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AKADÉMIAI KIADÓ

PERFECT SOLUTIONS TO PROBLEMS ON COMMON TRANSVERSALS AND SUBMODULAR FUNCTIONS FROM WELSH'S 1976 TEXT *MATROID THEORY*

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ABSTRACT

Problem 2 of Welsh's 1976 text *Matroid Theory*, asking for criteria telling when two families of sets have a common transversal, is solved.

Another unsolved problem in the text *Matroid Theory*, on whether the "join" of two non-decreasing submodular functions is submodular, is answered in the negative. This resolves an issue first raised by Pym and Perfect in 1970.

KEYWORDS

Independence space, independence structure, matroid, submodular, (common) transversal

MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 05D15, 05B35; Secondary

A University of Waterloo citation for Dominic Welsh reads, "James Anthony Dominic Welsh is Professor of Mathematics at the University of Oxford in England.... [H]e was Chairman of Mathematics at Oxford.... He has made significant contributions to matroid theory, including a text with that title, which held centre stage in that discipline for fifteen years, until the spotlight shifted to a text by one of his former research students – James Oxley." [16]

The *Mathematical Reviews* abstract for Welsh's book has appeared in 23 other *Mathematical Reviews* abstracts and has appeared as a reference for 467 works whose abstracts have appeared in *Mathematical Reviews.* Tutte himself wrote the *Mathematical Reviews* abstract for Welsh's book, saying, "The reviewer recommends this work to all students of matroid theory." [11]

Oxley wrote in *Combinatorics, Complexity, and Chance: A Tribute to Dominic Welsh*, "There seem to be three notable groups of people that one can distinguish when discussing the development of matroid theory.... Two people stand out as clear leaders of the third group, the late Gian-Carlo Rota and Dominic Welsh." [12]

This suggests that any unsolved problem in *Matroid Theory* has passed before the eyes of many in the field, including the field's preeminent leaders. Thus, a solution to such a problem, whether

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or not the solutions appear technically sophisticated to some readers, might be regarded as worth noting.

This article solves two such problems.

1. A SOLUTION TO PROBLEM 2 OF WELSH'S 1976 TEXT *MATROID THEORY*

We use terminology from [17, Chapter 7] for possibly infinite sets. Let E be a set. Let I be an index set. Let $\mathcal{A} = (A_i : i \in I)$ be a family of subsets of E (a function from I to $\mathcal{P}(E)$ such that $i \mapsto A_i$ for all $i \in I$). A family $(x_i : i \in I)$ is a *system of representatives* of $\mathcal{A}$ if there exists a bijection $\pi : I \longrightarrow I$ such that $x_i \in A_{\pi(i)}$ for all $i \in I$. A set T is a *transversal* of $\mathcal{A}$ if there is a bijection $\pi : T \longrightarrow I$ such that $x \in A_{\pi(x)}$ for all $x \in T$; it is a *partial transversal* of $\mathcal{A}$ if it is a transversal of a subfamily $(A_j : j \in J)$ of $\mathcal{A}$, where $J \subseteq I$.

Welsh [17, Chapter 20, Section 3] stated the following two problems in a section of his 1976 text *Matroid Theory* entitled, "Infinite Transversal Theory."

"*Problem* 1. Find necessary and sufficient conditions for a family of sets to have a transversal.

"*Problem* 2. Find necessary and sufficient conditions for two families of sets to have a common system of representatives (or a common transversal)."

Aharoni, Nash-Williams, and Shelah solved Problem 1 [1, Theorem 5.1], generalizing Hall's Marriage Theorem (where the index set corresponds to the set of "men" to be married off) [7, Theorem 1]. One way to deduce a criterion for the existence of a common transversal for two finite families of subsets of a finite set E is to note that the collection of partial transversals for one of the families yields a matroid structure on E ([5, p. 2]; see [10, Theorem 6.5.2]). Then one can use Rado's generalization of Hall's Marriage Theorem to matroids [10, Theorem 6.2.1] to deduce the following criterion ([6, Corollary]; see [10, Corollary 9.3.4]).

THEOREM 1.1 (Ford and Fulkerson). Let $n \in \mathbb{N}$. Let $\mathcal{A} = (A_1, \dots, A_n)$ and $\mathcal{B} = (B_1, \dots, B_n)$ be families of sets. Then $\mathcal{A}$ and $\mathcal{B}$ have a common transversal if and only if for all $I, J \subseteq \{1, \dots, n\}$

$$\left| \left(\bigcup_{i \in I} A_i \right) \cap \left(\bigcup_{j \in J} B_j \right) \right| \geq |I| + |J| - n. \qquad \square$$

We cannot use this method to solve Problem 2 since no one has published a matroid generalization of the theorem of Aharoni, Nash-Williams, and Shelah. (A matroid generalization of Hall's Marriage Theorem for countable families appears in [9, Theorem]; also see [15] and [8].) Perfect, however, proved Theorem 1 another way [13], and her approach does generalize to infinite sets.

THEOREM 1.2. Let E be a set. Let I be an index set. Let $J := I$. Let $\mathcal{A} = (A_i : i \in I)$ and $\mathcal{B} = (B_j : j \in J)$ be families of subsets of E. Assume $E \cap I = \emptyset$.

Let $\mathcal{X} = (X_k : k \in E \cup I)$ be a family of subsets of $E \cup J$ where for all $k \in E \cup I$

$$X_k := \begin{cases} \{k\} \cup \{j \in J : k \in B_j\} \text{ if } k \in E \\ A_k \text{ if } k \in I. \end{cases}$$

Let $\mathcal{Y} = (Y_\ell : \ell \in E \cup J)$ be a family of subsets of $E \cup I$ where for all $\ell \in E \cup J$

$$Y_\ell := \begin{cases} \{\ell\} \cup \{i \in I : \ell \in A_i\} \text{ if } \ell \in E \\ B_\ell \text{ if } \ell \in J. \end{cases}$$

Then $\mathcal{A}$ and $\mathcal{B}$ have a common transversal if and only if $\mathcal{X}$ has a transversal containing J and $\mathcal{Y}$ has a transversal containing I.

Proof. ($\Longleftarrow$) Suppose that $\mathcal{X}$ has a transversal containing J and $\mathcal{Y}$ has a transversal containing I. Let $(x_k : k \in E \cup I)$ be the transversal of $\mathcal{X}$ containing J. For $j \in J$, there is a (unique) $e_j \in E$ such that $j = x_{e_j} \in X_{e_j}$. Note that $e_j \in B_j$ for all $j \in J$.

For all $e \in E \setminus \{e_j : j \in J\}$, we cannot have $x_e \in J$, so $x_e = e$. Thus $\{x_i\}_{i \in I} \subseteq E \setminus (E \setminus \{e_j : j \in J\}) = \{e_j : j \in J\}$ – that is, for all $i \in I$, there exists $j_i \in J$ such that $x_i = e_{j_i} \in A_i \cap B_{j_i}$, and the map $i \mapsto j_i$ ($i \in I$) is 1-1. Thus $\mathcal{A}$ and a subfamily of $\mathcal{B}$ have a common transversal.

Similarly, the assumption on $\mathcal{Y}$ means that $\mathcal{B}$ and a subfamily of $\mathcal{A}$ have a common transversal. By [10, Theorem 10.4.4] ([3, Corollary 2]), $\mathcal{A}$ and $\mathcal{B}$ have a common transversal.

($\Longrightarrow$) Let $\sigma : I \longrightarrow J$ be a bijection and let $(x_i)_{i\in I}$ be a system of distinct representatives of $\mathcal{A}$ such that for all $i \in I$, $x_i \in A_i \cap B_{\sigma(i)}$.

We will define x_e for $e \in E$. For $i \in I$, let $x_{x_i} := \sigma(i)$ and for $e \in E \setminus \{x_i : i \in I\}$, let $x_e := e$.

We will show that $(x_k)_{k\in E\cup I}$ is a system of representatives for $\mathcal{X}$.

It is given that $x_i \in A_i$ for $i \in I$.

For $i \in I$, $x_{x_i} \in J$ and $x_i \in B_{\sigma(i)}$ so $x_{x_i} \in X_{x_i}$.

For $e \in E \setminus \{x_i : i \in I\}$, $e \in X_e$.

Now we show that $(x_k)_{k\in E\cup I}$ is a system of *distinct* representatives of $\mathcal{X}$.

For $i_1, i_2 \in I$ such that $i_1 \neq i_2$, it is given that $x_{i_1} \neq x_{i_2}$. Moreover, $x_{x_{i_1}} = \sigma(i_1) \neq \sigma(i_2) = x_{x_{i_2}}$. Also, for all $i \in I$, $x_{x_i} \in J$, so $x_{x_i} \notin E$. If $e \in E \setminus \{x_i : i \in I\}$, then for all $i \in I$, $x_i \neq e = x_e$. Of course, if $e_1, e_2 \in E \setminus \{x_i : i \in I\}$ but $e_1 \neq e_2$, then $x_{e_1} = e_1 \neq e_2 = x_{e_2}$.

Since σ is onto, $\{x_{x_i} : i \in I\} = \{\sigma(i) : i \in I\} = J$, so $J \subseteq \{x_k : k \in E \cup I\}$.

By symmetry, $\mathcal{Y}$ has a transversal containing I. □

LEMMA 1.3. Let $\mathcal{X}$ be as in Theorem 2. If $\mathcal{B}$ has a transversal and $\mathcal{X}$ has a transversal, then $\mathcal{X}$ has a transversal containing J.

Proof. Represent the set system $\mathcal{X}$ as a bipartite graph with $E \cup I$ on one side and a disjoint set in bijection with $E \cup J$ on the other side. (Denote the "twins" by $\tilde{E}$ and $\tilde{J}$ with $\tilde{j}$ corresponding to $j \in J$.)

Let $\tau : j \longmapsto b_j \in B_j$ be a map corresponding to a transversal for $\mathcal{B}$. This gives us a matching in the graph using the edges $(\tilde{j}, b_j)$ where $j \in J$. The transversal for $\mathcal{X}$ gives us a matching containing $E \cup I$. By the usual arguments or by [4, Theorem 4], the two matchings give us a graph whose components are one- or two-way infinite paths, finite paths, or finite cycles. (Because the author needed more thought than expected to understand one part of the proof of [4, Theorem 4], we elaborate here.)

We cannot have any components that are finite paths with an odd number of vertices starting in $E \cup I$ since we have a transversal for $\mathcal{X}$. We cannot have any components that are finite paths starting in $\tilde{J}$ with an odd number of vertices none of which are in $\tilde{E}$, because there is a matching covering $\tilde{J}$. Figure 1 shows the possibilities. The edges in the matching we will create are labeled "μ."

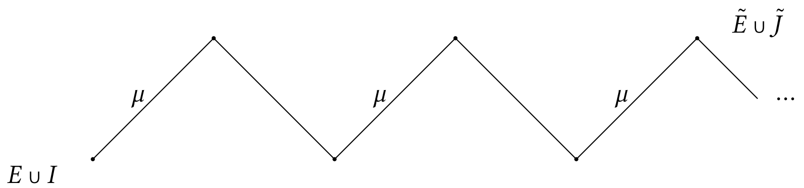

Figure 1a: A one-way infinite path starting in $E \cup I$

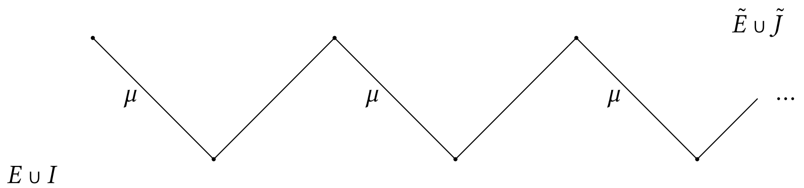

Figure 1b: A one-way infinite path starting in $\tilde{E} \cup \tilde{J}$

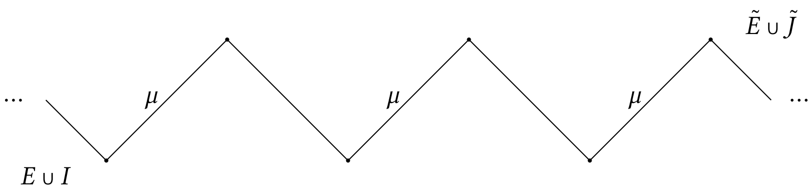

Figure 1c: A two-way infinite path

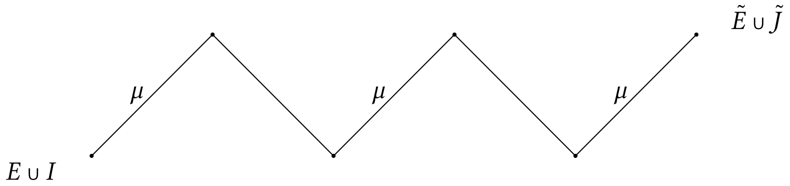

Figure 1d: A path with an even number of vertices

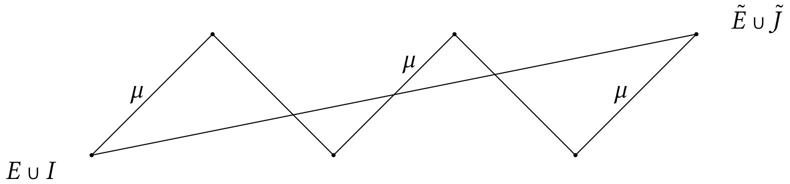

Figure 1e: A cycle

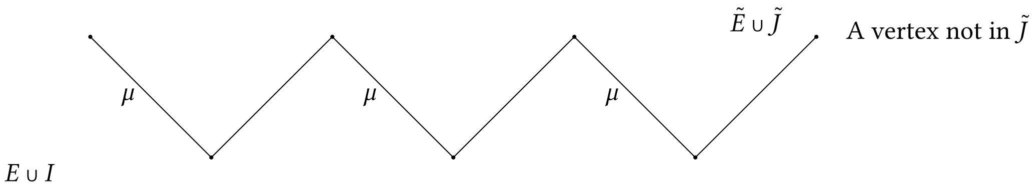

Figure 1f: A path with an odd number of vertices starting in $\tilde{E} \cup \tilde{J}$

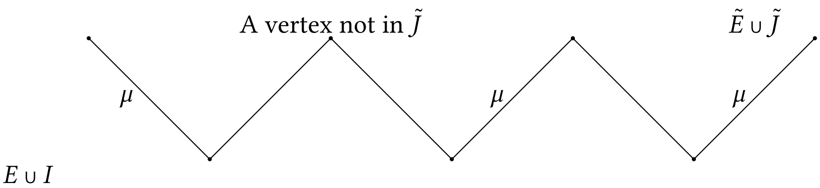

Figure 1g: Another path with an odd number of vertices starting in $\tilde{E} \cup \tilde{J}$

In Figures 1a to 1e, every member of $E \cup I$ is covered by the matching and so is every member of $\tilde{E} \cup \tilde{J}$. In Figure 1f, not every node in $\tilde{E} \cup \tilde{J}$ can be in $\tilde{J}$. For example, if one endpoint is not in $\tilde{J}$, We can choose a matching that includes the other endpoint and all internal path vertices, so our matching covers all of $E \cup I$ but also all of $\tilde{J}$.

If both endpoints are in $\tilde{J}$ (Figure 1g), we can choose the edges containing those endpoints and keep going until we get to the vertex not in $\tilde{J}$, as shown.

Note that an edge $(\tilde{j}, b_j)$ where $j \in J$ means $b_j \in B_j \subseteq E$, so $j \in X_{b_j}$. Thus the matching we have created does yield a transversal for $\mathcal{X}$ and we have ensured that it contains J. □

COROLLARY 1.4. Using the notation of Theorem 2, $\mathcal{A}$ and $\mathcal{B}$ have a common transversal if and only if $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{A}$, and $\mathcal{B}$ have transversals. □

To derive a criterion for having a common system of representatives, we employ the usual device: Let $\mathcal{A} = (A_i : i \in I)$ and $\mathcal{B} = (B_j : j \in J)$ be two families of sets. Let $\tilde{J}$ be a set in bijection with J but disjoint from I, where $\tilde{j}$ corresponds to $j \in J$. Build a bipartite graph Γ with parts I and $\tilde{J}$ where $(i, \tilde{j})$ is an edge if and only if $A_i \cap B_j \neq \emptyset$. Then $\mathcal{A}$ and $\mathcal{B}$ have a common system of representatives if and only if Γ has a perfect matching. By a theorem of Ore ([10, Theorem 1.3.4] for $X' = X$ and $Y' = Y$, derived in Mirsky's book as a consequence of a theorem of Perfect and Pym), this happens if and only if the society given by Γ where I is the set of "men" has a marriage and the dual society where $\tilde{J}$ is the set of "men" has a marriage.

2. A QUESTION IN WELSH'S 1976 TEXT *MATROID THEORY* BASED ON AN ISSUE PYM AND PERFECT RAISED IN 1970

Let S be a set. Consider functions $\mu, \mu_1, \mu_2 : \mathcal{P}(S) \longrightarrow \{r \in \mathbb{R} : r \geq 0\} \cup \{\infty\}$. The function μ is *non-decreasing* ([17, p. 119] also uses the term *increasing*) if for all $A \subseteq B \subseteq S$ implies $\mu(A) \leq \mu(B)$. It is *submodular* if for all $A, B \subseteq S$,

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B),$$

where $\infty + \infty = \infty$ and $r + \infty = \infty + r = \infty$ for all $r \in \mathbb{R}$.

Define $\mu_1 \sqcup \mu_2 : \mathcal{P}(S) \longrightarrow \{r \in \mathbb{R} : r \geq 0\} \cup \{\infty\}$ as follows: For $A \subseteq S$

$$(\mu_1 \sqcup \mu_2)(A) = \sup_{X \subseteq A} \{\mu_1(X) + \mu_2(A \setminus X)\}.$$

In their 1970 paper [14], Pym and Perfect focused on functions with range $\mathbb{N}_0 \cup \{\infty\}$, and considered the cardinality $|A|$ of a set A to take values in this range. They wrote "that if μ_1 and μ_2 are submodular [and non-decreasing], continuous from below [a term Pym and Perfect define on p. 10], and $\mu_1, \mu_2 \leq |\cdot|$, then $\mu_1 \sqcup \mu_2$ is submodular [and non-decreasing]. We do not know whether $\mu_1 \sqcup \mu_2$ is submodular [and non-decreasing] in general. . . ." [14, p. 23]

Referring to Pym and Perfect, Welsh poses a problem in his 1976 text where $\mu_1, \mu_2 : \mathcal{P}(S) \longrightarrow \{r \in \mathbb{R} : r \geq 0\}$. (The proof of [17, Section 7.2, Theorem 1] suggests that Welsh is assuming S is finite.) Welsh notes that $\delta = \mu_1 \sqcup \mu_2$ "is submodular when the μ_i are the rank functions of matroids." He then poses the unsolved problem [17, pp. 1,119]: "Is δ submodular when the μ_i are not the rank functions of matroids (see Pym and Perfect [70])?" (Number in brackets in the original.) He is assuming that μ_1 and μ_2 are submodular, non-decreasing, non-negative functions.

In fact, we can resolve the issue Pym and Perfect raise and we can answer negatively the question Welsh asks.

Let $S = \{a, b, c\}$. Then μ_1, μ_2, and δ are given by Figure 2.

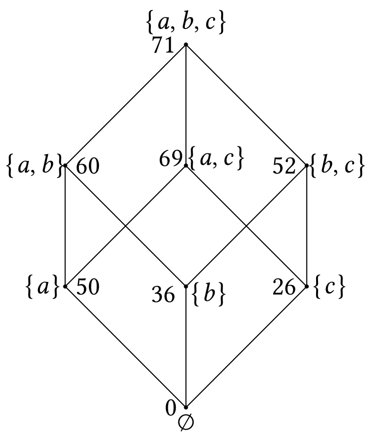

Figure 2a: μ_1

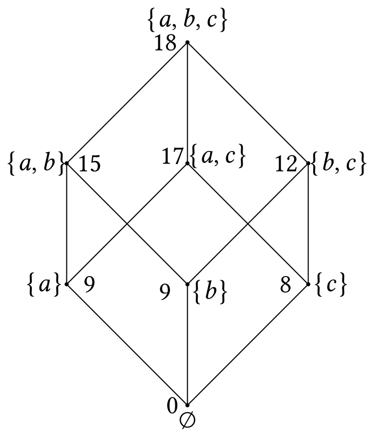

Figure 2b: μ_2

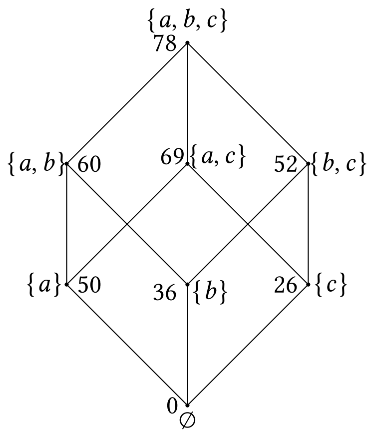

Figure 2c: $\delta = \mu_1 \sqcup \mu_2$

Obviously the submodular inequality holds when $A \subseteq B$ or $B \subseteq A$, so we can check that μ_1 is submodular by noting

$0 + 60 = 60 \leq 86 = 50 + 36$	$A = \{a\}$	$B = \{b\}$
$0 + 69 = 69 \leq 76 = 50 + 26$	$A = \{a\}$	$B = \{c\}$
$0 + 52 = 52 \leq 62 = 36 + 26$	$A = \{b\}$	$B = \{c\}$
$50 + 71 = 121 \leq 129 = 60 + 69$	$A = \{a, b\}$	$B = \{a, c\}$
$36 + 71 = 107 \leq 112 = 60 + 52$	$A = \{a, b\}$	$B = \{b, c\}$
$26 + 71 = 97 \leq 121 = 69 + 52$	$A = \{a, c\}$	$B = \{b, c\}$
$0 + 71 = 71 \leq 102 = 50 + 52$	$A = \{a\}$	$B = \{b, c\}$
$0 + 71 = 71 \leq 105 = 36 + 69$	$A = \{b\}$	$B = \{a, c\}$
$0 + 71 = 71 \leq 86 = 26 + 60$	$A = \{c\}$	$B = \{a, b\}$.

Also μ_1 is non-decreasing and $\mu_1(\emptyset) = 0$.

We can check that μ_2 is submodular by noting

$0 + 15 = 15 \leq 18 = 9 + 9$	$A = \{a\}$	$B = \{b\}$
$0 + 17 = 17 \leq 17 = 9 + 8$	$A = \{a\}$	$B = \{c\}$
$0 + 12 = 12 \leq 17 = 9 + 8$	$A = \{b\}$	$B = \{c\}$
$9 + 18 = 27 \leq 32 = 15 + 17$	$A = \{a, b\}$	$B = \{a, c\}$
$9 + 18 = 27 \leq 27 = 15 + 12$	$A = \{a, b\}$	$B = \{b, c\}$

$$
\begin{array}{lll}
8+18=26\le 29=17+12 & A=\{a,c\} & B=\{b,c\}\\
0+18=18\le 21=9+12 & A=\{a\} & B=\{b,c\}\\
0+18=18\le 26=9+17 & A=\{b\} & B=\{a,c\}\\
0+18=18\le 23=8+15 & A=\{c\} & B=\{a,b\}.
\end{array}
$$

It is non-decreasing and $\mu_2(\emptyset)=0$.

$$
\begin{aligned}
\delta(\emptyset) &= \sup\{\mu_1(\emptyset)+\mu_2(\emptyset)\} = 0+0=0\\
\delta(\{a\}) &= \sup\{\mu_1(\emptyset)+\mu_2(\{a\}), \mu_1(\{a\})+\mu_2(\emptyset)\}\\
&= \sup\{0+9, 50+0\} = \sup\{9,50\} = 50\\
\delta(\{b\}) &= \sup\{\mu_1(\emptyset)+\mu_2(\{b\}), \mu_1(\{b\})+\mu_2(\emptyset)\}\\
&= \sup\{0+9, 36+0\} = \sup\{9,36\} = 36\\
\delta(\{c\}) &= \sup\{\mu_1(\emptyset)+\mu_2(\{c\}), \mu_1(\{c\})+\mu_2(\emptyset)\}\\
&= \sup\{0+8, 26+0\} = \sup\{8,26\} = 26\\
\delta(\{a,b\}) &= \sup\{\mu_1(\emptyset)+\mu_2(\{a,b\}), \mu_1(\{a\})+\mu_2(\{b\}), \mu_1(\{b\})+\mu_2(\{a\}), \mu_1(\{a,b\})+\mu_2(\emptyset)\}\\
&= \sup\{0+15, 50+9, 36+9, 60+0\}\\
&= \sup\{15,59,45,60\} = 60\\
\delta(\{a,c\}) &= \sup\{\mu_1(\emptyset)+\mu_2(\{a,c\}), \mu_1(\{a\})+\mu_2(\{c\}), \mu_1(\{c\})+\mu_2(\{a\}), \mu_1(\{a,c\})+\mu_2(\emptyset)\}\\
&= \sup\{0+17, 50+8, 26+9, 69+0\}\\
&= \sup\{17,58,35,69\} = 69\\
\delta(\{b,c\}) &= \sup\{\mu_1(\emptyset)+\mu_2(\{b,c\}), \mu_1(\{b\})+\mu_2(\{c\}), \mu_1(\{c\})+\mu_2(\{b\}), \mu_1(\{b,c\})+\mu_2(\emptyset)\}\\
&= \sup\{0+12, 36+8, 26+9, 52+0\}\\
&= \sup\{12,44,35,52\} = 52\\
\delta(\{a,b,c\}) &= \sup\{\mu_1(\emptyset)+\mu_2(\{a,b,c\}), \mu_1(\{a\})+\mu_2(\{b,c\}), \mu_1(\{b\})+\mu_2(\{a,c\}),\\
&\quad \mu_1(\{c\})+\mu_2(\{a,b\}), \mu_1(\{a,b\})+\mu_2(\{c\}), \mu_1(\{a,c\})+\mu_2(\{b\}),\\
&\quad \mu_1(\{b,c\})+\mu_2(\{a\}), \mu_1(\{a,b,c\})+\mu_2(\emptyset)\}\\
&= \sup\{0+18, 50+12, 36+17, 26+15, 60+8, 69+9, 52+9, 71+0\}\\
&= \sup\{18,62,53,41,68,78,61,71\} = 78.
\end{aligned}
$$

But

$$
\begin{aligned}
\delta(\{a,b\}\cup\{b,c\})+\delta(\{a,b\}\cap\{b,c\}) &= \delta(\{a,b,c\})+\delta(\{b\})\\
&= 78+36=114\\
&\not\le 112 = 60+52 = \delta(\{a,b\})+\delta(\{b,c\}),
\end{aligned}
$$

so δ is not submodular. (It is non-decreasing and $\delta(\emptyset)=0$.)

While the answer to the question is “no” for $|S|\ge 3$, we do, however, show that the answer to the question is “yes” if $|S|\le 2$. We will now stop assuming $S=\{a,b,c\}$. If $|S|\le 1$, then $\mu_1\sqcup\mu_2$ is submodular since $\mathcal{P}(S)$ is a totally ordered set. If $S=\{a,b\}$, to check if $\delta=\mu_1\sqcup\mu_2$ is submodular when μ_1 and μ_2 are submodular and non-decreasing, we can “translate” the values of μ_1 and μ_2 so $\mu_1(\emptyset)=0=\mu_2(\emptyset)$. (See Figure 3.)

Submodularity means $z_1\le x_1+y_1$ and $z_2\le x_2+y_2$. Without loss of generality, $x_2\le x_1$.

Now $\delta(\emptyset)=0$, $\delta(\{a\})=x_1$, $\delta(\{b\})=\sup\{y_1,y_2\}$, and

$$
\begin{aligned}
\delta(\{a,b\}) &= \sup\{\mu_1(\emptyset)+\mu_2(\{a,b\}), \mu_1(\{a\})+\mu_2(\{b\}),\\
&\quad \mu_1(\{b\})+\mu_2(\{a\}), \mu_1(\{a,b\})+\mu_2(\emptyset)\}\\
&= \sup\{z_2, x_1+y_2, x_2+y_1, z_1\}.
\end{aligned}
$$

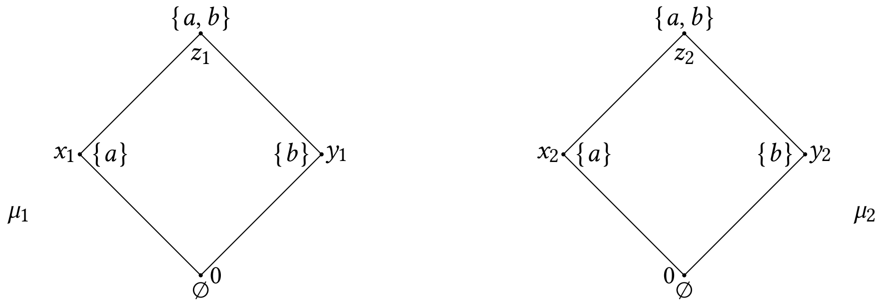

Figure 3

CASE 1. $\delta(\{b\}) = y_1$

This means $y_2 \le y_1$.

CASE 1a. $\delta(\{a, b\}) = z_2$.

Then $\delta(\emptyset) + \delta(\{a, b\}) = 0 + z_2 = z_2 \le x_2 + y_2 \le x_1 + y_1 = \delta(\{a\}) + \delta(\{b\})$.

CASE 1b. $\delta(\{a, b\}) = x_1 + y_2$.

Then $\delta(\emptyset) + \delta(\{a, b\}) = 0 + x_1 + y_2 = x_1 + y_2 \le x_1 + y_1 = \delta(\{a\}) + \delta(\{b\})$.

CASE 1c. $\delta(\{a, b\}) = x_2 + y_1$.

Then $\delta(\emptyset) + \delta(\{a, b\}) = 0 + x_2 + y_1 = x_2 + y_1 \le x_1 + y_1 = \delta(\{a\}) + \delta(\{b\})$.

CASE 1d. $\delta(\{a, b\}) = z_1$.

Then $\delta(\emptyset) + \delta(\{a, b\}) = 0 + z_1 = z_1 \le x_1 + y_1 = \delta(\{a\}) + \delta(\{b\})$.

CASE 2. $\delta(\{b\}) = y_2$.

This means $y_1 \le y_2$.

CASE 2a. $\delta(\{a, b\}) = z_2$.

Then $\delta(\emptyset) + \delta(\{a, b\}) = 0 + z_2 = z_2 \le x_2 + y_2 \le x_1 + y_2 = \delta(\{a\}) + \delta(\{b\})$.

CASE 2b. $\delta(\{a, b\}) = x_1 + y_2$.

Then $\delta(\emptyset) + \delta(\{a, b\}) = 0 + x_1 + y_2 = x_1 + y_2 = \delta(\{a\}) + \delta(\{b\})$.

CASE 2c. $\delta(\{a, b\}) = x_2 + y_1$.

Then $\delta(\emptyset) + \delta(\{a, b\}) = 0 + x_2 + y_1 = x_2 + y_1 \le x_1 + y_2 = \delta(\{a\}) + \delta(\{b\})$.

CASE 2d. $\delta(\{a, b\}) = z_1$.

Then $\delta(\emptyset) + \delta(\{a, b\}) = 0 + z_1 = z_1 \le x_1 + y_1 \le x_1 + y_2 = \delta(\{a\}) + \delta(\{b\})$.

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