = \varphi_0^{-1}\rho\left(\frac{\varphi_1^{-1}}{\varphi_0^{-1}}\right), \psi^{-1} = \psi_0^{-1}\rho\left(\frac{\psi_1^{-1}}{\psi_0^{-1}}\right), \tag{1.1}$$

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for some concave function ρ. For the precise references and proofs, see [3, 4, 5].

Note that (1.1) is only a sufficient condition, since concave function ρ does not always exist for two Young functions, for example

$$\varphi_0(s) = s, \varphi_1(s) = s^2, \varphi(s) = sln(1 + e^s). \tag{1.2}$$

The interpolation theorems for weak Orlicz martingale spaces are also obtained by some authors (cf. [6]).

Our result is the following:

THEOREM 1.1. Let φ_i $(i = 0, 1)$ be Ψ-functions and satisfy $\triangle_2$-condition (the definitions can be found in Section 2),

$$\frac{\varphi_0'(s)}{\varphi_0(s)} \le \alpha\frac{\varphi'(s)}{\varphi(s)}, \frac{\varphi'(s)}{\varphi(s)} \le \beta\frac{\varphi_1'(s)}{\varphi_1(s)}, \tag{1.3}$$

where $0 < \alpha, \beta < 1$.

Let T be a quasi-linear operator, i.e.

$$|T(f + g)(x)| \le C(|Tf(x)| + |Tg(x)|), \tag{1.4}$$

for every f, g in some function spaces. And it is a bounded operator in $L_w^{\varphi_0}$ and $L_w^{\varphi_1}$, i.e.

$$\sup \varphi_0(t)\mu_{Tf}(t) \le C \sup \varphi_0(t)\mu_f(t), \tag{1.5}$$

$$\sup \varphi_1(t)\mu_{Tf}(t) \le C \sup \varphi_1(t)\mu_f(t). \tag{1.6}$$

Then

$$\int_\Omega \varphi(|Tf(x)|dx) \le C \int_\Omega \varphi(|f(x)|dx). \tag{1.7}$$

The paper is organized as follows. In Section 2, we introduce some necessary definitions and auxiliary results. In Section 3, the proof of Theorem 1.1 is given. In the last Section, an existence result for a parabolic equation in divergence form is obtained by applying Theorem 1.1.

2. PRELIMINARIES

Let $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ be a C^1 function. We say that φ is a Ψ-function if it is convex and satisfies $\varphi(0) = 0$,$\varphi(s) > 0$ for $s > 0$.

LEMMA 2.1. If φ is a Ψ-function, then

$$\lim_{s\longrightarrow\infty} \varphi(s) = \infty \quad \text{and} \quad \frac{\varphi'(s)}{\varphi(s)} \ge \frac{1}{s}. \tag{2.1}$$

Proof. For any s, t with $0 < s < t$, by the definition of Ψ-function, we have

$$\varphi(s) = \varphi\left(\frac{s}{t}t + \left(1 - \frac{s}{t}\right)0\right) \le \frac{s}{t}\varphi(t). \tag{2.2}$$

This gives that $\frac{\varphi(s)}{s}$ is increasing for $s > 0$. □

Since φ is C^1, we see that

$$0 \le \frac{d}{ds}\frac{\varphi(s)}{s} = \frac{\varphi'(s)s - \varphi(s)}{s^2}, \tag{2.3}$$

and thus $\frac{\varphi'(s)}{\varphi(s)} \ge \frac{1}{s}$.

Let $s > 1$. From the above discussion,

$$\frac{\varphi(1)}{1} \le \frac{\varphi(s)}{s}, \tag{2.4}$$

which yields that $\lim_{s\longrightarrow\infty} \varphi(s) = \infty$.

A Ψ-function φ is said to satisfy $\triangle_2$-condition, if there is a constant $C > 0$ such that $\varphi(2s) \le C\varphi(s)$ for any $s > 0$.

A Ψ-function φ is said to satisfy $(\mathrm{Dec})_b$-condition, if there is $b > 1$ such that $\frac{\varphi(s)}{s^b}$ is decreasing.

As is well known, a Ψ-function φ satisfies $\triangle_2$-condition iff it satisfies $(\mathrm{Dec})_b$-condition for some $b > 1$.

LEMMA 2.2. If φ is a Ψ-function and satisfies $(Dec)_b$-condition, then

$$\varphi\left(\frac{s}{C}\right) \geq \inf\left\{\frac{1}{C}, \frac{1}{C^b}\right\}\varphi(s). \tag{2.5}$$

Let f a be measurable function. We say that $f \in K^{\phi}(\Omega)$ if

$$\int_\Omega \varphi\big(|f(x)|\big)\,dx < \infty. \tag{2.6}$$

The Orlicz space $L^\varphi(\Omega)$ is a Banach space with respect to the following Luxemburg norm

$$\|f\|_{L^\varphi} = \inf\left\{\lambda > 0; \int_\Omega \varphi\left(\frac{|f(x)|}{\lambda}\right)dx \leq 1\right\}. \tag{2.7}$$

If φ satisfies $\triangle_2$–condition, then $K^\varphi(\Omega) = L^\varphi(\Omega)$ (cf. [7]).

The distribution function of f is defined by

$$\mu_f(t) = meas\{x \in \Omega : |f(x)| > t\}. \tag{2.8}$$

The following lemma is well known.

LEMMA 2.3.

$$\int_\Omega \varphi(|f(x)|)dx = \int_0^\infty \varphi'(t)\mu_f(t)dt. \tag{2.9}$$

Weak Orlicz space is also defined,

$$\|f\|_{L^\varphi_w} = \inf\left\{\lambda > 0 : \varphi\left(\frac{t}{\lambda}\right)\mu_f(t) \leq 1\right\}. \tag{2.10}$$

$L^\varphi_w(\Omega)$ is a quasi-Banach space, see [8].

LEMMA 2.4. Let φ be a Ψ-function and satisfy $\triangle_2$-condition, then

$$\sup \varphi(t)\mu_f(t) \leq \int_\Omega \varphi(|f(x)|)dx. \tag{2.11}$$

Proof.

$$\varphi(t)\mu_f(t) \leq \int_{|f(x)|\geq t} \varphi(|f(x)|)dx \leq \int_\Omega \varphi(|f(x)|)dx. \qquad \square$$

For two Ψ-functions φ_i, $i = 1, 2$, we want to define that φ_1 is "smaller" than φ_2 iff $\frac{\varphi_2(s)}{\varphi_1(s)}$ is increasing on $(0, \infty)$. It is equivalent to $\frac{\varphi_1'(s)}{\varphi_1(s)} \leq \frac{\varphi_2'(s)}{\varphi_2(s)}$. For example, $\varphi_1(s) = s^p$, $\varphi_1(s) = s^q$ and $p \leq q$.

LEMMA 2.5. Let φ_1 and φ_2 be two Ψ-functions. If there is α such that $0 < \alpha < 1$ and

$$\frac{\varphi_1'(s)}{\varphi_1(s)} \leq \alpha\frac{\varphi_2'(s)}{\varphi_2(s)}, \tag{2.12}$$

then,

$$\lim_{s\to 0}\frac{\varphi_2(s)}{\varphi_1(s)} = 0, \lim_{s\to\infty}\frac{\varphi_1(s)}{\varphi_2(s)} = 0. \tag{2.13}$$

Proof.

$$\frac{d}{dx}\frac{\varphi_2^\alpha(s)}{\varphi_1(s)} = \frac{\alpha\varphi_2'(s)\varphi_1(s) - \varphi_1'(s)\varphi_2(s)}{\varphi_1^{3-\alpha}(s)} \geq 0, \tag{2.14}$$

then $\frac{\varphi_2^\alpha(s)}{\varphi_1(s)}$ is increasing on $(0, \infty)$.

If $s<1$, we have

$$\frac{\varphi_2(s)}{\varphi_1(s)}=\frac{\varphi_2^{\alpha}(s)}{\varphi_1(s)}\varphi_2^{1-\alpha}(s)\leq\frac{\varphi_2^{\alpha}(1)}{\varphi_1(1)}\varphi_2^{1-\alpha}(s). \tag{2.15}$$

Letting $s\longrightarrow 0$, then we have the former limit.

If $s>1$, we have

$$\frac{\varphi_2(s)}{\varphi_1(s)}=\frac{\varphi_2^{\alpha}(s)}{\varphi_1(s)}\varphi_2^{1-\alpha}(s)\geq\frac{\varphi_2^{\alpha}(1)}{\varphi_1(1)}\varphi_2^{1-\alpha}(s). \tag{2.16}$$

Letting $s\longrightarrow\infty$, then we obtain the latter limit. □

3. PROOF OF THEOREM

Proof. By Lemma 2.3,

$$\int_{\Omega}\phi(|Tf(x)|)dx=\int_0^{\infty}\varphi'(s)\mu_{Tf}(s)ds. \tag{3.1}$$

For any $s>0$, a 'big' part and a 'small' part of f are defined as follows:

$$f_0(x)=\begin{cases}f(x), & \text{if } |f(x)|\geq s,\\ 0, & \text{otherwise },\end{cases},\qquad f_1(x)=\begin{cases}f(x), & \text{if } |f(x)|<s,\\ 0, & \text{otherwise },\end{cases}. \tag{3.2}$$

Thanks to the bounded operator in $L_w^{\varphi_0}$ and $L_w^{\varphi_1}$, and Lemma 2.4,

$$\varphi_0(t)\mu_{T(f_0)}(t)\leq C\sup\varphi_0(t)\mu_{f_0}(t)\leq C\int_{\Omega}\varphi_0(|f_0(x)|)dx\leq C\int_{|f(x)|>s}\varphi_0(|f(x)|)dx, \tag{3.3}$$

$$\varphi_1(t)\mu_{T(f_1)}(t)\leq C\sup\varphi_1(t)\mu_{f_1}(t)\leq C\int_{\Omega}\varphi_1(|f_1(x)|)dx\leq C\int_{|f(x)|<s}\varphi_1(|f(x)|)dx. \tag{3.4}$$

If $|Tf(x)|>s$, then $|Tf_0(x)|>\frac{s}{2C}$ or $|Tf_1(x)|>\frac{s}{2C}$. Therefore,

$$\mu_{Tf}(s)\leq\mu_{Tf_0}\left(\frac{s}{2C}\right)+\mu_{Tf_1}\left(\frac{s}{2C}\right). \tag{3.5}$$

Combing (3.3) and (3.4),

$$\mu_{Tf}(s)\leq\frac{\int_{|f(x)|>s}\varphi_0(|f(x)|)dx}{\varphi_0(\frac{s}{2C})}+\frac{\int_{|f(x)|<s}\varphi_0(|f(x)|)dx}{\varphi_1(\frac{s}{2C})}. \tag{3.6}$$

Recalling (3.1), Fubini Theorems and $\triangle_2$–condition, we obtain

$$\begin{aligned}&\int_{\Omega}\varphi(|Tf(x)|)dx\\ &\leq\int_0^{\infty}\frac{\varphi'(s)}{\varphi_0(\frac{s}{2C})}ds\int_{|f(x)|>s}\varphi_0(|f(x)|)dx+\int_0^{\infty}\frac{\varphi'(s)}{\varphi_1(\frac{s}{2C})}ds\int_{|f(x)|<s}\varphi_1(|f(x)|)dx\\ &=\int_{\Omega}dx\int_0^{|f(x)|}\frac{\varphi_0(|f(x)|)\varphi'(s)}{\varphi_0(\frac{s}{2C})}ds+\int_{\Omega}dx\int_{|f(x)|}^{\infty}\frac{\varphi_1(|f(x)|)\varphi'(s)}{\varphi_1(\frac{s}{2C})}ds\\ &\leq\sup\{2C,(2C)^b\}\left(\int_{\Omega}dx\int_0^{|f(x)|}\frac{\varphi_0(|f(x)|)\varphi'(s)}{\varphi_0(s)}ds+\int_{\Omega}dx\int_{|f(x)|}^{\infty}\frac{\varphi_1(|f(x)|)\varphi'(s)}{\varphi_1(s)}ds\right).\end{aligned}$$

$$\begin{aligned}
&\int_0^{|f(x)|} \frac{\varphi_0(|f(x)|)\varphi'(s)}{\varphi_0(s)} ds \\
&= \int_0^{|f(x)|} \frac{\varphi_0(|f(x)|)}{\varphi_0(s)} d\varphi(s) \text{ (by integrating by part and Lemma 2.5)} \\
&= \varphi(|f(x)|) + \int_0^{|f(x)|} \frac{\varphi(s)\varphi(|f(x)|)\varphi_0'(s)}{\varphi_0^2(s)} ds \\
&\le \varphi(|f(x)|) + \alpha \int_0^{|f(x)|} \frac{\varphi(|f(x)|)\varphi'(s)}{\varphi_0(s)} ds \\
&\le \frac{\varphi(|f(x)|)}{1-\alpha}.
\end{aligned}$$

$$\begin{aligned}
&\int_{|f(x)|}^{\infty} \frac{\varphi_1(|f(x)|)\varphi'(s)}{\varphi_1(s)} ds \\
&= \int_{|f(x)|}^{\infty} \frac{\varphi_1(|f(x)|)}{\varphi_1(s)} d\varphi(s) \text{ (by integrating by part and Lemma 2.5)} \\
&= \varphi(|f(x)|) + \int_{|f(x)|}^{\infty} \frac{\varphi(s)\varphi(|f(x)|)\varphi_1'(s)}{\varphi_1^2(s)} ds \\
&\le \varphi(|f(x)|) + \beta \int_{|f(x)|}^{\infty} \frac{\varphi(|f(x)|)\varphi'(s)}{\varphi_1(s)} ds \\
&\le \frac{\varphi(|f(x)|)}{1-\beta}.
\end{aligned}$$

Hence

$$\int_\Omega \varphi(|Tf(x)|)dx \le C\left(\frac{1}{1-\alpha} + \frac{1}{1-\beta}\right)\int_\Omega \varphi(|f(x)|)dx \qquad \square$$

4. AN APPLICATION TO REGULARITY FOR A PARABOLIC EQUATION IN DIVERGENCE FORM

As an application, we consider the following nonhomogeneous second order parabolic differential equation in divergence form:

$$\frac{\partial u}{\partial t} + \triangle u = \operatorname{div} f, \text{ in } \Omega \times (0,T), \tag{4.1}$$

where f is a given function. While it is equipped with the following initial/boundary-value conditions:

$$u(x,0) = 0 \text{ on } \Omega, \tag{4.2}$$

and

$$u(x,t) = 0 \text{ on } \partial\Omega \times [0,T]. \tag{4.3}$$

The $W^{1,p}$ regularity theory can be found in [9],

THEOREM 4.1. For any $f \in L^p$ $(p > 1)$, there exists a unique weak solution u of parabolic systems and satisfies

$$\int_0^T \int_\Omega |u|^p dxdt + \int_0^T \int_\Omega |\nabla u|^p dxdt \le C \int_0^T \int_\Omega |f|^p dxdt. \tag{4.4}$$

In the remains, we will give a new proof to the following theorem.

THEOREM 4.2. If φ is a Ψ-function, and there exist $a > 1$ and $b > 1$, such that

$$\frac{\varphi(s)}{s^a} \text{ is increasing and } \frac{\varphi(s)}{s^b} \text{ is decreasing,} \tag{4.5}$$

then, the solution u of the parabolic systems satisfies

$$\int_0^T \int_\Omega \varphi(|\nabla u|)dxdt \le C \int_0^T \int_\Omega \varphi(|f|)dxdt. \tag{4.6}$$

Proof. For a given measurable function f, a linear operator $T : f \longrightarrow \nabla u$ is defined by the above parabolic systems.

Denote by $\varphi_0(s) = s^{a-\varepsilon}$, $\varphi_1(s) = s^{b+\varepsilon}$, where $\varepsilon > 0$ and $a - \varepsilon > 1$, then, by Theorem 4.1, T is of strong (φ_0, φ_0) and of strong (φ_1, φ_1). Therefore, T is of weak (φ_0, φ_0) and of weak (φ_1, φ_1).

Since $\frac{\varphi(s)}{s^a}$ is increasing, and we have

$$\frac{\varphi'(s)}{\varphi(s)} \ge \frac{a}{s} = \frac{a}{a-\varepsilon}\frac{a-\varepsilon}{s} = \frac{a}{a-\varepsilon}\frac{\varphi_0'(s)}{\varphi_0(s)}. \tag{4.7}$$

Since $\frac{\varphi(s)}{s^b}$ is decreasing, and we have

$$\frac{\varphi'(s)}{\varphi(s)} \le \frac{b}{s} = \frac{b}{b+\varepsilon}\frac{b+\varepsilon}{s} = \frac{b}{b+\varepsilon}\frac{\varphi_1'(s)}{\varphi_1(s)}. \tag{4.8}$$

And the conditions of Theorem 1.1 hold for $\alpha = \frac{a-\varepsilon}{a}$, $\beta = \frac{b}{b+\varepsilon}$. The theorem follows from Theorem 1.1. □

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