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BIPARTITE DOMINATION IN GRAPHS

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ABSTRACT

The bipartite domination number of a graph is the minimum size of a dominating set that induces a bipartite subgraph. In this paper we initiate the study of this parameter, especially bounds involving the order, the ordinary domination number, and the chromatic number. For example, we show for an isolate-free graph that the bipartite domination number equals the domination number if the graph has maximum degree at most 3; and is at most half the order if the graph is regular, 4-colorable, or has maximum degree at most 5.

KEYWORDS

graph, domination, bipartite domination

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1. INTRODUCTION

The idea of domination with constraints on the dominating set has been discussed multiple times in the literature. The most studied examples are where the dominating set S must be an independent set, that is, independent domination, and where S must have no isolated vertex, that is total domination. But further examples include connected domination [16], acyclic domination [10] and path-free domination [8]. There are also general results where the induced subgraph $G[S]$ is required to satisfy some (hereditary) property; see for example, [5].

In this paper, we investigate the case that S must induce a bipartite subgraph. We define a *bipartite dominating set* of a graph G as a set S such that the induced subgraph $G[S]$ is bipartite and every vertex not in S has a neighbor in S. Then we define the *bipartite domination number* $\gamma_{bip}(G)$ of graph G as the minimum size of a bipartite dominating set. We note that the term "bipartite domination" has also been used to describe the domination problem in bipartite graphs, such as in [11]. On the other hand, it is used in our way in [9].

We use $\gamma(G)$ to denote the domination number and $i(G)$ to denote the independent domination number of a graph G. The bipartite domination number is sandwiched between them: that is,

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$\gamma(G) \le \gamma_{bip}(G) \le i(G)$. Note, however, that unlike with independent domination, it is not true that every bipartite subgraph that dominates is maximal bipartite.

We proceed as follows. In Section 2 we provide some initial observations. Then in Section 3, we develop (upper) bounds on the bipartite domination number in terms of the order and chromatic number, while in Section 4 we develop bounds in terms of degrees and the ordinary domination number. Finally in Section 5 we provide some results on graph operations and maximal planar graphs and in Section 6 conclude with a couple of thoughts on future study.

2. INITIAL OBSERVATIONS

It is well-known (and originally shown by Ore) that a graph without isolated vertices has domination number at most half its order. However, the bipartite domination number of a graph can be much larger, The extreme value is a special case of Theorem 4 of [5]:

THEOREM 1 ([5]). If G is a graph with order n and no isolated vertex, then $\gamma_{bip}(G) \le n + 4 - 2\sqrt{2n}$, and the result is best possible

To describe the extremal graphs we introduce a recurring family of graphs. For integer $a \ge 1$, the *generalized corona* $\mathrm{cor}(G, a)$ is the graph obtained from graph G by adding a pendant vertices to each vertex: that is, for each vertex v of G, one adds a new vertices and an edge from each new vertex to the vertex v. For example, the graph $\mathrm{cor}(K_5, 3)$ is illustrated in Figure 1. The graphs that show equality in Theorem 1 are $\mathrm{cor}(K_{2a+2}, a)$.

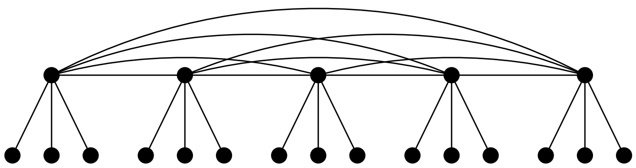

FIGURE 1. The generalized corona $\mathrm{cor}(K_5, 3)$

We noted above that there is the chain $\gamma(G) \le \gamma_{bip}(G) \le i(G)$. Cycles and complete graphs are examples where all three parameters are equal. It is straightforward to find graphs where $\gamma(G) = \gamma_{bip}(G)$ but $i(G) \gg \gamma_{bip}(G)$: for example, consider a tree of diameter three where both central vertices have high degree.

One can also find graphs where $i(G) = \gamma_{bip}(G)$ but $\gamma(G) \ll \gamma_{bip}(G)$. For example, consider the cartesian product $K_3 \square K_{1,m}$ with $m \ge 3$. This graph has domination number 3: one can take the three vertices of the K_3-fiber corresponding to the central vertex of the star. But for a bipartite dominating set, there must be some $K_{1,m}$-fiber, call it H, where one does not take its central vertex. The vertices in H have disjoint closed neighborhoods if one ignores its central vertex. So the bipartite domination number of this product is at least m. Further, the value m can be achieved by taking one vertex from each noncentral K_3-fiber, subject to the constraint that one takes at least one vertex from each star fiber. Thus the independent and bipartite domination numbers of $K_3 \square K_{1,m}$ are both m.

Allan and Laskar [1] showed that if a graph G is claw-free, then one can choose the minimum dominating set to be independent. Thus for such graphs it holds that $\gamma_{bip}(G) = \gamma(G) = i(G)$. It is not true, however, that $K_{1,4}$-free graphs G have $\gamma_{bip}(G) = \gamma(G)$. For example, $\gamma(\mathrm{cor}(K_3, 2)) = 3$ while $\gamma_{bip}(\mathrm{cor}(K_3, 2)) = 4$.

Finally in this section we note that computing the bipartite domination number of a graph is NP-hard. The simplest way to see this is that ordinary domination is known to be NP-hard when restricted to bipartite graphs, as shown by several authors. This fact was also noted at the end of [5].

3. BOUNDS INVOLVING THE CHROMATIC NUMBER

In this section we consider bounds on the bipartite domination number involving the chromatic number. If the graph is bipartite, then the bipartite domination number is just the domination number. So consider graphs with chromatic number at least 3. For both 3- and 4-colorable graphs, the bipartite domination number is at most half the order (which is best possible because it is best possible for domination due to the coronas cor$(H, 1)$; see [14]). The bound is a consequence of the following theorem:

THEOREM 2. If G is a 4-colorable graph with no isolated vertex, then there exist two disjoint bipartite dominating sets of G.

Proof. By combining the colors in pairs, it follows that there exists a partition of the graph G into two bipartite subgraphs. Out of all such partitions (R, B), consider the one that has the minimum number of monochromatic edges (that is, edges whose two ends have the same color). We claim that in such a partition both colors form a dominating set of G. For, if say vertex $v \in R$ has only vertices of R as neighbors, one can move v to B and still have a bipartite partition but with less monochromatic edges. □

COROLLARY 3. If G is a 4-colorable graph of order n with no isolated vertex, then $\gamma_{bip}(G) \le n/2$.

For larger chromatic number we provide a result motivated by that of MacGillivray and Seyffarth [12]. They provided a sharp upper bound on the independent domination number of k-chromatic graphs:

THEOREM 4 ([12]). If G is a graph with order n, chromatic number $k \ge 3$ and no isolated vertex, then $i(G) \le (k-1))n/k - (k-2)$.

THEOREM 5. For $k \ge 4$, if graph G has order n, chromatic number k and no isolated vertex, then

$$\gamma_{bip}(G) \le \frac{k-2}{k} n - (k-4).$$

Proof. The proof uses a similar approach to the proof of Theorem 4 that was given in the survey [6]. Assume first that graph G is connected.

Consider a coloring of the vertices of G using colors 1 up to k. Define a vertex as *insular* if all its neighbors have the same color. Amongst all such colorings, choose one that has the minimum number of insular vertices. We claim that then every insular vertex has at least one neighbor that is not insular. For, suppose v and every vertex in its neighborhood $N(v)$ is insular. Then changing the color of v to a third color (which exists since $k \ge 3$) cannot increase the number of insular vertices, and indeed decreases the number unless every vertex in $N(v)$ has only v as its neighbor. But in that case, the graph is a star, which is not possible.

We partition the non-insular vertices according to their color: let N_i denote the set of non-insular vertices of color i. And we partition the insular vertices according to their neighbor's color: let X_i denote the set of insular vertices whose neighbors have color i. (Some of these sets might be empty.)

Now, create a set B as follows. Choose two colors at random, say r and s, and call these colors *base*. Start with the set $N_r \cup N_s$ and add all of $Y_{r,s} = \bigcup_{i \notin \{r,s\}} X_i$. Then, take undominated vertices one at a time until every vertex is dominated. The resultant set B dominates G.

Now, consider the subgraph $G[B]$. There is no edge in $G[B]$ joining $N_r \cup N_s$ to $Y_{r,s}$ while the undominated vertices that get added are isolated in $G[B]$. The subgraph induced by $N_r \cup N_s$ is clearly bipartite. Suppose there is a cycle C in the subgraph induced by $Y_{r,s}$. Then, since each vertex v of C is insular, the two neighbors of v on C have the same color. By repeated application it follows that the vertices of C alternate colors, and in particular, the cycle C has even length. It follows that $G[B]$ is bipartite. We need to bound the expected size of B.

Consider a vertex u that is insular. Then we observed above that it has a non-insular neighbor, say v. If v's color is base, vertex u is dominated by $v \in B$; and if v's color is not base, vertex u is in $Y_{r,s}$. In short, vertex u is in B if and only if its neighbor color is not base. Thus the probability that u is in B equals $(k-2)/k$.

Consider a vertex v that is not insular; say $v \in N_i$. Suppose first that at least one of v's neighbors is insular, say w. If the color i is not base, then vertex w will be in $Y_{r,s}$ and v will be dominated by $w \in B$. Thus, vertex v is in B if and only if its color is base. Hence the probability that v is in B equals $2/k$ (which is at most $(k-2)/k$ since $k \ge 4$).

Suppose second that none of v's neighbors is insular. If color i is base, then v is in B. If color i is not base, then v can be in B only if none of its neighbors' colors is base. Since v is not insular there are at least two colors present in its neighborhood, say j_1 and j_2. There are $k-1$ unordered pairs of colors containing i, and $\binom{k-3}{2}$ unordered pairs containing none of i, j_1, j_2. Thus the probability that v is in B is at most

$$\frac{(k-1)+\binom{k-3}{2}}{\binom{k}{2}} = \frac{(2k-2)+(k-3)(k-4)}{k(k-1)} = \frac{k^2-5k+10}{k(k-1)}.$$

It can be checked that this quantity is at most $(k-2)/k$, since $k \geq 4$. Thus we have shown that for every vertex, the probability it is in B is at most $(k-2)/k$.

Now, define a vertex as *rainbow* if it has a non-insular neighbor of every other color. The minimality of the number k of colors means that each color is necessary. It follows that for each color i there exists a vertex v_i of color i that has neighbors of every other color. Indeed, vertex v_i must have a non-insular neighbor of each color, because an insular neighbor can be recolored. That is, there are at least k rainbow vertices. A rainbow vertex is in B if and only if its color is base. That is, the probability a rainbow vertex is in B is at most $2/k$.

It follows that the expected size of B is at most

$$\frac{k-2}{k}(n-k)+\frac{2}{k}(k) = \frac{kn-2n-k^2+2k+2k}{k} = \frac{k-2}{k}n-(k-4),$$

and thus there exists a bipartite dominating set of G of at most this size.

If G is not connected, then the above argument is valid for every component with chromatic number k (of which there is at least one). If a component H has chromatic number less than k and is not a star, then one can follow the above argument except for the existence of rainbow vertices and show that the expected size of B restricted to H is at most $(k-2)n_H/k$, where n_H is the order of H. A bound of $n_H/2$ is trivial if H is a star. The theorem follows. □

The bound in Theorem 5 is sharp. It is achieved by the corona $\mathrm{cor}(K_k, a)$ for $k \geq 4$ and $a \geq 1$.

4. BOUNDS INVOLVING THE DEGREES

4.1. Maximum degree

Let G be a graph and let D be a set of vertices in G. Then the *D-external private neighborhood* $\mathrm{epn}[v, D]$ of v is the set of vertices of $V - D$ dominated by v but by no other vertex of D.

THEOREM 6. If G is a graph with maximum degree at most r for $r \geq 3$, then

$$\gamma_{bip}(G) \leq \left(\frac{r^2-5r+8}{r-1}\right)\gamma(G).$$

Proof. Among all minimum dominating sets of G, let D be chosen so that $G[D]$ has the minimum number of edges. Let v be a non-isolated vertex in $G[D]$ (if any). By the minimality of D it follows that $\mathrm{epn}[v, D]$ is nonempty. If $\mathrm{epn}[v, D] = \{v'\}$, then replacing v in D with the vertex v' produces a minimum dominating set of G that induces a subgraph with fewer edges, a contradiction. Hence, $|\mathrm{epn}[v, D]| \geq 2$.

It follows that the subgraph $G[D]$ has maximum degree at most $r-2$. Thus it is $(r-1)$-colorable. Let A be a maximum bipartite subset of D. The colorability of $G[D]$ implies that $|A| \geq \frac{2}{r-1}|D|$. Let $X = V(G) - N[A]$. Note that by the maximality of A, every vertex of $D - A$ (if any) has at least two neighbors in A. In particular, the set X is disjoint from D, and every vertex of $D - A$ has at most $r-2$ neighbors in X. Thus $|X| \leq (r-2)(|D| - |A|)$.

Now, let Y be a bipartite dominating set of $G[X]$. Since $A \cup Y$ is a bipartite dominating set of G, it follows that

$$\begin{aligned}\gamma_{bip}(G) &\leq |A \cup Y| \\ &\leq |A| + (r-2)(|D| - |A|) \\ &= (r-2)|D| - (r-3)|A| \\ &\leq (r-2)|D| - \frac{2(r-3)}{r-1}|D| \\ &= \frac{r^2 - 5r + 8}{r-1}|D|,\end{aligned}$$

as required. □

COROLLARY 7. (a) If graph G has maximum degree at most 3, then $\gamma_{bip}(G) = \gamma(G)$.
(b) If graph G has maximum degree at most 4, then $\gamma_{bip}(G) \leq \frac{4}{3}\gamma(G)$.

The bounds in Corollary 7 are sharp. For example, consider the graph $\mathrm{cor}(K_3, 2)$. However, for $r \geq 5$ the bound in Theorem 6 is not sharp, and we provide next an asymptotic better bound. The proof builds on the ideas given above.

THEOREM 8. If G is a graph with maximum degree at most r for $r \geq 3$, then

$$\gamma_{bip}(G) \leq (r - 2\sqrt{2r} + 3)\,\gamma(G).$$

Proof. Let $D = \{w_1, \dots, w_\gamma\}$ be a minimum dominating set of G. Choose some partition $(Y_1, \dots, Y_\gamma)$ of $V(G)$ such that, for each i, the set Y_i contains w_i and every other vertex of Y_i is adjacent to w_i.

Construct a coloring of the subgraph $G[D]$ as follows. Use as colors the positive integers. Pick a *random* ordering of D. Then color each vertex of D in order with the smallest color that does not appear on its already colored neighbors. (That is, the result is a Grundy coloring of D.) Let A be the subset of D consisting of colors 1 and 2. It clearly induces a bipartite subgraph. Let $X = V(G) - N[A]$. Note that every vertex of $D - A$ has at least two neighbors in A; thus X is disjoint from D. In particular, by considering the union of A with any bipartite dominating set of X, it follows that $\gamma_{bip}(G) \leq |A| + |X|$.

Let q_i denote the expected size of $Y_i \cap (A \cup X)$. Say vertex w_i has d_i neighbors in D. If $d_i \leq 1$, then w_i will always be in A and no vertex of Y_i will be in X; that is, $q_i = 1$. Otherwise, let p_i be the probability that $w_i \in A$. If w_i is in A, then no other vertex of Y_i will be in X; otherwise there is a possibility of up to $r - d_i$ vertices of Y_i in X. That is,

$$q_i \leq p_i + (1 - p_i)(r - d_i).$$

By the minimality of D, it holds that $d_i < r$. Thus the bound on q_i is non-increasing in p_i. By the construction of the coloring, vertex w_i will always be in A if at most one neighbor comes before it in the random ordering of D. That is, it is sufficient that in the random ordering restricted to w_I and its d_i neighbors in D, the vertex w_i is first or second. Hence

$$p_i \geq \frac{2}{d_i + 1}.$$

Thus

$$q_i \leq r - d_i - (r - d_i - 1)\frac{2}{d_i + 1} = r + 3 - (d_i + 1) - \frac{2r}{d_i + 1}.$$

By calculus the maximum value of this expression is attained at $d_i = \sqrt{2r} - 1$, and so $q_i \leq r - 2\sqrt{2r} + 3$. That is, by linearity of expectation, the expected size of $|A| + |X|$ is at most $(r - 2\sqrt{2r} + 3)\gamma$. The result follows. □

Theorem 8 is best possible. Equality is obtained whenever $2r$ is a perfect square by the corona $\mathrm{cor}(K_a, a^2/2 - a + 1)$ for even a.

Along the same lines, we have the following result.

THEOREM 9. If G is a $K_{1,s}$-free graph with $s \geq 3$ and $\gamma(G) \geq 2$, then

$$\gamma_{bip}(G) \leq (s-2)\gamma(G) - 2(s-3).$$

Proof. Let D be a minimum dominating set of G, and let A be a maximal subset of D such that $G[A]$ is a bipartite graph. We note that $|A| \geq 2$. Let $X = V(G) - N[A]$. Since the set A dominates the set D, we note that $X \subseteq V(G) - D$ and no vertex in X has a neighbor in A. Let Z be a maximal independent set of $G[X]$.

By the maximality of the set A, each vertex in $D - A$ has a neighbor in A (in fact, at least two such neighbors). Since G is $K_{1,s}$-free, each vertex in $D - A$ therefore has at most $s - 2$ neighbors in Z. Since $D - A$ dominates Z, this implies that $|Z| \leq (s-2)(\gamma(G) - |A|)$. The set $A \cup Z$ is a bipartite dominating set of G. Thus

$$\gamma_{bip}(G) \leq |A| + |Z| \leq (s-2)\gamma(G) - (s-3)|A| \leq (s-2)\gamma(G) - 2(s-3),$$

as required. □

The bound in Theorem 9 is sharp. For example, for integers $k \geq 3$ and $s \geq 3$, if G is the corona $\mathrm{cor}(K_k, s-2)$, then G is $K_{1,s}$-free. Moreover, $\gamma(G) = k$ and

$$\gamma_{bip}(G) = 2 + (k-2)(s-2) = (s-2)\gamma(G) - 2(s-3).$$

We note also that in the special case when G is a claw-free graph, by Theorem 9 we have that $\gamma_{bip}(G) \leq \gamma(G)$, as we observed earlier.

We conclude this subsection with an upper bound for graphs with maximum degree 5. The proof of the result is a refinement of the technique used in the proof of Theorem 2.

THEOREM 10. If G is a graph of order n and maximum degree at most 5 with no isolated vertex, then $\gamma_{bip}(G) \leq n/2$.

Proof. It suffices to show the result for a connected graph. If the graph G is K_6, then the result is trivial. So we may assume that G is 5-colorable. Consider a 5-coloring $(C_1, C_2, C_3, C_4, C_5)$ of G. Out of all such colorings, take one that minimizes the quantity $|C_5|$ and, subject to that, minimizes the quantity $|[C_1, C_2]| + |[C_3, C_4]|$: that is, the number of edges with one end of color 1 and one of color 2 or one end of color 3 and one of color 4.

Let X_5 be the set of vertices all of whose neighbors are color 5. Since one can recolor the vertices of X_5 to any of color 1 through 4, the choice of coloring implies that each vertex $v_5 \in C_5$ must have a neighbor in $C_i - X_5$ for $1 \leq i \leq 4$. Since the maximum degree is at most 5, the vertex v_5 can therefore have at most one neighbor in X_5. That is, it holds that $|X_5| \leq |C_5|$.

Now, consider the set $B = (C_1 - X_5) \cup (C_2 - X_5) \cup X_5 = C_1 \cup C_2 \cup X_5$. This set induces a bipartite subgraph (note that the vertices of X_5 are isolated in this subgraph). We claim that B dominates G. By the above discussion, every vertex of C_5 has in fact two neighbors in this set. Consider a vertex v_3 of C_3 and suppose it is not dominated by B. Then v_3 is not in X_5 and all its neighbors have color 4 or 5. That is, vertex v_3 has at least one neighbor in C_4. So, by changing v_3 to have color 1, one reduces the number of edges in $[C_3, C_4]$ (while keeping $[C_1, C_2]$ and C_5 unchanged), a contradiction to the choice of coloring. The argument for a vertex of color 4 is similar. Thus, B is a bipartite dominating set of G.

Similarly, the set $B' = C_3 \cup C_4 \cup X_5$ is also a bipartite dominating set of G. Since the sets $C_1 - X_5$, $C_2 - X_5$, $C_3 - X_5$, $C_4 - X_5$, X_5 and C_5 are all disjoint, by the above it follows that $|B| + |B'| \leq n$. The result follows. □

It would be interesting to determine the smallest c_k such that $\gamma_{bip}(G) \leq c_k n$ for all graphs of order n with maximum degree k and no isolated vertex. The above theorem shows that $c_5 = 1/2$. The corona $\mathrm{cor}(K_5, 2)$ shows that $c_6 \geq 8/15$; further, similar examples (or equivalently Theorem 1) show that $c_k \longrightarrow 1$ as k grows.

4.2. Regular graphs

We next consider regular graphs. It is well known that $i(G) \leq n/2$ for a regular graph G of order n (that is not just a collection of isolates). It follows that $\gamma_{bip}(G) \leq n/2$ for such a graph. The bound for independent domination number is sharp because of the complete bipartite graphs, but it is unclear the correct asymptotics for bipartite domination. If G is a cubic graph, then we have from Observation 6 of [5] (and Theorem 6) that $\gamma_{bip}(G) = \gamma(G)$ and thus the upper bound is the same as it is for ordinary domination. For 4-regular graphs, recently Cho et al. [4] showed that if $G \neq K_{4,4}$ is a 4-regular graph of order n, then $i(G) \leq 3n/7$. As a consequence, we have that if G is a 4-regular graph, then $\gamma_{bip}(G) \leq 3n/7$, since $\gamma_{bip}(K_{4,4}) = 2$.

One can also consider the relationship between the ordinary and bipartite domination numbers. We already noted that they are equal for cubic graphs. For a 4-regular graph G, Corollary 7 implies that $\gamma_{bip}(G) \leq \frac{4}{3}\gamma(G)$. This value is definitely not obtainable: indeed we can show that there exists some $\varepsilon > 0$ such that $\gamma_{bip}(G) \leq (\frac{4}{3} - \varepsilon)\gamma(G)$. We do not include the proof since first it is messy and second it seems far from optimal. Indeed, it is hard to find a 4-regular graph with $\gamma_{bip}(G) > \gamma(G)$. But an example, attributed to Fricke, is given in Figure 4 of [10] of a 4-regular graph F with acyclic domination number bigger than domination number. It can readily be checked that every minimum dominating set contains all three vertices of the bottom triangle, and so $\gamma(F) = 21$ and $\gamma_{bip}(F) = 22$.

4.3. Large minimum degree

Finally in this section we consider the other end of the spectrum and show that large minimum degree does not imply that the bipartite domination number and domination number are equal.

LEMMA 11. For all $\varepsilon > 0$ there is a graph G such that $\delta(G) \geq (1 - \varepsilon)|V(G)|$ and $\gamma_{bip}(G) > \gamma(G)$.

Proof. Let $p = 1 - \varepsilon/2$. Take a set R of $n - 3$ vertices and choose three disjoint subsets A_1, A_2, A_3 each of size $\varepsilon n/2$. Add edges randomly between vertices of R independently with probability p. Then add three vertices v_1, v_2, v_3 that form a triangle and join each v_i to all vertices of $A_i \cup (R - (A_1 \cup A_2 \cup A_3))$. Let G be the resultant graph. The minimum degree of G is (almost surely) at least $(1 - \varepsilon)n$.

The three new vertices dominate G; so $\gamma(G) \leq 3$. Now a bipartite dominating set can contain at most two of the added vertices. So without loss of generality assume that A_1 needs to be dominated. It is well-known that the domination number of the random graph is logarithmic in the order. One can, by the same techniques show that, given a constant fraction X of the random graph, that to dominate X also requires logarithmic vertices. It follows that $\gamma_{bip}(G) \gg 3$. □

5. OPERATIONS AND FAMILIES

5.1. Operations

It is easy to see that the join of two graphs has bipartite domination number 2 except when one of the constituent graphs has domination number 1. For the disjoint union, the resultant graph has bipartite domination number the sum of the constituent bipartite domination numbers. Another common operation is duplicating a vertex. In particular, consider the family of expansions; that is, the composition $G[mK_1]$. It turns out that the bipartite domination problem in an expansion is equivalent to a weighted version in the original.

THEOREM 12. If G is a graph and $m \geq 1$, then $\gamma_{bip}(G[mK_1])$ is the minimum weight of a bipartite dominating set B of G, where nonisolated vertices of $G[B]$ have weight 1 and isolated vertices of $G[B]$ have weight m.

Proof. Consider a minimal bipartite dominating set B' of $G[mK_1]$, and let B be the projection of B' onto $V(G)$. If vertex v is isolated in $G[B]$, then B' must contain all vertices in $G[mK_1]$ corresponding to v. If vertex v is not isolated in $G[B]$, then the minimality of B' means it contains only one vertex corresponding to v. The result follows. □

We consider next the result of edge changes. The bipartite domination number can increase by at most one on edge removal or equivalently decrease by at most one on edge addition. In contrast:

THEOREM 13. If G is a graph of order n, then the bipartite domination number can increase by at most $n/3 - 2$ on the addition of an edge, and this is sharp.

Proof. Let G' denote the graph formed from graph G by the addition of edge e. Consider a minimum bipartite dominating set D of G and assume $G'[D]$ is not bipartite. Then $G'[D]$ is 3-chromatic, say with color classes (C_1, C_2, C_3). One can build a bipartite dominating set B_i of G' by starting with the set $D - C_i$ and then adding the undominated vertices as needed.

We seek an upper bound on $|B_1| + |B_2| + |B_3|$. The worst that can happen for a vertex outside D is that it contributes once, since its neighbor in D is in at least two of the $D - C_i$. Since $G'[D]$ is not 2-colorable, there must exist for each color class C_i a vertex v_i that has neighbors of both other colors. That vertex v_i contributes at most twice to the sum. Every other vertex of D contributes at most thrice. Thus $|B_1| + |B_2| + |B_3| \leq n - 3 + 2|D|$. And so $\gamma_{bip}(G') \leq n/3 - 1 + 2|D|/3$.

To make the difference between the $\gamma_{bip}(G) = |D|$ and $\gamma_{bip}(G')$ as large as possible, one makes D as small as possible, namely $|D| = 3$. The result follows. □

Equality in Theorem 13 requires $|D| = 3$ and every other vertex to have exactly one neighbor in D. Such a graph is the corona $\mathrm{cor}(K_3, m)$. Some related results on graph operations are given by Samodivkin [15].

5.2. Planar triangulations

Finally in this section we consider triangulations. Matheson and Tarjan [13] proved that every planar triangulation has 3 disjoint dominating sets. This we generalize. Indeed, we show that a planar triangulation has 3 disjoint acyclic dominating sets.

THEOREM 14. A planar triangulation G with order $n \geq 3$ has 3 disjoint bipartite dominating sets.

Proof. The result is trivial if $n = 3$. So assume $n \geq 4$. Then every vertex has degree at least 3. The vertices of G can be partitioned into 3 sets each of which induces a forest. (That is, the arboricity is at most 3: see [3].) Now, out of all such partitions, take the partition (C_1, C_2, C_3) that has the *least number of monochromatic edges* (edges whose two ends are in the same class of the partition).

Suppose that some class, say C_1, does not dominate the graph G. Assume it does not dominate vertex v. Say $v \in C_2$. Since the subgraph $G[N(v)]$ induced by the neighborhood of v contains a cycle, it cannot be that all neighbors of v are in C_3. So vertex v must have a neighbor in C_2. If we now move vertex v to C_1, it does not create a cycle in the graph induced by C_1, and we have reduced the number of monochromatic edges, a contradiction.

That is, each of C_1, C_2, C_3 is a dominating set of G. □

COROLLARY 15. If G is a planar triangulation with order $n \geq 3$, then $\gamma_{bip}(G) \leq n/3$.

In a slightly different direction, we [7] conjectured that every planar triangulation can be partitioned into two disjoint bipartite sets each of which is a total dominating set of G.

6. FUTURE WORK

Apart from the conjectures and questions posed in the preceding sections, it would be interesting to determine further results on graph families, for example planar graphs in general or subsets thereof. In another direction, one could examine the "bipartite perfect graphs" where $\gamma(H) = \gamma_{bip}(H)$ for all induced subgraphs H.

REFERENCES

[1] Allan, R. B. and Laskar, R. On domination and independent domination numbers of a graph. *Discrete Math. 23* (1978), 73–76.

[2] Borowiecki, M., Michalak, D. and Sidorowicz, E. Generalized domination, independence and irredundance. *Discuss. Math. Graph Theory 17* (1997), 143–153.

[3] Chartrand, G., Kronk, H. V. and Wall, C. E. The point-arboricity of a graph. *Israel J. Math. 6* (1968), 169–175.

[4] Cho, E.-K., Choi, I. and Park, B. On independent domination of regular graphs. arXiv:2107.00295.

[5] Goddard, W., Haynes, T. W. and Knisely, D. Hereditary domination and independence parameters. *Discuss. Math. Graph Theory 24* (2004), 239–248.

[6] Goddard, W. and Henning, M. A. Independent domination in graphs: A survey and recent results. *Discrete Math. 313* (2013), 839–854.

[7] Goddard, W. and Henning, M. A. Thoroughly dispersed colorings. *J. Graph Theory 88* (2018), 174–191.

[8] Haynes, T. W. and Henning, M. A. Path-free domination. *J. Combin. Math. Combin. Comput. 33* (2000), 9–21.

[9] Hedetniemi, J. T., Hedetniemi, K. D., Hedetniemi, S. M. and Hedetniemi, S. T. Secondary and internal distances of sets in graphs. *AKCE Int. J. Graphs Comb. 6* (2009), 239–266

[10] Hedetniemi, S. M., Hedetniemi, S. T. and Rall, D. F. Acyclic domination. *Discrete Math 222* (2000), 151–165.

[11] Ko, C. W. and Shepherd, F. B. Bipartite domination and simultaneous matroid covers. *SIAM J. Discrete Math. 16* (2003), 517–523.

[12] MacGillivray, G. and Seyffarth, K. Bounds for the independent domination number of graphs and planar graphs. *J. Combin. Math. Combin. Comput. 49* (2004), 33–55.

[13] Matheson, L. R. and Tarjan, R. E. Dominating sets in planar graphs. *European J. Combin. 17* (1996), 565–568.

[14] Payan, C. and Xuong, N. H. Domination-balanced graphs. *J. Graph Theory 6* (1982), 23–32.

[15] Samodivkin, V. D. Domination with respect to nondegenerate and hereditary properties. *Mathematica Bohemica 133* (2008), 167–178.

[16] Sampathkumar, E. and Walikar, H. B. The connected domination number of a graph. *J. Math. Phys. Sci. 13* (1979), 607–613.