ON QUASI $I$-STATISTICAL CONVERGENCE OF TRIPLE SEQUENCES IN CONE METRIC SPACES

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ABSTRACT

Fast [12] is credited with pioneering the field of statistical convergence. This topic has been researched in many spaces such as topological spaces, cone metric spaces, and so on (see, for example [19, 21]). A cone metric space was proposed by Huang and Zhang [17]. The primary distinction between a cone metric and a metric is that a cone metric is valued in an ordered Banach space. Li et al. [21] investigated the definitions of statistical convergence and statistical boundedness of a sequence in a cone metric space. Recently, Sakaoğlu and Yurdakadim [29] have introduced the concepts of quasi-statistical convergence. The notion of quasi $I$-statistical convergence for triple and multiple index sequences in cone metric spaces on topological vector spaces is introduced in this study, and we also examine certain theorems connected to quasi $I$-statistically convergent multiple sequences. Finally, we will provide some findings based on these theorems.

KEYWORDS
Cone metric, multiple sequence, convergence, ideal convergence

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1. INTRODUCTION

The concept of statistical convergence of sequences in real numbers was introduced in 1951 by Fast in [12]. This concept has been studied under different names (see, [2, 4, 5, 6, 8, 13, 24, 25, 29, 30, 33, 34, 35]). Cone metric spaces were actually defined by several authors many years ago and took place under different names in the literature (see, for example [1, 3, 7, 17, 21, 22, 23]). On the other hand, in [15], the authors defined the concept of a quasi-statistical filter. Recently, Sakaoğlu and Yurdakadim [29] defined the notations of quasi-statistical convergence and strongly-Cesàro summability by relying on [8, 9] and [15], and they found some inclusion theorems between these concepts for single sequences. Ganguly and Dafadar [14] established some results relating to statistical convergence and quasi statistical convergence for double sequences.

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The idea of statistical convergence has been further extended to $I$-convergence in [19] using the notion of ideals of natural numbers, with many interesting consequences. Further investigations in this direction and more applications of ideals can be found in [10, 16, 27, 28]. In another direction, a new type of convergence was introduced in [11] called $I$-statistical convergence.

In this article’s Section 2, we will introduce the reader to the basic concepts of $I$-statistical convergence for single and triple sequences, give some consequences of this convergence, definitions and properties of cone metric spaces and, the concept of quasi-statistical convergence. In Section 3, we will introduce the quasi $I$-statistical convergence and quasi $I'$-statistically convergence for triple and multiple index sequences in topological vector space (tvs) valued cone metric space (cms).

2. PRELIMINARIES

The set of all positive integers is denoted by $\mathbb{N}$ throughout this article, and the set of all real numbers is denoted by $\mathbb{R}$. Readers can refer to $[12, 13, 14, 17, 19, 21, 25, 29, 36]$ for any concepts in this article that aren’t defined and for a subset $A$ of $\mathbb{N}$, $|A|$ stands for the cardinality of $A$.

**DEFINITION 2.1.** Let $A \subset \mathbb{N}$ and $A(n) = \{k \in A : k \leq n\}$, $\forall n \in \mathbb{N}$. Then

$$
\delta(A) := \limsup_{n \to \infty} \frac{|A(n)|}{n}
$$

are termed as set $A$’s upper and lower asymptotic densities, respectively. The natural (or asymptotic) density of the set $A$ is indicated by the symbol

$$
\delta(A) = \delta(A) := \lim_{n \to \infty} \frac{|A(n)|}{n}.
$$

If any, all three densities are in $[0, 1]$. Also, if $\delta(A) = 1$, then $S$ is called statistically dense. It should be easily obtained that $\delta(\mathbb{N} - A) = 1 - \delta(A)$ for each $A \subset \mathbb{N}$.

**DEFINITION 2.2.** With the use of the information above, we can conclude that $\{x_k\}_{k \in \mathbb{N}}$ is statistically convergent to $x$, provided that for $\forall \varepsilon > 0$,

$$
\delta(\{k \in \mathbb{N} : |x_k - x| \geq \varepsilon\}) = 0.
$$

We write $st$-$\lim x_k = x$ if $\{x_k\}_{k \in \mathbb{N}}$ is statistically convergent to $x$.

Additionally, $I$-convergence in a metric space was introduced by Kostyrko et al. [19]. This definition is depending upon the definition of an ideal $I$ in $\mathbb{N}$.

**DEFINITION 2.3.** A family $I \subset 2^\mathbb{N}$ is called an ideal if the following properties hold true:

(i) $\emptyset \not\in I$.

(ii) For each $P, R \in I$ we have $P \cup R \in I$.

(iii) For each $P \in I$ and each $R \subset P$ we have $R \in I$.

An ideal $I$ on $\mathbb{N}$ for which $I \neq 0(\mathbb{N})$ is called a proper ideal. A proper ideal $I$ is called admissible if $I$ contains all finite subsets of $\mathbb{N}$.

A family of sets $F \subset 2^\mathbb{N}$ is a filter in $\mathbb{N}$ iff (i) $\emptyset \not\in F$; (ii) for each $P, R \in F$ we have $P \cap R \in F$; and (iii) for each $P \in F$ and each $R \supset P$ we have $R \in F$.

If $I$ is proper ideal of $\mathbb{N}$ ($\mathbb{N} \not\in I$), then the family of sets

$$
F(I) = \{S \subset \mathbb{N} : 3P \in I, S = \mathbb{N} \setminus P\}
$$

is a filter of $\mathbb{N}$, it is called the filter associated with the ideal.

An admissible ideal $I \subset 2^\mathbb{N}$ is said to hold the property $(AP)$ if for every family $\{P_n\}_{n \in \mathbb{N}}$ with $P_n \cap P_k = \emptyset$ $(n \neq k)$, $P_\infty \in I$ ($n \in \mathbb{N}$) there is a family $\{R_n\}_{n \in \mathbb{N}}$ such that $(P_\infty \setminus R_n) \cup (R_n \setminus P_k)$ for $\forall k \in \mathbb{N}$ and a limit set $R = \bigcup_{k=1}^{\infty} R_k \in I$ ([19]).

**DEFINITION 2.4 ([19]).** A sequence of reals $\{x_n\}_{n \in \mathbb{N}}$ is called $I$-convergent to $L$ if for $\forall \varepsilon > 0$ the set

$$
A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in I.
$$
See the references in [26, 27] for more information on $I$-convergent.

A nontrivial ideal $I$ of $\mathbb{N}^2$ is called strongly admissible if $\{i\} \times \mathbb{N} \in I$ and $\mathbb{N} \times \{i\} \in I$ for $\forall i \in \mathbb{N}$. Obviously, a strongly admissible ideal is also admissible. Let

$$I_0 = \{ A \in \mathbb{N}^2 : (\exists m(A) \in \mathbb{N}) (i, j \geq m(A) \Rightarrow (i, j) \notin A) \} .$$

So $I_0$ is a nontrivial strongly admissible ideal and clearly an ideal $I$ is strongly admissible iff $I_0 \subseteq I$.

We now remember back to the following fundamental concepts from [18, 22] which are necessary for the remainder of the article.

**DEFINITION 2.5.** Let $E$ be a Hausdorff tvs with the zero vector $0$. A subset $P$ of $E$ is called a (convex) cone if it satisfies the following conditions:

(i) $P \neq \{0\}, P \neq \emptyset$ and $P$ is closed;
(ii) $\lambda P \subset P$ for $\forall \lambda \geq 0$ and $P + P \subset P$;
(iii) $\{0\} = P \cap (-P)$.

Given a $P \subset E$ cone, we can define a partial ordering $\preceq$ with respect to $P$ by defining $x \preceq y \iff y - x \in P$. We shall write $x < y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$, where int$P$ represent the set of the interior points of $P$. The sets of the form $[x, y]$ are named order-intervals and are defined as the follows:

$$[x, y] = \{ z \in E : x \preceq z \preceq y \} .$$

Order-intervals are observed to be convex. If $[x, y] \subset A$ while $x, y \in A$ and $x \preceq y$, then $A \subset E$ is called order-convex.

It is order-convex if ordered tvs $(E, P)$ has a neighborhoods’ base of 0 that are made up of order-convex sets. Accordingly, the cone $P$ is called a normal cone. Considering the normed space, this condition means that the unit ball is order-convex; it is equivalent to the condition that $\exists k$ with $x, y \in E$ and $0 \leq x \preceq y \Rightarrow |x| \leq k |y|$. The smallest constant $k$ is called the normal constant of $P$ [18].

If each of the increasing sequence that is bounded in $P$ is convergent then, we describe to $P$ as a regular cone. To put it another way, if a sequence $\{x_n\}$ exists such that

$$x_1 \preceq x_2 \preceq \ldots \preceq x_n \preceq \ldots \preceq y,$$

then $\exists x \in E$ such that $\lim_{n \to \infty} |x_n - x| = 0$. Similarly, the $P$ cone is regular, if all decreasing sequences that are bounded from below converges. If $P$ is a regular cone, it is known to be a normal cone.

Let $E$ be a tvs, $V \subset E$ is an absolutely convex and absorbent subset, the corresponding Minkowski functional $f_V : E \to \mathbb{R}$ is defined

$$x \mapsto f_V(x) = \inf \{ \lambda > 0 : x \in \lambda V \} .$$

It is a semi-norm on $E$. If $V$ is an absolutely convex neighborhood of $0 \in E$, then $f_V$ is continuous and

$$\{ x \in E : f_V(x) < 1 \} = \text{int} V \subset V \subset \overline{V} = \{ x \in E : f_V(x) \leq 1 \} .$$

Let $e \in \text{int} P$ and $(E, P)$ be an ordered tvs. After that

$$[-e, e] = (P - e) \cap (e - P) = \{ z \in E : -e \preceq z \preceq e \}$$

is an absolutely convex neighborhood of 0. We denote the corresponding Minkowski functional $f_{[-e]}$ by $f_e$. It can be verified that $\text{int} [-e, e] = (\text{int} P - e) \cap (e - \text{int} P)$. If $P$ is normal and solid, then the Minkowski functional $f_e$ is the norm on $E$. Furthermore, it is an increasing function on $P$. In fact, for $0 \leq x_1 \preceq x_2$ the set $\{ \lambda : x_1 \in \lambda [-e, e] \}$ is the subset of $\{ \lambda : x_2 \in \lambda [-e, e] \}$ and it follows that $f_e(x_1) \preceq f_e(x_2)$.

**DEFINITION 2.6.** Let $X \neq \emptyset$. Suppose that $d : X \times X \to E$ satisfies

(i) $d(x, y) = 0$ iff $x = y$ and $0 \leq d(x, y)$ for $\forall x, y \in X$;
(ii) $d(y, x) = d(x, y)$ for $\forall x, y \in X$;
(iii) $d(x, z) + d(z, y) \preceq d(x, y)$ for $\forall x, y, z \in X$.

Then $d$ is called a cone metric on $X$. $(X, d)$ is called a cone metric space (cms). Obviously, the notion of cone metric spaces generalizes the notion of metric spaces.
**Definition 2.7.** Let \((X, d)\) be a cms. \(\{x_n\}_{n \in \mathbb{N}}\) be a sequence in cms \(X\) and let \(x \in X\). If for \(\forall c \in E\) with \(0 < c\) there is \(N \in \mathbb{N}\) such that for all \(n > N\), \(d(x_n, x) < c\), then \(\{x_n\}_{n \in \mathbb{N}}\) is called convergent to \(x\) and it is called the limit of the sequence \(\{x_n\}_{n \in \mathbb{N}}\).

**Definition 2.8.** Let \((X, d)\) be a cms. \(\{x_n\}_{n \in \mathbb{N}}\) be a sequence in cms \(X\). If for any \(c \in E\) with \(0 < c\) there is \(N \in \mathbb{N}\) such that for all \(n, m > N\), \(d(x_n, x_m) < c\), then \(\{x_n\}_{n \in \mathbb{N}}\) is called a Cauchy sequence in \(X\). All Cauchy sequences in \(X\) are convergent in \(X\), and \(X\) is called a complete cms [22].

**Definition 2.9.** A sequence \(\{x_n\}_{n \in \mathbb{N}}\) in \(X\) is said to be \(I\)-convergent to \(\xi \in X\) if there is a set \(M \in \mathcal{P}(I), M = \{m_1 < m_2 < ... < m_j < ...\}\) such that \(\lim_{j \to \infty} x_{m_j} = \xi\), that is for \(\forall c \in E\) with \(c < 0\), there is \(p \in \mathbb{N}\) such that \(c = d(x_m, \xi) \in \text{int} P\), for \(\forall j \geq p\).

**Definition 2.10.** Let \(\{x_n\}_{n \in \mathbb{N}}\) be a sequence which is called to be \(I\)-statistically convergent to a point \(\xi \in X\) provided that for \(\epsilon > 0, y > 0\)

\[
\left\{ n \in N : \frac{1}{n} \sum_{k \leq n} |x_k - \xi| \geq \epsilon, \frac{\delta_I(A(\epsilon))}{n} \right\} \in I
\]

or, equivalently if for \(\forall \epsilon > 0\),

\[
\delta_I(A(\epsilon)) = I - \lim_{n \to \infty} \frac{|A_n(\epsilon)|}{n} = 0,
\]

where \(A_n(\epsilon) = \{k \leq n : |x_k - \xi| \geq \epsilon\}\).

Firstly, [31] introduced the concepts of triple sequences and statistically convergent triple sequences. The following definition of the extension of \(I\)-convergence to triple sequences in the tvs-cms are taken from the [34] research.

**Definition 2.11.** Let \((X, d)\) be a tvs-cms. A triple sequence \(x = \{x_{i,j,k}\}_{i,j,k \in \mathbb{N}}\) in \(X\) is called \(I_3\)-convergent to \(\xi \in X\) if for \(\forall c \in E\) with \(c > 0\), there is \(N \in \mathbb{N}\) such that

\[
A(c) = \{(i,j,k) \in \mathbb{N}^3 : d(x_{i,j,k}, \xi) \leq c\} \in I_3.
\]

It is denoted by \(I_3 - \lim_{i,j,k \to \infty} x_{i,j,k} = \xi\).

In [29], the notations of quasi-statistical convergence and quasi-density were defined as follows.

**Definition 2.12.** Let \(c = \{c_n\} \in \mathbb{R}^+\) be a sequence with

\[
\lim_{n \to \infty} c_n = \infty \quad \text{and} \quad \limsup_{n \to \infty} \frac{c_n}{n} < \infty.
\]

The quasi density of a subset \(K \subset \mathbb{N}\) with respect to the sequence \(c = \{c_n\}\) is defined by

\[
\delta_{c}(K) = \lim_{n \to \infty} \frac{1}{n} \sum_{k \leq n : k \in K} |c_n|.
\]

**Definition 2.13.** A sequence \(\{x_n\}_{n \in \mathbb{N}}\) in \(\mathbb{R}\) is named quasi-statistical convergent to \(x\) provided that for \(\forall \epsilon > 0\) the set \(K_{\epsilon} = \{k \in \mathbb{N} : |x_k - x| \geq \epsilon\}\) has quasi-density zero. It is denoted by \(\text{sl}_{\epsilon} - \lim_{\epsilon \to \infty} x_n = x\).

Since it is known [37] that any cone metric space is a first countable Hausdorff topological space with the topology induced by the open balls defined naturally for each element \(z\) in \(X\) and for each element \(c\) in \(\text{int} P\). So as in [20] we can show that \(I^\ast\)-convergence always implies \(I\)-convergence but the converse is not true. The two concepts are equivalent iff the ideal \(I\) satisfies condition \((AP)\).

### 3. New Results on Triple Sequences

We define the concepts of quasi \(I_1\)-statistical convergence and quasi \(I_3\)-statistical convergence of triple sequences in tvs-cms in this section.

Throughout this paper, we assume that \(c = \{c_{mno}\} \in \mathbb{R}^+\) be a triple sequence with

\[
\lim_{m,n,o \to \infty} c_{mno} = \infty \quad \text{and} \quad \limsup_{m,n,o \to \infty} \frac{c_{mno}}{mno} < \infty.
\]
DEFINITION 3.1. Let $(X, d)$ be a tvs-cms. A triple sequence $\{v_{i,j,k}\}_{i,j,k \in \mathbb{N}}$ in a tvs-cms $(X, d)$ is called quasi $I_3$-statistically convergent to a point $\xi \in X$ provided that for $\forall \alpha, \gamma \in E$ with $\alpha \geq 0, \gamma \geq 0$

$$\left\{(m, n, o) \in \mathbb{N}^3 : \frac{1}{c_{mno}} \left\{i \leq m, j \leq n, k \leq o : d(v_{i,j,k}, \xi) \geq \alpha\right\} \geq \gamma \right\} \in I_3,$$

or, equivalently if for $\forall \alpha, \gamma \in E$ with $\alpha \gg 0, \gamma \gg 0$,

$$\delta^I_3(A(\alpha, \gamma)) = (I_3)_q - \lim_{m,n,o \to \infty} \frac{|A_{mno}(\alpha, \gamma)|}{c_{mno}} = 0,$$

where $A_{mno}(\alpha, \delta) = \{i \leq m, j \leq n, k \leq o : d(v_{i,j,k}, \xi) \geq \alpha\}$.

If a triple sequence $\{v_{i,j,k}\}_{i,j,k \in \mathbb{N}}$ is quasi $I_3$-statistically convergent to $\xi$ in a tvs-cms $(X, d)$ then we write

$$I_3 - s_q - \lim_{i,j,k \to \infty} v_{i,j,k} = \xi,$$

$\xi$ is called quasi $I_3$-statistical limit of the sequence $\{v_{i,j,k}\}_{i,j,k \in \mathbb{N}}$.

If we take $\{c_{mno}\} = \{mno\}$, then we obtain that $\{v_{i,j,k}\}_{i,j,k \in \mathbb{N}}$ is $I_3$-statistical convergent given in [32].

DEFINITION 3.2. A triple sequence $v = \{v_{i,j,k}\}_{i,j,k \in \mathbb{N}}$ in a tvs-cms $(X, d)$ is called quasi $I_3$-statistical bounded if there exists a positive number $M$ such that for any $\gamma \in E, \gamma \geq 0$ the set

$$\left\{(m, n, o) \in \mathbb{N}^3 : \frac{1}{c_{mno}} \left\{i \leq m, j \leq n, k \leq o : d(v_{i,j,k}, 0) \geq M\right\} \geq \gamma \right\} \in I_3,$$

EXAMPLE 3.3. Let $I_3$ be an ideal $(I_3)_0$ of $\mathbb{N}^3$ and $d : R^3 \times R^3 \to (E, P)$ be a cone metric ($P \subset E; E$ is tvs and $P$ is cone). If we define triple sequence $\{v_{i,j,k}\}_{i,j,k \in \mathbb{N}}$ by

$$v_{i,j,k} = \begin{cases} (i, 1, 1), & \text{if } j, k = 2, i \in \mathbb{N} \\ \left(\frac{1}{\sqrt{i+j+k}}, \frac{1}{j+k}, \frac{1}{i+k}\right), & \text{otherwise} \end{cases}$$

then $\{v_{i,j,k}\}_{i,j,k \in \mathbb{N}}$ is unbounded but this sequence is quasi $I_3$-statistical convergent.

DEFINITION 3.4. A triple sequence $v = \{v_{i,j,k}\}_{i,j,k \in \mathbb{N}}$ in a tvs-cms $(X, d)$ is called a quasi $I_3$-statistical Cauchy sequence if $\forall \epsilon > 0, \gamma > 0$ there exists $(p, q, r) \in \mathbb{N}^3$ such that

$$\left\{(m, n, o) \in \mathbb{N}^3 : \frac{1}{c_{mno}} \left\{i \leq m, j \leq n, k \leq o : d(v_{i,j,k}, v_{i+j+k, q+r}) \geq \epsilon\right\} \geq \gamma \right\} \in I_3.$$

From Definitions 3.1 and 3.4, we can give the following result.

COROLLARY 3.5. Let $v = \{v_{i,j,k}\}_{i,j,k \in \mathbb{N}}$ be a triple sequence in a tvs-cms $(X, d)$, and $e \in \text{int } P$ and $f_\epsilon$ be Minkowski functional of $[-\epsilon, \epsilon]$ and $d_f = f_\epsilon \circ d$. Then

(i) A triple sequence $v$ is in tvs cone that is quasi $I_3$-statistically convergent to $\xi$ iff $d_f(v_{i,j,k}, \xi)$ is quasi $I_3$-statistically convergent to $0$ ($i, j, k \to \infty$).

(ii) $v$ is a quasi $I_3$-statistically Cauchy sequence iff $d_f(v_{i,j,k}, v_{i+p, q+r})$ is quasi $I_3$-statistically convergent to $0$ ($i, j, k, p, q, r \to \infty$).

THEOREM 3.6. Let $(X, d)$ be a tvs-cms. If a triple sequence $v = \{v_{i,j,k}\}_{i,j,k \in \mathbb{N}}$ in $(X, d)$ is a quasi $I_3$-statistical convergent, then $v$ is a quasi $I_3$-statistical Cauchy sequence.

Proof. Let $\{v_{i,j,k}\}_{i,j,k \in \mathbb{N}}$ be a quasi $I_3$-statistically convergent to $\xi$. Then for $\forall \epsilon \in \text{int } P$, $\epsilon > 0, \gamma > 0$, there is the set

$$\left\{(m, n, o) \in \mathbb{N}^3 : \frac{1}{c_{mno}} \left\{i \leq m, j \leq n, k \leq o : d(v_{i,j,k}, \xi) \geq \epsilon\right\} \geq \gamma \right\} \in I_3.$$

Select $R, S, U$ such that $(R, S, U) \notin \omega_k$, where

$$\omega_k = \left\{(i, j, k) : d(v_{i,j,k}, \xi) \geq \frac{\epsilon}{2}\right\}.$$
Then, 
\[ d(v_{jk}, v_{RSU}) \leq d(v_{jk}, \xi) + d(\xi, v_{RSU}). \]

Now, we identify
\[ \psi_c = \{ (i, j, k) : d(v_{jk}, v_{RSU}) \gg c \} \quad \text{and} \quad \chi_c = \{ (i, j, k) : d(\xi, v_{RSU}) \gg \frac{c}{2} \}. \]

Hereby, \( \psi_c \subseteq \omega_c \cup \chi_c \) and from this \( v \) is a quasi \( I_3 \)-statistical Cauchy. \( \square \)

**THEOREM 3.7.** Assume that \((X, d)\) be a tvs-cms, \( c \in \text{int} \, P \), \( f \) be Minkowski functional of \([-e, e]\) and \( d_f = f \circ d \). Let \( v = \{ v_{ijk} \}_{i,j,k}^{\infty} \) be a two triple sequence in \((X, d)\). \( v \) is quasi \( I_3 \)-statistical convergent to \( \xi_1 \) and \( y \) is quasi \( I_3 \)-statistical convergent to \( \xi_2 \) in a tvs-cms. Then \( \{ d_f(v_{ijk}, y_{ijk}) \}_{i,j,k}^{\infty} \) is quasi \( I_3 \)-statistical convergent to \( d_f(\xi_1, \xi_2) \) as \( i, j, k \to \infty \).

**Proof.** For each \( \varepsilon > 0 \), we get 
\[ d(v_{ijk}, y_{ijk}) \leq d(v_{ijk}, \xi_1) + d(y_{ijk}, \xi_2) + d(\xi_1, \xi_2). \]

Hereby, 
\[ d(v_{ijk}, y_{ijk}) - d(\xi_1, \xi_2) \leq d(v_{ijk}, \xi_1) + d(y_{ijk}, \xi_2) \]

so 
\[ \{ (i, j, k) \in \mathbb{N}^3 : f_c (d(v_{ijk}, y_{ijk}) - d(\xi_1, \xi_2)) \gg \varepsilon \} \subset \{ (i, j, k) \in \mathbb{N}^3 : d_f(v_{ijk}, \xi_1) \gg \varepsilon \} \cup \{ (i, j, k) \in \mathbb{N}^3 : d_f(y_{ijk}, \xi_2) \gg \varepsilon \} \]

and hence the result is: 
\[ \left\{ (m,n,o) \in \mathbb{N}^3 : \frac{1}{\varepsilon_{mno}} \left| \left\{ i \leq m, j \leq n, k \leq o : f_c (d(v_{ijk}, y_{ijk}) - d(\xi_1, \xi_2)) \gg \varepsilon \right\} \right| \right\} \in I_3. \]

\( \square \)

**THEOREM 3.8.** Let \((X, d)\) be a tvs-cms. \( v = \{ v_{ijk} \}_{i,j,k}^{\infty} \) is a triple sequence in \((X, d)\). If \( v \) is a quasi \( I_3 \)-statistical convergent then \( v \)'s quasi \( I_3 \)-statistical limit point is unique.

**Proof.** Let \( c \gg 0, y \gg 0 \). It is enough to indicate that for \( \forall Y_1, Y_2 \in I_3 \), we get 
\[ (\mathbb{N}^3 \setminus Y_1) \cap (\mathbb{N}^3 \setminus Y_2) = \emptyset, \]

since the sets \( \{ \mathbb{N}^3 \setminus Y_1 \} \) and \( \{ \mathbb{N}^3 \setminus Y_2 \} \) belong to the filter related to \( I_3 \). If \( \xi_1, \xi_2 \) two limits exist which are \( \xi_1 \neq \xi_2 \). Select \( c, y \) with \( 0 < c < \frac{d(\xi_1, \xi_2)}{2}, 0 < y < \frac{d(\xi_1, \xi_2)}{2} \) so

\[ Y_1 = \left\{ (m,n,o) \in \mathbb{N}^3 : \frac{1}{\varepsilon_{mno}} \left| \left\{ i \leq m, j \leq n, k \leq o : d(v_{ijk}, \xi_1) \gg c \right\} \right| \right\} \in I_3 \]

\[ Y_2 = \left\{ (m,n,o) \in \mathbb{N}^3 : \frac{1}{\varepsilon_{mno}} \left| \left\{ i \leq m, j \leq n, k \leq o : d(v_{ijk}, \xi_2) \gg c \right\} \right| \right\} \in I_3. \]

Since the sets \( \{ \mathbb{N}^3 \setminus Y_1 \} \) and \( \{ \mathbb{N}^3 \setminus Y_2 \} \) are in the filter of \( I_3 \), the intersection of these two sets cannot be empty. Therefore, this contradicts the disjunction of \( Y_1 \)'s and \( Y_2 \)'s neighborhoods. \( \square \)

**DEFINITION 3.9.** Let \((X, d)\) be a tvs-cms. A triple sequence \( \{ v_{ijk} \}_{i,j,k}^{\infty} \) in \( X \) is said to be \( I_3^* \)-convergent to \( \xi \in X \) if there exists a set \( L \subseteq \mathcal{P}(I_3) \), i.e., \( \mathbb{N}^3 \setminus L \in I_3 \) such that \( \lim_{i,j,k \to \infty} v_{ijk} = \xi \) and we write
\[ I_3^* - s_{q} = \lim_{i,j,k \to \infty} v_{ijk} = \xi \]

**THEOREM 3.10.** Assume that \( I_3 \) be a strongly admissible ideal. If \( I_3^* - s_{q} = \lim_{i,j,k \to \infty} v_{ijk} = \xi \), then \( I_3 - s_{q} = \lim_{i,j,k \to \infty} v_{ijk} = \xi \).
THEOREM 3.13. \( (\text{If an} \, \mathbb{Z} \, \mathcal{A}) \) Then there is \( n_0 \in \mathbb{N} \) such that \( d\left(v_{ijk}, \xi\right) \leq \epsilon \) for all \( i, j, k \) such that \( (i, j, k) \in \mathcal{Z} \) and \( i, j, k \geq n_0 \). Then
\[
\mathcal{A}(c, \rho) = \left\{ (m, n, o) \in \mathbb{N}^3 : \frac{1}{c_{mno}} \left\{ i \leq m, j \leq n, k \leq o : d\left(v_{ijk}, \xi\right) \leq \epsilon \right\} \right\} \subseteq \mathbb{Z} \cap \left( (\mathbb{Z} \times \{1, 2, \ldots, (n_0 - 1)\} \times \mathbb{N}^2) \cup (\mathbb{N}^2 \times \{1, 2, \ldots, (n_0 - 1)\}) \right) \right\}.
\]
Now
\[
\mathcal{I} = \left\{ m, o \in \mathbb{N}^2 : \frac{1}{c_{m}} \left\{ | j \leq m : d\left(v_{ij}, \xi\right) \leq \epsilon \right\} \right\} \in \mathcal{I}_m.
\]
We also say that \( \{v_{ij}\}_{j \in \mathbb{N}^2} \) is quasi \( \mathcal{I}_m \)-statistically convergent to \( \xi \) in \( \mathbb{N}^3 \) if \( \forall \epsilon > 0, \) \( \forall \rho > 0 \) there is a nontrivial quasi ideal \( \mathcal{I}_n \) of \( \mathbb{N}^3 \times \mathbb{N}^2 \) called strongly admissible if \( \{j\} \times \mathbb{N}^{n-1} \) and \( \mathbb{N}^{n-1} \times \{j\} \) belong to \( \mathcal{I}_n \) for \( \forall i \in \mathbb{N} \).

Without proof, we give results for \( n \)-tuple sequences, which are just a generalization of results for triple sequences.

THEOREM 3.14. If an \( n \)-tuple sequence \( \{v_{ij}\}_{j \in \mathbb{N}^2} \) is quasi \( \mathcal{I}_m \)-statistically convergent in tvs-cms \((X, d)\), then \( \{v_{ij}\}_{j \in \mathbb{N}^2} \) is quasi \( \mathcal{I}_m \)-statistical Cauchy sequence.

PROOF. Let \( \epsilon > 0, \rho > 0 \). Since \( \mathcal{I}_n \rightarrow \mathcal{I}_m \), \( \lim_{j \rightarrow \infty} v_{ijk} = \xi \), so there exists a set \( L \in \mathbb{P}(\mathcal{I}_3) \) such that for \( \mathcal{Z} = \mathbb{N}^3 \setminus L \), \( \mathcal{Z} \in \mathcal{I}_3 \), we have
\[
\mathcal{I}_n \rightarrow \mathcal{I}_m \lim_{j \rightarrow \infty} v_{ijk} = \xi.
\]
Let \( \epsilon > 0 \). Then there is \( n_0 \in \mathbb{N} \) such that \( d\left(v_{ijk}, \xi\right) \leq \epsilon \) then for all \( i, j, k \) such that \( (i, j, k) \in \mathcal{Z} \) and \( i, j, k \geq n_0 \). Then
\[
A(c, \rho) = \left\{ (m, n, o) \in \mathbb{N}^3 : \frac{1}{c_{mno}} \left\{ i \leq m, j \leq n, k \leq o : d\left(v_{ijk}, \xi\right) \leq \epsilon \right\} \right\} \subseteq \mathbb{Z} \cap \left( (\mathbb{Z} \times \{1, 2, \ldots, (n_0 - 1)\} \times \mathbb{N}^2) \cup (\mathbb{N}^2 \times \{1, 2, \ldots, (n_0 - 1)\}) \right) \right\}.
\]
This indicates that \( A(c, \rho) \in \mathcal{I}_3 \). Therefore \( \mathcal{I}_3 \rightarrow \mathcal{I}_m \lim_{j \rightarrow \infty} v_{ijk} = \xi \).

REFERENCES


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