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POLYNOMIAL INTERPOLATION ON SEQUENCES

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ABSTRACT

This short note deals with polynomial interpolation of complex numbers verifying a Lipschitz condition, performed on consecutive points of a given sequence in the plane. We are interested in those sequences which provide a bound of the error at the first uninterpolated point, depending only on its distance to the last interpolated one.

KEYWORDS

polynomial interpolation, Lipschitz condition, contractive sequence

MATHEMATICS SUBJECT CLASSIFICATION (2020)

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1. INTRODUCTION

The interpolation of any sequence of complex numbers $\Gamma = (\eta_i)_{i \in \mathbb{N} \cup \{0\}}$ on a sequence of points $\mathcal{Z} = (z_i)_{i \in \mathbb{N} \cup \{0\}}$ in an open set U of the complex plane $\mathbb{C}$ without limit points in U, by a holomorphic function f on U, that is, $f(z_i) = \eta_i$ for all i, is always possible (it is a consequence of combining the Weierstrass factorization theorem and the Mittag-Leffler theorem). However, if it is imposed that the values of Γ satisfy certain conditions, then the problem arises of knowing for which sequences $\mathcal{Z}$ interpolation by a holomorphic function in an appropriate space is possible (they are called *interpolating sequences*).

The characterization of interpolating sequences for different function spaces is a topic of the Complex Analysis, initiated in the mid-twentieth century, with a lot of contributions to date. In one of the first, Carleson ([2]) proved for $U = \mathbb{D}$ (the unit disc) that given any bounded sequence Γ, there exists a holomorphic bounded function on $\mathbb{D}$ interpolating Γ on $\mathcal{Z}$ if and only if $\mathcal{Z}$ is a *(C)-sequence*, that is,

$$\prod_{\substack{j \in \mathbb{N} \\ j \neq k}} \left| \frac{z_k - z_j}{1 - \overline{z}_k z_j} \right| \geq \gamma > 0.$$

In [4], there is an explicit form for the function performing this interpolation.

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Let n be a positive integer. For finite interpolation, Pick and Nevanlinna had independently proved in 1916 and 1919, respectively, that given distinct points $a_1, \dots, a_n$ in $\mathbb{D}$ and arbitrary values $w_1, \dots, w_n$ in $\mathbb{D}$, there exists a holomorphic function f with $|f| \le 1$ on $\mathbb{D}$ and $f(a_j) = w_j$, $1 \le j \le n$, if and only if the matrix

$$\left(\frac{1 - \overline{w}_k w_j}{1 - \overline{a}_k a_j} \right)_{1 \le j,k \le n}$$

is positive semidefinite (see [3]).

When $U = \mathbb{C}$, we are going to replace infinite interpolation with finite interpolation and restrict holomorphic interpolation to a polynomial interpolation; in addition, we consider it on a finite subset of consecutive points of $\mathcal{Z}$, taking $\mathcal{Z}_n = \{z_0, \dots, z_n\}$ without loss of generality. Now, a well-known result of Lagrange, published in 1795, states that there exists a polynomial P_n of degree n such that $P_n(z_i) = \eta_i$, $0 \le i \le n$. We take P_n in the Newton's form ([1]), that is,

$$P_n(z) = [z_n] + \sum_{k=1}^{n} [z_n, \dots, z_{n-k}] \prod_{j=0}^{k-1} (z - z_{n-j}),$$

where $[z_i] = \eta_i$, and recursively:

$$[z_j, \dots, z_i] = \frac{[z_j, \dots, z_{i+1}] - [z_{j-1}, \dots, z_i]}{z_j - z_i}, \quad j > i.$$

The motivation of this paper is to answer the question: what happens with the error at non-interpolated points of $\mathcal{Z}$? For this we take the first non-interpolated point z_{n+1}. We write E_n for the error at z_{n+1}, that is,

$$E_n = |P_n(z_{n+1}) - \eta_{n+1}|.$$

Although it is not our case, if $x_i \in [a, b]$, $0 \le i \le n$, and $y_i = f(x_i)$ for $f \in C^{n+1}([a, b])$, then for $x \in [a, b]$ ([1]):

$$|P_n(x) - f(x)| \le \frac{\max_{t \in [a,b]} |f^{(n+1)}(t)|}{(n+1)!} \prod_{i=0}^{n} |x - x_i|. \tag{1.1}$$

We note that the term on the right depends on the distances of x to each point x_i. If x is such that $|x - x_i| \le |x - x_l|$ for a certain l and all i, then $\prod_{i=0}^{n} |x - x_i| \le |x - x_l|^{n+1}$ and the dependence is reduced to a power of the distance to a single point. In the case at hand, we want to avoid a global dependence as in (1.1) and only take into account the distance of z_{n+1} to a single point of $\mathcal{Z}_n$ without raising to any power; we choose as such a point the last interpolated one to forget about the previous ones. More specifically, we want $E_n \le c_n |z_{n+1} - z_n|$ for a certain positive constant c_n. The reason for this is to have an easier and immediate control of the error (there is no previous result in this regard).

For our purpose, we need to explicitly calculate E_n and since there are distances between values of Γ, it seems natural to take them in such a way that they are related to the distances between the corresponding points of $\mathcal{Z}_n$. Thus, we suppose that the values of Γ satisfy a Lipschitz condition of order one on $\mathcal{Z}_n$, that is, we consider the space $\Lambda_n(\mathcal{Z})$ of all sequences Γ verifying

$$|\eta_j - \eta_k| \le \delta |z_j - z_k|, \quad \text{for all } j, k \le n$$

and a certain $\delta > 0$.

We point out that the interpolating sequences for the corresponding infinite interpolation, when $U = \mathbb{D}$ and the interpolating functions f verify

$$|f(z) - f(w)| \le \beta |z - w|, \quad \text{for all } z, w \in \mathbb{D} \quad (\beta > 0),$$

must be an union of two (C)-sequences ([6]); on the other hand, this condition is also sufficient if they are in a Stolz angle $\{z \in \mathbb{D} / |z - \tau| < \mu(1 - |z|^2)\}$, $\tau \in \partial\mathbb{D}$ and $\mu > 1$ ([5]).

Finally, we introduce the following sequences.

DEFINITION 1.1. We say that $\mathcal{Z}$ is a n-memoryless sequence if for each $m \le n$ and any Γ in $\Lambda_{m+1}(\mathcal{Z})$, there is a constant $c(m)$ such that

$$E_m \le c(m)|z_{m+1} - z_m|. \tag{1.2}$$

Note that all sequences $\mathcal{Z}$ are 1-memoryless, because there is a bound of the error at z_2 depending on the distance from z_2 to z_1, but not to z_0:

$$E_1 = |\eta_1 - \eta_2| + |[z_1, z_0](z_2 - z_1)| \le 2\delta|z_2 - z_1|.$$

Our aim is to relate this property to the separation between the points of $\mathcal{Z}$.

2. STATEMENT OF RESULTS

First, we consider some conditions on distances in $\mathcal{Z}$.

DEFINITION 2.1. We say that $\mathcal{Z}$ is a n-previous sequence if

$$|z_{n+1} - z_n| \le |z_{n+1} - z_j|, \quad \text{for all } j < n. \tag{2.1}$$

DEFINITION 2.2. We say that $\mathcal{Z}$ is a n-positioned sequence if for each $m \le n$, there is a constant $K(m)$ such that

$$\prod_{i=0}^{m-1} |z_{m+1} - z_i| \le K(m)|z_{m+1} - z_m| \prod_{i=0}^{m-2} |z_m - z_i|. \tag{2.2}$$

$\mathcal{Z} = (2^i)_i$ is an example of n-previous and n-positioned sequence for any n ($K(m) = (2^{m+1} - 1)/2$ works in (2.2)).

DEFINITION 2.3. We say that $\mathcal{Z}$ is a n-posterior sequence if for all $i \le n$,

$$|z_i - z_{i+1}| \le |z_i - z_j|, \quad j \ne i.$$

DEFINITION 2.4. A sequence $\mathcal{Z}$ is said α-contractive ($0 < \alpha < 1$) if

$$|z_{i+2} - z_{i+1}| \le \alpha|z_{i+1} - z_i|, \quad \text{for any } i.$$

$\mathcal{Z} = (2^{-i})_i$ is an example of n-posterior sequence for any n, and also (1/2)-contractive.

Our results are the following Propositions.

PROPOSITION 2.5. Let $\mathcal{Z}$ be a n-previous sequence. $\mathcal{Z}$ is n-memoryless if and only if it is n-positioned.

Proof. Suppose $\mathcal{Z}$ is n-memoryless and $m \le n$. Let Γ be defined by: $\eta_m = z_m - z_{m-1}$, and $\eta_i = 0$, otherwise. Clearly, Γ is in $\Lambda_{m+1}(\mathcal{Z})$, by (2.1). Taking into account the Lagrange's form of the interpolating polynomial P_m ([1]) and (1.2), we have:

$$\left|\frac{(z_{m+1} - z_0)\dots(z_{m+1} - z_{m-1})}{(z_m - z_0)\dots(z_m - z_{m-1})}\eta_m\right| \le c(m)|z_{m+1} - z_m|,$$

and (2.2) follows.

Suppose $\mathcal{Z}$ is n-positioned. In order to avoid cumbersome notation and over-calculation, we confine ourselves to $n = 3$, although the procedure applies to any n. So, we prove that $\mathcal{Z}$ is 3-memoryless. We have:

$$\begin{aligned} E_2 &\le |\eta_3 - \eta_2| + |[z_2, z_1](z_3 - z_2)| + |[z_2, z_1, z_0](z_3 - z_1)(z_3 - z_2)| \\ &\le 2\delta\left(1 + \frac{|z_3 - z_1|}{|z_2 - z_0|}\right)|z_3 - z_2|. \end{aligned} \tag{2.3}$$

By (2.2) and (2.1),

$$\frac{|z_3 - z_1|}{|z_2 - z_0|} \le K(2)\frac{|z_3 - z_2|}{|z_3 - z_0|} \le K(2) \tag{2.4}$$

and (1.2) follows.

On the other hand, we have:

$$\begin{aligned} E_3 \le& |\eta_4 - \eta_3| + |[z_3, z_2](z_4 - z_3)| + |[z_3, z_2, z_1](z_4 - z_2)(z_4 - z_3)| \\ &+ |[z_3, z_2, z_1, z_0](z_4 - z_1)(z_4 - z_2)(z_4 - z_3)|. \end{aligned} \tag{2.5}$$

As above for E_2, the first three terms in (2.5) are bounded by

$$2\delta\left(1+\frac{|z_4-z_2|}{|z_3-z_1|}\right)|z_4-z_3|. \tag{2.6}$$

By (2.2), (2.1) and the triangle inequality,

$$\begin{aligned}\frac{|z_4-z_2|}{|z_3-z_1|} &\le K(3)\frac{|z_3-z_0||z_4-z_3|}{|z_4-z_0||z_4-z_1|} \le K(3)\frac{|z_3-z_0|}{|z_4-z_0|}\\ &\le K(3)\frac{|z_4-z_3|+|z_4-z_0|}{|z_4-z_0|} \le 2K(3).\end{aligned}$$

Similarly, the fourth term in (2.5) is bounded by

$$2\delta\left(\frac{1}{|z_3-z_1|}+\frac{1}{|z_2-z_0|}\right)\frac{|z_4-z_1||z_4-z_2|}{|z_3-z_0|}|z_4-z_3|. \tag{2.7}$$

By (2.2) and (2.1),

$$\frac{|z_4-z_1||z_4-z_2|}{|z_3-z_1||z_3-z_0|} \le K(3)\frac{|z_4-z_3|}{|z_4-z_0|} \le K(3).$$

By (2.2), (2.1) and (2.4),

$$\begin{aligned}\frac{|z_4-z_1||z_4-z_2|}{|z_2-z_0||z_3-z_0|} &\le K(3)\frac{|z_3-z_1||z_4-z_3|}{|z_2-z_0||z_4-z_0|}\\ &\le K(3)\frac{|z_3-z_1|}{|z_2-z_0|} \le K(3)K(2),\end{aligned}$$

and (1.2) follows. □

PROPOSITION 2.6. If $\mathcal{Z}$ is a n-posterior sequence, then it is n-memoryless.

Proof. For the same reason as in the proof of Proposition 2.5, we only consider $n=3$. In (2.3), we have:

$$\frac{|z_3-z_1|}{|z_2-z_0|} \le \frac{|z_1-z_2|+|z_2-z_3|}{|z_2-z_0|} \le \frac{|z_1-z_0|+|z_2-z_0|}{|z_2-z_0|} \le 2$$

and thus,

$$E_2 \le 6\delta|z_3-z_2|.$$

In (2.6), we have:

$$\frac{|z_4-z_2|}{|z_3-z_1|} \le \frac{|z_4-z_3|+|z_3-z_2|}{|z_3-z_1|} \le \frac{|z_3-z_1|+|z_2-z_1|}{|z_3-z_1|} \le 2.$$

In (2.7), we have:

$$\frac{|z_4-z_1||z_4-z_2|}{|z_3-z_1||z_3-z_0|} \le 2\frac{|z_4-z_1|}{|z_3-z_0|} \le 2\frac{|z_4-z_3|+|z_3-z_0|+|z_0-z_1|}{|z_3-z_0|} \le 6$$

and

$$\begin{aligned}\frac{|z_4-z_1||z_4-z_2|}{|z_2-z_0||z_3-z_0|} &\le 3\frac{|z_4-z_2|}{|z_2-z_0|} \le 3\frac{|z_4-z_3|+|z_3-z_2|}{|z_2-z_0|}\\ &\le 6\frac{|z_3-z_2|}{|z_2-z_0|} \le 6.\end{aligned}$$

So, it follows:

$$E_3 \le 30\delta|z_4-z_3|.$$ □

Finally, we proclaim that if $\mathcal{Z}$ is α-contractive, then it is n-memoryless for any n and certain values of α. We prove it for $n=3$ and observe the behavior of $c(2)$ and $c(3)$. In the following Lemma we collect the estimates we need.

LEMMA 2.7. Let $\mathcal{Z}$ be an α-contractive sequence.

(i) If $k \le j < m$, then

$$|z_m - z_j| \le \frac{\alpha^{j-k} - \alpha^{m-k}}{1-\alpha} |z_{k+1} - z_k|.$$

(ii) If $k \le j - 2$ and $\alpha^{j-k} - 2\alpha + 1 > 0$, then

$$|z_j - z_k| \ge \frac{\alpha^{j-k} - 2\alpha + 1}{1-\alpha} |z_{k+1} - z_k|.$$

Proof.

$$|z_m - z_j| \le \sum_{i=j}^{m-1} |z_{i+1} - z_i| \le \left(\sum_{i=j}^{m-1} \alpha^{i-k} \right) |z_{k+1} - z_k|$$

and the bound in (i) follows. For (ii), we write:

$$|z_j - z_k| \ge |z_{k+1} - z_k| - |z_j - z_{k+1}| \ge |z_{k+1} - z_k| - \sum_{i=k+1}^{j-1} |z_{i+1} - z_i|$$
$$\ge \left(1 - \sum_{i=k+1}^{j-1} \alpha^{i-k} \right) |z_{k+1} - z_k| = \left(1 - \frac{\alpha - \alpha^{j-k}}{1-\alpha} \right) |z_{k+1} - z_k|$$

and the inequality holds. □

PROPOSITION 2.8. Let $\alpha < (\sqrt{5} - 1)/2$. If $\mathcal{Z}$ is an α-contractive sequence, then it is 3-memoryless.

Proof. By Lemma 2.7, we have:

$$\frac{|z_3 - z_1|}{|z_2 - z_0|} \le \frac{\alpha - \alpha^3}{(1-\alpha)^2},$$

and from (2.3) it follows:

$$c(2) = 2\delta \frac{1 + \alpha^2}{1 - \alpha}.$$

By Lemma 2.7, we have:

$$\frac{|z_4 - z_2|}{|z_3 - z_1|} \le \frac{\alpha - \alpha^3}{(1-\alpha)^2}$$
$$\frac{|z_4 - z_2|}{|z_2 - z_0|} \le \frac{\alpha^2 - \alpha^4}{(1-\alpha)^2},$$

and if $\alpha < (\sqrt{5} - 1)/2$, then $\alpha^3 - 2\alpha + 1 > 0$ and

$$\frac{|z_4 - z_1|}{|z_3 - z_0|} \le \frac{\alpha - \alpha^4}{\alpha^3 - 2\alpha + 1}$$

So, from (2.6) and (2.7),

$$E_3 \le 2 \left[1 + \frac{\alpha - \alpha^3}{(1-\alpha)^2} + \frac{(\alpha - \alpha^4)(\alpha + \alpha^2 - \alpha^3 - \alpha^4)}{(1-\alpha)^2(\alpha^3 - 2\alpha + 1)} \right] \delta |z_4 - z_3|,$$

and it follows:

$$c(3) = 2\delta \frac{\alpha^6 + 3\alpha^5 + 3\alpha^4 + 2\alpha^3 + \alpha^2 - \alpha + 1}{\alpha^3 - 2\alpha + 1}.$$ □

EXAMPLE 2.9. For the (1/2)-contractive sequence $(2^{-i})_i$, we have $c(2) = 5\delta$ and $c(3) = 20.75\delta$. Taking $\Gamma = (3^{-i})_i$, then easily $\delta = 4/3$ and

$$P_2(z) = 0.5\widehat{92}z^2 + 0.\hat{4}z + 0.\widehat{037} \Longrightarrow P_2(z_3) = 0.02\hat{7}$$
$$P_3(z) = -0.22575z^3 + 0.98765z^2 + 0.24691z - 0.00882 \Longrightarrow P_3(z_4) = 0.01047$$

Thus,

$$E_2 = 0.00926 \leq c(2)|z_3 - z_2| = 0.8\hat{3}$$
$$E_3 = 0.00193 \leq c(3)|z_4 - z_3| = 1.72917$$

It turns out that $c(2)$ and $c(3)$ are increasing functions in α, $c_2 < c_3$, and both approach 2δ when $\alpha \longrightarrow 0$. Note that this convergence is uniform, since

$$\sup_{\delta>0} |c(2) - 2\delta| = 2\delta\alpha\frac{1+\alpha}{1-\alpha} \longrightarrow 0, \text{ when } \alpha \longrightarrow 0$$
$$\sup_{\delta>0} |c(3) - 2\delta| = 2\delta\alpha\frac{1+\alpha+\alpha^2+3\alpha^3+3\alpha^4+\alpha^5}{1-\alpha} \longrightarrow 0, \text{ when } \alpha \longrightarrow 0$$

On the other hand, value $\alpha < (\sqrt{5}-1)/2$ in Proposition 2.8 is not sharp, because $\mathcal{Z} = (\alpha^i)_i$, with $0 < \alpha < 1$, is an α-contractive sequence that is n-memoryless for any n (by Proposition 2.6).

The reciprocal of Proposition 2.8 is not true; for example, $\mathcal{Z} = (i)_i$ is a n-posterior sequence for all n and so it is n-memoryless (by Proposition 2.6), but it is not α-contractive for any $0 < \alpha < 1$. Nevertheless, we have the following more restrictive result.

PROPOSITION 2.10. If $\mathcal{Z}$ is a n-posterior sequence and there exists $0 < \alpha < 1$ such that

$$E_n \leq \alpha|z_{n+1} - z_n| \quad \text{for all } n,$$

then it is α-contractive.

Proof. For a fixed $n \geq 2$, let Γ be defined by $\eta_n = z_{n+1} - z_n$, and $\eta_i = 0$ otherwise. Since $\mathcal{Z}$ is n-posterior, then Γ is in $\Lambda_{n+1}(\mathcal{Z})$. Using that $P_{n-1}(z) \equiv 0$ and the hypothesis about E_n, we have:

$$|z_{n+1} - z_n| = |P_{n-1}(z_n) - \eta_n| = E_{n-1} \leq \alpha|z_n - z_{n-1}|$$

and it follows that $\mathcal{Z}$ is α-contractive. □

REMARK 2.11. Is there an explicit expression for $c(m)$ for some type of these sequences? We think that the difficulty in answering this question is of a computational nature. The only existing results in this regard are those obtained in the proofs of the above Propositions for $m = 2$ and $m = 3$.

REMARK 2.12. This topic can be extended to a Hermite interpolation, which consists of interpolating values $(\eta_i)_i$ and $(\lambda_i)_i$ in $\Lambda_{n+1}(Z)$ on $\mathcal{Z}_n$ by a polynomial Q_{2n+1} of degree $2n+1$ and its first derivative, respectively, that is:

$$Q_{2n+1}(z_j) = \eta_j, Q'_{2n+1}(z_j) = \lambda_j, 0 \leq j \leq n.$$

The requirement for a sequence to be n-memoryless is that for each $m \leq n$, there are constants $c(m)$ and $d(m)$ such that

$$\begin{cases} |Q_{2m+1}(z_{m+1}) - \eta_{m+1}| \leq c(m)|z_{m+1} - z_m| \\ |Q'_{2m+1}(z_{m+1}) - \lambda_{m+1}| \leq d(m)|z_{m+1} - z_m| \end{cases}$$

To summarize, we have obtained two criteria for a sequece is n-memoryless: be n-previous and n-positioned, and be n-posterior; possibly, be α-contractive is another criterion, but we have only gotten a partial result.

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