CONFLUENT KAMPÉ DE FÉRIET SERIES ARISING IN THE
SOLUTIONS OF CAUCHY PROBLEM
FOR THE DEGENERATE HYPERBOLIC EQUATION OF THE
SECOND KIND WITH THE SPECTRAL PARAMETER

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ABSTRACT
We define the order of the double hypergeometric series, investigate the properties of the new confluent Kampé de Fériet series, and build systems of partial differential equations that satisfy the new Kampé de Fériet series. We solve the Cauchy problem for a degenerate hyperbolic equation of the second kind with a spectral parameter using the high-order Kampé de Fériet series. Thanks to the properties of the introduced Kampé de Fériet series, it is possible to obtain a solution to the problem in explicit forms.

KEYWORDS
Confluent Kampé de Fériet series, convergence domain, degenerate hyperbolic equation with spectral parameter, Cauchy problem, Riemann function

MATHEMATICS SUBJECT CLASSIFICATION (2020)
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1. INTRODUCTION
A great deal of interest in the theory of hypergeometric functions (that is, hypergeometric functions of one, two, and more variables) stems from the fact that hypergeometric (higher and special or transcendental) functions can be used to solve many applied problems involving thermal conductivity and dynamics, electromagnetic oscillation and aerodynamics, and quantum mechanics and potential

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theory [2, 20, 23]. Such kinds of functions are often referred to as special functions of mathematical physics. It is known that hypergeometric series \( F(a, b; c; z) \) (cf. equation (2.2) infra) were studied by Leonhard Euler, but the first full systematic treatment was given by Carl Friedrich Gauss (1813). Studies in the nineteenth century included those of Ernst Kummer (1836) and the fundamental characterization by Bernhard Riemann (1857) of the hypergeometric function utilizing the differential equation it satisfies. The great success of the theory of hypergeometric series in one variable has stimulated the development of a corresponding theory in two and more variables. Appell has defined, in 1885, four series, \( F_1 \) to \( F_4 \) which are all analogous to Gauss’ \( F(a, b; c; z) \). Picard has pointed out that one of these series is intimately related to a function studied by Pochhammer in 1870, and Picard and Goursat also constructed a theory of Appell’s series which is analogous to Riemann’s theory of Gauss’ hypergeometric series. P. Humbert has studied confluent hypergeometric series in two variables. An expansion of the results of the French school together with references to the original literature are to be found in the monograph by Appell and Kampe de Fériet [1], which is the standard work on the subject. This work also contains an extensive bibliography of all relevant papers up to 1926. A great merit in the further development of the theory of the hypergeometric series in two variables belongs to Horn, who gave a general definition and order classification of double hypergeometric series. He has investigated the convergence of hypergeometric series of two variables and established the systems of partial differential equations which they satisfy. Horn investigated in particular hypergeometric series of order two and found that, apart from certain series which are either expressible in terms of one variable or are products of two hypergeometric series, each in one variable, there are essentially 34 distinct convergent series of order two. The four Appell series were unified and generalized by Kampe de Fériet [15] who defined a general hypergeometric series in two variables. The notation introduced by Kampe de Fériet for his double hypergeometric series of superior order was subsequently abbreviated by Burchnell and Chaundy [3]. Srivastava and Panda [28] gave the definition of a more general double hypergeometric series (than the one defined by Kampe de Fériet) in a slightly modified notation and announced some groups of conditions on the parameters under which the Kampe de Fériet series converges in a non-empty set. Interesting results in this direction have been obtained, for example, in the works [4, 5, 7, 10, 17, 18, 19, 24, 26]. Further study of the properties of hypergeometric Kampe de Fériet series when applied to the solution of applied problems showed (see, for example, [9]) that, it turns out, the well-known assertion about the convergence domain of these series does not cover all classes of Kampe de Fériet series, i.e., there is a certain set of series whose convergence domain cannot be used to draw conclusions based on the theorem given in [28] (see, also [27, p. 27]). Here, in particular, we are talking about some confluent series, which have applications in solving boundary value problems for the degenerating hyperbolic equation of the second kind. The degenerate hyperbolic equations are encountered in the solution of various problems of gas dynamics [2], in computer tomography [22], etc. In the scientific literature, degenerate hyperbolic equations are usually divided into the first and second kinds. If a hyperbolic equation degenerates along a straight line, which is at the same time a characteristic, then the such equation is a degenerate equation of the second kind, in contrast to equations of the first kind, when the degeneracy line consists of cusp points of the family of characteristics of the degenerate hyperbolic equation. Therefore, equations of the second kind are difficult to study and they are relatively little studied concerning equations of the first kind. In the present paper, we study the properties of the higher-order confluent Kampe de Fériet hypergeometric series and apply the results obtained to find an explicit regular solution to the Cauchy problem for a degenerate hyperbolic equation of the second kind with a spectral parameter. The plan of this paper is as follows. In Section 2 we briefly give some preliminary information, which will be used later. In Section 3 we define the order of the double hypergeometric series and study the properties of the new confluent Kampe de Fériet series. In Section 4 we will construct systems of partial differential equations that are satisfied by the new Kampe de Fériet series. In section 5 using the Riemann method we solve the Cauchy problem for a degenerate hyperbolic equation of the second kind with a spectral parameter. Thanks to the properties of the introduced functions, it is possible to obtain a solution to the problem in explicit form.
2. DEFINITIONS AND PRELIMINARIES

In the usual notation \((\lambda)_n\) stands for the Pochhammer symbol defined by
\[
(\lambda)_n = \Gamma(\lambda + n)/\Gamma(\lambda) = \lambda(\lambda + 1)\cdots(\lambda + n - 1), \quad n = 1, 2, \ldots; \quad (\lambda)_0 = 1
\]
where
\[
(1)_n = n!, \quad (\lambda + k)_n = (\lambda)_n(\lambda + n)_k.
\]
where \(\Gamma(z)\) is a famous Euler’s gamma-function. The hypergeometric function of Gauss is defined inside the circle \(|z| < 1\) as the sum of the hypergeometric series [8]:
\[
F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \quad c \neq 0, -1, -2, \ldots
\]
where \(a, b, c\), are independent of \(z\). We shall call \(a, b, c\) the parameters of the hypergeometric function; they are arbitrary complex numbers. If \(\Re(c) > \Re(b) > 0\), we have Euler’s formula [8]
\[
F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^c} \, dt.
\]

Here the right-hand side is a one-valued analytic function of \(z\) within the domain \(|\arg(1-z)| < \pi\); therefore (2.3) gives also the analytic continuation of \(F(a, b; c; z)\). The integral representation (2.3) allows to derive the Boltz formula [8]
\[
F(a, b; c; z) = (1-z)^{-a}F\left(c-a, b; c; \frac{z}{z-1}\right),
\]
the autotransformation formula [8]
\[
F(a, b; c; z) = (1-z)^{-a-b}F(c-a, c-b; c; z)
\]
and get the value of the Gaussian function in unity (the summation formula) [8]
\[
F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{(c-a)\Gamma(c-b)} \quad \Re(c-a-b) > 0, \quad c \neq 0, -1, -2, \ldots
\]
The following differentiation formulas
\[
\frac{d}{dz} F(a, b; c; z) = \frac{ab}{c} F(a+1, b+1; c+1; z)
\]
\[
\frac{d}{dz} (1-z)^{a} F(a, b; c; z)] = \frac{a(c-b)}{c} (1-z)^{a-1} F(a+1, b; c+1; z)
\]
are valid. In the Gaussian hypergeometric series \(F(a, b; c; z)\) there are two numerator parameters \(a, b\), and one denominator parameter \(c\). A natural generalization of this series is accomplished by introducing any arbitrary number of numerator and denominator parameters. The resulting series
\[
\pFq{p}{q}{\alpha_1, \ldots, \alpha_p}{\beta_1, \ldots, \beta_q}{z} = \sum_{\alpha, \beta} \frac{(\alpha)_\alpha (\beta)_\beta}{(\alpha)_\alpha (\beta)_\beta} \frac{z^\alpha}{\alpha!}
\]
is known as the generalized Gauss series [8], or simply, the generalized hypergeometric series. Here \(p\) and \(q\) are positive integers or zero (interpreting an empty product as 1), and we assume that the variable \(z\), the numerator parameters \(\alpha_1, \ldots, \alpha_p\), and the denominator parameters \(\beta_1, \ldots, \beta_q\) take on complex values, provided that \(\beta_j \neq 0, -1, -2, \ldots; j = 1, \ldots, q\). Gauss’ series (2.2) in the present notation is
\[
\pFq{1}{1}{\alpha}{\beta}{z} = \left[\begin{array}{c} a \b \end{array} c \mid z \right] .
\]
The Bessel function is defined as a special case of the generalized hypergeometric series
\[
J_\nu(z) = \frac{1}{\Gamma(\alpha + 1)} \left(\frac{z}{2}\right)^\alpha \pFq{1}{0}{\alpha}{\beta}{-\frac{z^2}{4}} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m + \alpha + 1)} \left(\frac{z}{2}\right)^{2m + \alpha}.
\]
Horn [12] gave the following general definition: the double power series

$$\sum_{m,n=0}^{\infty} A(m, n)x^m y^n$$

(2.10)

is a hypergeometric series if the two quotients

$$f(m, n) = \frac{A(m+1, n)}{A(m, n)}, \quad g(m, n) = \frac{A(m, n+1)}{A(m, n)}$$

are rational functions of $m$ and $n$. Horn puts

$$f(m, n) = \frac{F(m, n)}{F'(m, n)}, \quad g(m, n) = \frac{G(m, n)}{G'(m, n)}$$

(2.11)

where $F, F', G, G'$ are polynomials in $m, n$, of respective degrees $p, p', q, q'$. $F'$ is assumed to have a factor $m + 1$, and $G'$ a factor $n + 1$; $F$ and $F'$ have no common factor except, possibly, $m + 1$; and $G$ and $G'$ no common factor except possibly $n + 1$. The highest of the four numbers $p, p', q, q'$, is the order of the hypergeometric series. Horn distinguished double series into complete and confluent series, that is, according to Horn’s definition, a second-order double hypergeometric series is called complete if $p = p' = q = q' = 2$, otherwise confluent. Horn [13] defined ten hypergeometric series in two variables and denoted them by $G_1, G_2, G_3, H_1, \ldots, H_7$; he thus completed the set of all possible second-order (complete) hypergeometric series in two variables [8, pp. 224–228]. Seven confluent forms of the four Appell series were defined by Humbert [14], and he denoted these confluent hypergeometric series in two variables by $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1, \Xi_2$. In addition, there exist 13 confluent forms of the Horn series, which are denoted by $\Gamma_1, \Gamma_2, H_1, H_2$. The work of Humbert has been described reasonably fully by Appell and Kampé de Fériet [1, pp. 124–135], and the definitions and convergence conditions of all of these 20 confluent hypergeometric series in two variables are given also in [8, pp. 225–228]. The definitions of $\Phi_1, \Phi_2$ and $\Xi_2$, given in [8, p. 225, equations (20), (21), and p. 226, equation (26)] are in error; these errors were first discovered and corrected by Marićev [21], and Srivastava and Karlsson [27, pp. 25–26, equations (16), (17) and (24)]; we recall here the corrected definitions of the following Humbert series that we need in further research:

$$\Phi_1 (a, b; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m}{(c)_{m+n} m! n!} x^m y^n, \quad |x| < 1,$$

$$\Psi_1 (a, b, c, c'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_{m+n} m! n!} x^m y^n, \quad |x| < 1,$$

$$\Xi_1 (a, a', b; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(a')_n(b)_m}{(c)_{m+n} m! n!} x^m y^n, \quad |x| < 1,$$

$$\Xi_2 (a, b; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m}{(c)_{m+n} m! n!} x^m y^n, \quad |x| < 1.$$  

(2.12)

Just as the Gaussian series $F(a, b; c; z)$ was generalized to $_pF_q$ by increasing the numbers of the numerator and denominator parameters, the four Appell series were unified and generalized by Kampé de Fériet [15] who defined a general hypergeometric series in two variables. The notation introduced by Kampé de Fériet for his double hypergeometric series of superior order was subsequently abbreviated by Burchnell and Chaundy [3]. Srivastava and Panda [28] (see, also [27, Section 3.1]) defined the more general double hypergeometric series (than the one defined by Kampé de Fériet) in a slightly modified notation

$$F_{A^1B^1; C^1D^1}^{\mathcal{B}^1; \mathcal{D}^1} [(a): (b); (b'); (c): (d); (d'); x, y]_{m+n} = \sum_{m,n=0}^{\infty} \frac{A^1_{m+n} A^1_{m+n}}{B^1_{m+n} B^1_{m+n}} \prod_{j=1}^{+\infty} (a_j)_{m+n} \prod_{j=1}^{+\infty} (b_j)_{m+n} \frac{b'_{m+n}}{d'_{m+n}} x^m y^n.$$  

(2.13)
For the double hypergeometric series (2.13), Srivastava and Daoust [26] deduced from much more general results in three cases (proved by them) that (i) if \( A + B > C + D + 1 \) and \( A + B' > C + D' + 1 \), then the series (2.13) diverges whenever \( x \neq 0 \) and \( y \neq 0 \); (ii) if \( A + B = C + D + 1 \) and \( A + B' = C + D' + 1 \), then the series (2.13) converges absolutely, provided that

\[
|x|^{1/(A-C)} + |y|^{1/(A-C)} < 1, \quad \text{if} \ A > C,
\]

\[
\max \{|x|, |y|\} < 1, \quad \text{if} \ A \leq C.
\]

(iii) if \( A + B < C + D + 1 \) and \( A + B' < C + D' + 1 \), then the series (2.13) converges absolutely for all \( x, y \in \mathbb{C} \). Further study of the properties of the series (2.13) showed that it is possible to find the region of convergence of the series (2.13) in several other cases: (iv) if \( A + B = C + D + 1 \) and \( A + B' < C + D' + 1 \), then the series (2.13) converges absolutely for all \( y \in \mathbb{C} \) and \( |x| < 1 \); (v) if \( A + B = C + D + 1 \) and \( A + B' > C + D' + 1 \), then the series (2.13) diverges whenever \( y \neq 0 \); (vi) if \( A + B > C + D + 1 \) and \( A + B' = C + D' + 1 \), then the series (2.13) diverges whenever \( x \neq 0 \); it is understood (in each situation) that no zeros appear in the denominator of (2.13). The proof of the convergence of the series (2.13) in the cases (iv)–(vii) does not differ significantly from the proof of the cases (i)–(iii) (see, [26]). Although the double hypergeometric series defined by (2.13) reduces to the Kampé de Fériet series in the special case: \( B = B' = D' \), it is usually referred to in the literature as the Kampé de Fériet series. This paper aims at presenting a discussion of the complexity of the problems when the order of the double hypergeometric series exceeds two. We shall study some higher-order Kampé de Fériet series and apply the obtained results to the solution of the Cauchy problem for a degenerate hyperbolic equation of the second kind with the spectral parameter.

3. THE CONFLUENT KAMPÉ DE FÉRIET SERIES

Horn’s definition of the order of double series applies to the second-order double hypergeometric series. When the order of the hypergeometric series in two variables exceeds two, a new definition will be required.

**DEFINITION 1.** If the polynomials \( F, F', G \) and \( G' \), defined in (2.11) have degrees \( p, p', q \) and \( q' \), respectively, and \( P = \max(p,p') \), \( Q = \max(q,q') \), a pair of numbers \((P,Q)\) is defined to be the order of the double hypergeometric series (2.10).

Consider the following Kampé de Fériet hypergeometric series:

\[
F_{1}^{1}: 1.21 \quad \alpha : b; a'; \quad \beta : -; c'; \quad x, y,
\]

\[
= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m} \, x^{m} \, y^{n}}{(c)_{m+n} \, (c')_{n} \, m! \, n!}.
\]  
(3.1)

\[
F_{1}^{1}: 2.21 \quad \alpha : a; b'; \quad \beta : -; c'; \quad x, y,
\]

\[
= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m} \, x^{m} \, y^{n}}{(c)_{m+n} \, (c')_{n} \, m! \, n!}.
\]  
(3.2)

\[
F_{1}^{1}: 1.10 \quad \alpha : b; -; \quad \beta : a'; \quad c : -; c'; \quad x, y,
\]

\[
= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m} \, x^{m} \, y^{n}}{(c)_{m+n} \, (c')_{n} \, m! \, n!}.
\]  
(3.3)

\[
F_{1}^{1}: 1.11 \quad \alpha : b; a'; \quad \beta : -; c'; \quad x, y,
\]

\[
= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m} \, x^{m} \, y^{n}}{(c)_{m+n} \, (c')_{n} \, m! \, n!}.
\]  
(3.4)

\[
F_{1}^{1}: 0.12 \quad \alpha : b; a'; \quad \beta : c; \quad c' \, d'; \quad x, y,
\]

\[
= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m} \, x^{m} \, y^{n}}{(c)_{m+n} \, (c' \, d')_{n} \, m! \, n!}.
\]  
(3.5)

\[
F_{1}^{1}: 1.12 \quad \alpha : b; a'; \quad \beta : -; c'; \quad x, y,
\]

\[
= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m} \, x^{m} \, y^{n}}{(c)_{m+n} \, (c' \, d')_{n} \, m! \, n!}.
\]  
(3.6)

By virtue of Definition 1, the series (3.1)–(3.5) have the order \((2,3)\) and the series (3.6) has \((2,4)\). The order of the function suggests the order of the equations included in the system of partial differential equations that the given hypergeometric function satisfies (see, equations (4.4)–(4.9)).
The series (3.1)–(3.6) converge absolutely for \(|x| < 1\) and \(|y| < \infty\), which follows from the conditions (iv). Following the work [10] it is to see that for \(|x| = 1\) and \(|y| < \infty\) we have: absolute convergence for \(|x| = 1\) and \(|y| < \infty\) if \(\Re(a + b - c) < 0\), conditional convergence for \(|x| = 1, x \neq 1\) and \(|y| < \infty\) if \(0 \leq \Re(a + b - c) < 1\), divergence for \(|x| = 1\) and \(|y| < \infty\) if \(\Re(a + b - c) \geq 1\). We list some properties of the introduced functions.

1. Using the famous properties (2.1) of Pochhammer symbol \((\lambda)_n\), it is easily to obtain the following expansion formulas:

\[
f_1; 0.1^0 \left[ \frac{a, b; a'; c; -; c'}{x, y} \right] = \sum_{m=0}^{\infty} \frac{(a)m(b)m}{(c)m} F_2(a'; c + m, c'; y) \frac{x^m}{m!} = \sum_{n=0}^{\infty} \frac{(d')_n}{(c)_n (c')_n} F(a, b; c + n; x) \frac{y^n}{n!};
\]

\[
f_1; 0.1^0 \left[ \frac{a, b; a'; b'; c; -; c'}{x, y} \right] = \sum_{m=0}^{\infty} \frac{(a)m(b)m}{(c)m} F_2(a', b'; c + m, c'; y) \frac{x^m}{m!} = \sum_{n=0}^{\infty} \frac{(d')_n}{(c)_n (c')_n} F(a, b; c + n; x) \frac{y^n}{n!};
\]

\[
f_1; 0.1^1 \left[ \frac{a; b; c; -; c'}{x, y} \right] = \sum_{m=0}^{\infty} \frac{(a)m(b)m}{(c)m} F_2(a + m, a'; c + m, c'; y) \frac{x^m}{m!} = \sum_{n=0}^{\infty} \frac{(d')_n}{(c)_n (c')_n} F(a + n, b; c + n; x) \frac{y^n}{n!};
\]

\[
f_1; 0.1^1 \left[ \frac{a; b; a'; c; -; c', d'}{x, y} \right] = \sum_{m=0}^{\infty} \frac{(a)m(b)m}{(c)m} F_3(a + m, a'; c + m, c', d'; y) \frac{x^m}{m!} = \sum_{n=0}^{\infty} \frac{(d')_n}{(c)_n (d')_n} F(a + n, b; c + n; x) \frac{y^n}{n!};
\]

\[
f_1; 0.1^1 \left[ \frac{a; b; c; c', d'}{x, y} \right] = \sum_{m=0}^{\infty} \frac{(a)m(b)m}{(c)m} F_2(a + m, a', c', d'; y) \frac{x^m}{m!} = \sum_{n=0}^{\infty} \frac{(d')_n}{(c')_n (d')_n} F(a + n, b; c; y) \frac{y^n}{n!}.
\]

2. Certain cases where the order of the Kampe de Feriet hypergeometric series is lowered:

\[
f_1; 0.1^0 \left[ \frac{a, b; p; c; -; c'}{x, y} \right] = \Xi_2(a, b; c; x, y);
\]

\[
f_1; 0.1^0 \left[ \frac{a, b; a'; p; c; -; c'}{x, y} \right] = \Xi_1\left(a, a'; b; c; x, y\right);
\]

\[
f_1; 0.1^1 \left[ \frac{a; b; p; c; -; p'; c'}{x, y} \right] = \Phi_3(a, b; c, x, y);
\]

\[
f_1; 0.1^1 \left[ \frac{a; b; p; c; c', d'}{x, y} \right] = \Psi_1(a, b; c, c', x, y);
\]

\[
f_1; 0.1^1 \left[ \frac{a; b; p; c; c', c'}{x, y} \right] = \Phi_4(a, b; c; c', x, y);
\]

\[
f_1; 0.1^0 \left[ \frac{a; b; p; c; -; c', p'; c'}{x, y} \right] = \Phi_5(a, b; c; c', x, y);
\]

\[
f_1; 0.1^0 \left[ \frac{a; b; p; c; -; c'}{x, y} \right] = \Phi_6(a, b; c; x, y);
\]
\[(c - b - 1)F_{1}^{a} \quad (1, 3, 1) \left[ a : b, c - b; \right]
\quad \left( \begin{array}{c}
\quad \left( \begin{array}{c}
\quad a : b, c - b; \\
\quad c : \quad c - b - 1, c'; \\
\end{array} \right) \right] x, y \right]
\quad \right] (3.7) = (c - 1)F_{1}^{\left(1, 1, 0 \right)} \left[ a : b, c - b; \right]
\quad \left( \begin{array}{c}
\quad \left( \begin{array}{c}
\quad a : b, c - b; \\
\quad c : \quad c - b - 1, c'; \\
\end{array} \right) \right] x, y \right]
\quad \right] (3.8)\]

\[c(c - 2b)F_{1}^{\left(1, 1, 0 \right)} \left[ a : b, c - b; \right]
\quad \left( \begin{array}{c}
\quad \left( \begin{array}{c}
\quad a : b, c - b; \\
\quad c : \quad c - b - 1, c'; \\
\end{array} \right) \right] x, y \right]
\quad \right] (3.9) = (1 + x)yF_{1}^{\left(1, 1, 0 \right)} \left[ a : b, c - b; \right]
\quad \left( \begin{array}{c}
\quad \left( \begin{array}{c}
\quad a : b, c - b; \\
\quad c : \quad c - b - 1, c'; \\
\end{array} \right) \right] x, y \right]
\quad \right] (3.10)\]

3. Analogues of the summation formula (2.6):

\[
\begin{align*}
F_{0}^{\left(2, 2, 2 \right)} \left[ a : b, a'; c; \right]
\quad \left( \begin{array}{c}
\quad \left( \begin{array}{c}
\quad a : b, a'; c; \\
\quad c : \quad c - c', c'; \\
\end{array} \right) \right] x, y \right]
\quad \right] A.F_{3}^{\left(1, 1, 1 \right)} \left[ a : b, c - b, c'; \right]
\quad \left( \begin{array}{c}
\quad \left( \begin{array}{c}
\quad a : b, c - b, c'; \\
\quad c : \quad c - c, c', c'; \\
\end{array} \right) \right] x, y \right]
\quad \right] = \left( \begin{array}{c}
\quad \left( \begin{array}{c}
\quad a : b, c - b, c'; \\
\quad c : \quad c - c, c', c'; \\
\end{array} \right) \right] x, y \right]
\quad \right] A \quad (3.11) \end{align*}
\]
4. Analogues of the Bolts formula (4.4):

\[ \Xi_2(a, b; c; x, y) = (1 - x)^{-b} F_{1:1}^{0:1,0} \left[ \begin{array}{c}
\frac{c - a}{c - a'}; \\
\frac{x}{x - 1}
\end{array} \middle| 
\begin{array}{c}
a : b ; c ' \\
\end{array} \right] ; \\
\]

\[ F_{1:0}^{0:1,2} \left[ \begin{array}{c}
\frac{c : a, b ; c }{c : - a', c '}
\end{array} \middle| 
\begin{array}{c}
x, y
\end{array} \right] = (1 - x)^{-b} F_{1:0}^{0:1,2} \left[ \begin{array}{c}
\frac{c - a}{c - a'}; \\
\frac{x}{x - 1}
\end{array} \middle| 
\begin{array}{c}
a : b , a ' ; c ' \\
\end{array} \right] ; \quad (3.13)
\]

5. Using differentiation formulas (2.7) and (2.8), it is easy to obtain the following formulas

\[ \frac{\partial}{\partial x} \left( (1 - x)^b F_{1:0}^{0:1,0} \left[ \begin{array}{c}
\frac{c - a}{c - a'}; \\
\frac{x}{x - 1}
\end{array} \middle| 
\begin{array}{c}
a : b ; c ' \\
\end{array} \right] \right) = \frac{(c - a)b}{c} (1 - x)^{(b - 1)} F_{1:0}^{0:1,0} \left[ \begin{array}{c}
\frac{c - a}{c - a'}; \\
\frac{x}{x - 1}
\end{array} \middle| 
\begin{array}{c}
\cdot : b ; c ' \\
\end{array} \right] , \quad (3.14)
\]

\[ \frac{\partial}{\partial y} \Xi_2(a, b; c; x, y) = \frac{1}{c} \Xi_2(a, b; c + 1; x, y) . \quad (3.15)
\]

6. Using the Bolts formula (2.4) and the summation formula (2.6), we obtain the following confluence formulas:

\[ \lim_{x \to 1} (1 - x)^{-b} \Xi_2(a, b; c; x, y) = \frac{\Gamma(c) \Gamma(a - b)}{\Gamma(a)} y^{(a - b)/2} \frac{\zeta_{1,0}(2 \sqrt{y})}{G(a - b, c)} ; \quad (3.16)
\]

\[ \lim_{x \to 1} (1 - x)^b F_{1:0}^{0:1,0} \left[ \begin{array}{c}
\frac{c - a}{c - a'}; \\
\frac{x}{x - 1}
\end{array} \middle| 
\begin{array}{c}
a : b ; c ' \\
\end{array} \right] = \frac{\Gamma(c) \Gamma(a - b)}{\Gamma(a) \Gamma(c - b)} F_{2:0}^{0:1,0} \left[ a - b ; b , c ' ; x, y \right] . \quad (3.17)
\]

7. Under (2.5) it is easy to get the autotransformation formula for the Kampé de Fériet hypergeometric series:

\[ F_{1:0}^{0:1,2} \left[ \begin{array}{c}
\frac{c : a, b ; c }{c : - a', c '}
\end{array} \middle| 
\begin{array}{c}
x, y
\end{array} \right] = (1 - x)^{-a} F_{1:0}^{0:1,2} \left[ \begin{array}{c}
\frac{c : a - a', d '}{c : - a - b', c '}
\end{array} \middle| 
\begin{array}{c}
x, y
\end{array} \right] . \quad (3.18)
\]

4. SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

The series

\[ \sum_{m,n=0}^{\infty} A(m,n)x^m y^n , \]

where

\[ \frac{A(m + 1, n)}{A(m, n)} = \frac{F(m, n)}{F'(m, n)} \frac{A(m, n + 1)}{A(m, n)} = \frac{G(m, n)}{G'(m, n)} \]

and \( F, F', G, G' \), are polynomials as in (2.11), which satisfy a system of linear differential equations which can be written in terms of differential operators

\[ \delta = x \frac{\partial}{\partial x}, \quad \delta' = y \frac{\partial}{\partial y} \]

as

\[ \begin{bmatrix}
F' (\delta, \delta') x^{-1} - F (\delta, \delta') & 0 \\
G' (\delta, \delta') y^{-1} - G (\delta, \delta') & 0
\end{bmatrix} z = 0 . \quad (4.1) \]

Consider

\[ z = F_{1:0}^{0:1,2} \left[ \begin{array}{c}
\frac{c : a, b ; c }{c : - a', c '}
\end{array} \middle| 
\begin{array}{c}
x, y
\end{array} \right] = \sum_{m,n=0}^{\infty} A(m,n)x^m y^n , \]

where

\[ A(m,n) = \frac{(a)m(b)n(a')}{(c)m+n(c')n} m! n! . \]

By virtue of (2.11), we obtain

\[ F(m, n) = (a + m)(b + m), \quad F'(m, n) = (c + m + n)(m + 1) ; \quad (4.2) \]
$G(m,n) = a' + n, \quad G'(m,n) = (c + m + n)(c' + n)(n + 1). \tag{4.3}$

Substituting (4.2) and (4.3) into the system (4.1), we get

$$x(1 - x)z_{xx} + yz_{xy} + [c - (a + b + 1)x]z_x - abz = 0, \tag{4.4}$$

$$y^2z_{yyy} + xy z_{xyy} + (c + c' + 1 - y)yz_{yy} + c'xz_{xy} + (c' - a' + b + 1)y] z_y - a'b'z = 0, \tag{4.5}$$

Similarly, we have

$$x(1 - x)z_{xx} + (1 - x)y z_{xy} + [c - (a + b + 1)x]z_x - byz_y - abz = 0, \tag{4.6}$$

$$y^2z_{yyy} + xy z_{xyy} + (c + c' + 1 - y)yz_{yy} + (c' - y)xz_{xy} - d'z_x \quad \left\{ F_{1:1:1}^{1:0:1}, \right. \tag{4.7}$$

$$x(1 - x)z_{xx} - y z_{xy} + [c - (a + b + 1)x]z_x - byz_y - abz = 0, \tag{4.8}$$

$$y^2z_{yyy} + (c' + d' + 1 - y)yz_{yy} - yz_{xy} - d'x_x + [c' + d' + b + 1 - y] z_y - a'd'z = 0, \tag{4.9}$$

5. APPLICATION TO SOLVING BOUNDARY VALUE PROBLEMS

Consider the following degenerating hyperbolic equation of the second kind with spectral parameter

$$L(u) = U_{xx} - y^m U_{yy} - \lambda^2 U = 0 \quad (0 < m < 1) \tag{5.1}$$

in a finite simply connected domain $D$, bounded by characteristics $AB : \xi = 0, BC : \eta = 1$ and $AB : y = 0$ of equation (5.1) for $y \geq 0$, where

$$\frac{\xi}{\eta} = x \mp \frac{2}{2 - m} y^{(2-m)/2}. \tag{5.2}$$

Here $\lambda = \lambda_1$ or $\lambda = \lambda_2$, where $\lambda_1$ is a real number.

**CAUCHY PROBLEM.** Find a function $U(x,y) \in C^2(D) \cap C(\bar{D})$ satisfying equation (5.1) and the following initial-value conditions

$$U(x,0) = \tau(x), \quad 0 \leq x \leq 1; \tag{5.3}$$

$$U_y(x,0) = \nu(x), \quad 0 < x < 1, \tag{5.4}$$

where $\tau(x)$ and $\nu(x)$ are sufficiently smooth given functions and $\tau(x) \in C[0,1], \nu(x) \in C^1[0,1]$. In the characteristic coordinates (5.2) at $\xi \geq \eta$ the equation (5.1) goes over into the Euler–Poisson–Darboux type equation

$$u_{\eta} - \frac{\beta}{\eta - \xi} (u_{\eta} - u_{\xi}) - \frac{\lambda^2}{4} u = 0. \tag{5.5}$$

Under the change (5.2), the domain $D$ is mapped into a triangle $\Delta$ in the plane $\xi \Omega \eta$ with sides $\xi = 0, \quad 0 \leq \eta \leq 1; \quad \eta = 0 \leq \xi \leq 1; \quad \xi = \xi$ whose vertices are at the points $A(0,0), B_1 (1,1)$ and $C_1 (0,1)$, and the conditions (5.3) and (5.4) take the forms

$$u(\xi, \xi) = \tau(\xi), \quad 0 \leq \xi \leq 1; \tag{5.6}$$

$$\lim_{\eta \to \xi} \left( \frac{\eta - \xi}{2(1 - 2\beta)} \right)^{2\beta} \left( \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \right) = \nu(\xi), \quad 0 < \xi < 1, \tag{5.7}$$
where

\[ u(\xi, \eta) = U \left[ \frac{\eta + \xi}{2}, \left( \frac{\eta - \xi}{2(1 - 2\beta)} \right)^{1 + 2\beta} \right] : \beta = \frac{m}{2(m - 2)}, -1 < 2\beta < 0. \]

Let’s solve the Cauchy problem. The Riemann function for the equation (5.5) is known [6, 11]

\[ R(\xi, \eta; \xi_0, \eta_0) = (\eta - \xi)^\beta (\eta_0 - \xi_0)^\beta \Xi_2(\beta, 1 - \beta; 1; \omega; \rho), \quad (5.8) \]

where \( \Xi_2 \) is Humbert hypergeometric series, defined in (2.12), and

\[ \omega = \frac{(\eta - \eta_0)(\xi - \xi_0)}{(\eta - \xi)(\eta_0 - \xi_0)}, \quad \rho = -\frac{1}{4} l^2 (\eta_0 - \eta)(\xi - \xi_0). \]

Using the formula (3.13), the Riemann function (5.8) can be rewritten in a form convenient for further research

\[ R(\xi, \eta; \xi_0, \eta_0) = \left( \frac{\eta - \xi}{\eta_0 - \xi_0} \right)^\beta (1 - \theta)^\beta F[1; 1; 0; 0; 0] \left( \beta, \beta; 1; \beta; \theta, \rho \right), \quad (5.9) \]

where

\[ \theta = \frac{\omega}{\omega - 1} = \frac{(\eta_0 - \eta)(\xi - \xi_0)}{(\eta_0 - \xi)(\eta - \xi_0)}. \]

Applying the Riemann method to the domain \( \Delta \), bounded by line segments \( MQ : \xi = \xi_0, MP : \eta = \eta_0 \) and \( P_1Q_2 : \eta = \xi + \epsilon, \epsilon > 0 \), we obtain

\[ u(\xi_0, \eta_0) = \frac{1}{2}(uR)_{\xi_0} + \frac{1}{2}(uR)_{\eta_0} + J_{1\xi}(\xi_0, \eta_0) + J_{2\xi}(\xi_0, \eta_0), \quad (5.10) \]

where

\[ J_{1\xi}(\xi_0, \eta_0) = \frac{1}{2} \int_{\xi_0}^{\xi_0 + \epsilon} \left[ R_{\xi} - R_{\eta} + \frac{4\beta}{\eta - \xi} R \right] u |_{\eta = \xi + \epsilon} d\xi, \]

\[ J_{2\xi}(\xi_0, \eta_0) = -\frac{1}{2} \int_{\xi_0}^{\xi_0 + \epsilon} R \left( \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \right) |_{\eta = \xi + \epsilon} d\xi. \]

It follows from the formula (5.9) that the function \( R \) on the line \( \eta = \xi \) tends to infinity of order \(-2\beta \) \(( \theta < -2\beta < 1) \). Therefore, unlike the degenerate hyperbolic equations of the first kind [16], the terms in (5.10) containing \( u(\eta_0 - \epsilon, \eta_0) \) and \( u(\xi_0, \xi_0 + \epsilon) \) do not disappear at \( \epsilon \to 0 \) (and even more: they are infinitely large). Moreover, \( J_{1\xi} \) becomes a divergent integral. Therefore, we first transform the integrand in \( J_{1\xi} \). Under differentiation formulas (3.14) and (3.15) it is easy to get

\[ R_{\xi} - R_{\eta} + \frac{4\beta}{\eta - \xi} R = 2\beta(\eta - \xi_0)^{-\beta}(\eta - \xi)^{-1 + 2\beta} \Xi_2(\beta, 1 - \beta; 1; \omega, \rho) \]

\[ + \frac{l^2}{4} (\eta_0 - \xi_0)^{-\beta}(\eta - \xi)^{\beta}(\eta - \xi_0 + \xi_0) \Xi_2(\beta, 1 - \beta; 2; \omega, \rho) \]

\[ - \beta(1 - \beta)(1 - \theta)^{-1 + 2\beta} (\eta - \xi) (\eta_0 - \xi_0) \frac{\partial \theta}{\partial \xi} - \frac{\partial \theta}{\partial \eta} F[1; 1; 0; 0; 0] \left( \beta, \beta; 1; \beta; \theta, \rho \right). \]

Using the identity

\[ \frac{\partial \theta}{\partial \xi} - \frac{\partial \theta}{\partial \eta} = \frac{\eta_0 - \xi_0}{(\eta_0 - \xi)(\eta_0 - \xi_0)} \left[ 2 - \frac{\eta_0 - \xi_0}{(\eta_0 - \xi)(\eta_0 - \xi_0)} - \frac{(\eta - \xi)^2}{(\eta_0 - \xi)(\eta_0 - \xi_0)} \right] \]

we single out terms that have a finite limit:

\[ R_{\xi} - R_{\eta} + \frac{4\beta}{\eta - \xi} R = i_1 + i_2, \]
\[ i_1 = \frac{\lambda^2}{4}(\eta_0 - \xi_0)^{-\beta}(\eta - \xi)^{1+\beta} \Xi_2(\beta, 1 - \beta; 2; \omega, \rho) + \frac{\beta(1 - \beta)(\eta - \xi)^{1+\beta}}{(\eta_0 - \xi_0)^{1+\beta}(\eta - \xi_0)^{1+\beta}} F_1^{(1, 0)} \left[ \frac{\beta}{2} : 1 + \beta; -; \theta, \rho \right] \]

\[ i_2 = -\frac{\lambda^2}{4}(\eta_0 - \xi_0)^{-\beta}(\eta - \xi)^{1+\beta} \Xi_2(\beta, 1 - \beta; 2; \omega, \rho) + \beta(\eta_0 - \xi_0)^{1-\beta}(\eta - \xi)^{1+\beta}(\eta - \xi_0)^{-1+\beta}. (i_3 + i_4), \]

\[ i_3 = 2F_1^{(1, 0)} \left[ 1 - \beta : 1 - \beta; -; \theta, \rho \right] - 2(1 - \beta)F_1^{(1, 0)} \left[ 1 - \beta : 2 - \beta; -; \theta, \rho \right], \]

\[ i_4 = (1 - \beta)(1 - \theta)F_1^{(1, 0)} \left[ 2 - \beta; 2 - 1 - \beta; -; \theta, \rho \right]. \]

To simplify the expression \(i_3\), we use the formula (3.7):

\[ i_3 = 2\beta F_1^{(1, 0)} \left[ 1 - \beta : 1 - \beta; 1 + \beta; \theta, \rho \right], \]

Applying the formula (3.8) to the sum: \(i_{34} = i_3 + i_4\), we get

\[ i_{34} = (1 + \beta)F_1^{(1, 0)} \left[ 2 - \beta : 2 - 1 - \beta; -; \theta, \rho \right] + \frac{1}{2}(1 + \theta)\rho F_3^{(1, 0)} \left[ 2 - \beta : 1 - \beta; -; \theta, \rho \right]. \]

Applying the formula (3.9) to the first term in \(i_{34}\), we have

\[ i_{34} = (1 + \beta)F_1^{(1, 0)} \left[ 2 - \beta : 2 - 1 - \beta; -; \theta, \rho \right] + \rho F_1^{(1, 0)} \left[ 2 - \beta : 1 - \beta; -; \theta, \rho \right]. \]

Substituting the last expression of \(i_{34}\) into \(i_2\), taking into account the formula (3.13), after bringing similar ones we have

\[ i_2 = i_{21} + i_{22}, \]

where

\[ i_{21} = -\frac{\lambda^2}{4}(\eta_0 - \xi_0)^{-1+\beta}(\eta_0 - \xi)^{1+\beta}(\eta - \xi_0)^{1+\beta} \Xi_2(\beta, 1 - \beta; 2; \omega, \rho), \]

\[ i_{211} = (1 - \theta)F_1^{(1, 0)} \left[ 2 - \beta : 2 - 1 - \beta; -; \theta, \rho \right] + \beta \rho F_1^{(1, 0)} \left[ 2 - \beta : 1 - \beta; -; \theta, \rho \right], \]

\[ i_{22} = \frac{\beta(1 + \beta)(\eta_0 - \xi_0)^{1-\beta}}{(\eta_0 - \xi)^{1-\beta}(\eta - \xi_0)^{1+\beta}} F_1^{(1, 0)} \left[ 2 - \beta ; 1 - \beta; -; \theta, \rho \right], \]

\[ i_{23} = \frac{\beta^2}{4}(\eta_0 - \xi_0)^{-1+\beta}(\eta - \xi)^{1+\beta} \Xi_2(\beta, 1 - \beta; 2; \omega, \rho) + \rho F_3^{(1, 0)} \left[ 2 - \beta : 1 - \beta; 3; \theta, \rho \right]. \]

The formulas (3.10) and (3.17) allow to transform \(i_{23}\) into an expression that has a finite limit:

\[ i_{23} = \frac{\lambda^2}{4}(\eta_0 - \xi_0)^{-1+\beta}(\eta - \xi)^{1+\beta} \Xi_2(\beta, 1 - \beta; 2; \omega, \rho). \]

Now let’s transform \(i_{22}\). To this aim we introduce auxiliary functions

\[ \varphi(\xi, \eta) = \frac{(\eta - \xi)^{\beta}(\eta - \xi_0)^{\beta}}{(\eta_0 - \xi_0)^{\beta}} F_1^{(1, 0)} \left[ \frac{\beta}{2} : 1 + \beta; 3; \theta, \rho \right], \]

\[ \psi(\xi, \eta) = \frac{\eta(\eta + \xi)(\eta + \xi_0)(\eta - \xi_0)}{\eta_0(\eta + \xi - \xi_0)^2}. \]

Using the expression of the total derivative \(\frac{d}{d\eta} + \frac{d}{d\xi}\) \(\varphi(\xi, \eta)\) on the line \(\eta = \xi\), we can represent \(i_{22}\) as a following way:

\[ i_{22} = i_{33} + \frac{1}{2} \psi(\xi, \eta) d\varphi(\xi, \eta). \]
where
\[
i_{35} = \frac{\beta(1 + \beta)}{2} \left\{ \frac{8\xi(\eta_0 - \xi_0)}{[\eta(\eta_0 - \xi) + \xi(\eta - \xi_0)]^2} + \psi(\xi, \eta) \frac{(\eta_0 - \xi_0 + \xi - \eta)(\eta - \xi)}{(\eta + \xi)(\eta_0 - \xi)(\eta_0 - \xi_0)} \right\} \times
\]
\[
\times \left( \frac{\eta_0 - \xi_0 - \eta}{(\eta_0 - \xi_0)^{3/2}} \right)_{\Gamma(1 + 1.0)} F\left[ 1.0; 2.0; -1; \beta, \rho \right] + \psi(\xi, \eta) \frac{\lambda^2}{8} \frac{(\eta_0 - \xi_0 - \eta)(\eta_0 - \xi_0 - \eta)}{(\eta_0 - \xi)(\eta_0 - \xi_0)^{3/2}} F\left[ 1.0; 2.0; -1; \beta, \rho \right].
\]
(5.13)

So we get
\[
R_\xi - R_\eta + \frac{4\beta}{\eta - \xi} R = A(\xi_0, \eta_0; \xi, \eta) + \frac{1}{2} \psi(\xi, \eta) \eta_\xi(\xi, \eta),
\]
where a function \(A(\xi_0, \eta_0; \xi, \eta)\) is determined from the formulas (5.11), (5.12), (5.13) and has a finite limit at \(\varepsilon \to 0\). Substituting the result in (5.10), we find that
\[
u(\xi_0, \eta_0) = \frac{1}{2} (uR)_Q + \frac{1}{2} (uR)_P + \frac{1}{4} (u \cdot \varphi \cdot \psi)_Q
\]
\[-\frac{1}{2} \int_0^1 R (u_\xi - u_\eta) d\xi + \frac{1}{2} \int_0^1 u \cdot A d\xi - \frac{1}{2} \int_0^1 \varphi d(u\psi).
\]
(5.14)

It is easy to show that
\[
\lim_{\xi \to 0} \left( (uR)_Q + (uR)_P + \frac{1}{2} (u \cdot \varphi \cdot \psi)_Q \right) = 0.
\]

Passing to the limit in (5.14) at \(\varepsilon \to 0\) and using the formulas (3.11), (3.12), (3.16), we obtain the formula for solving the Cauchy problem for an equation of the Euler–Poisson–Darboux type (5.5) with data (5.6) and (5.7) in the form
\[
u(\xi_0, \eta_0) = \kappa_1 \left( \lambda^2 / 2 \right)^{-\beta} (\eta_0 - \xi_0)^{-2\beta - 1} \int_{\xi_0}^{\eta_0} \left[ z^{\beta} f_\beta(z) - \frac{1}{1 + 2\beta} z^{1 + \beta} f_{1 + \beta}(z) \right] r(\xi) d\xi
\]
\[-\frac{\kappa_1}{2(1 + 2\beta)} \left( \lambda^2 / 2 \right)^{-\beta} (\eta_0 - \xi_0)^{-2\beta - 1} \int_{\xi_0}^{\eta_0} (\eta_0 + \xi_0 - 2\xi) z^{\beta} f_\beta(z) r(\xi) d\xi
\]
\[-(2(1 - 2\beta))^{2\beta - 1} \kappa_2 \left( \lambda^2 / 2 \right)^{\beta} \int_{\xi_0}^{\eta_0} z^{\beta} f_{-\beta}(z) r(\xi) d\xi,
\]
(5.15)

where
\[
\kappa_1 = \frac{\Gamma(2 + 2\beta)}{\Gamma(1 + \beta)}, \quad \kappa_2 = \frac{\Gamma(2 - 2\beta)}{\Gamma(1 - \beta)}, \quad z = \lambda \sqrt{(\eta_0 - \xi)(\xi - \xi_0)};
\]
\(J_\lambda(z)\) is a Bessel function, defined in (2.9). Passing in (5.15) to the old variables \(x, y\), after performing some transformations, we obtain the solution of the Cauchy problem for the equation (5.1) in the half-plane \(y > 0\) with initial data (5.3) and (5.4) as follows
\[
U(x, y) = \kappa_1 \int_0^1 s^\beta(1 - s)^{\beta - 2\beta} \frac{d}{dr} \left[ r^{1 + 2\beta} f_\beta(r) \right] r(t) ds
\]
\[+ \frac{1}{1 - m} \kappa_1 y^{(2 - m)/2} \int_0^1 s^\beta(1 - s)^{\beta - 2\beta} f_\beta(r) r(t) ds + \kappa_2 y \int_0^1 s^{\beta - 2\beta} f_{-\beta}(r) r(t) ds,
\]
(5.16)

where
\[
r = \frac{4\lambda}{2 - m} y^{(2 - m)/2} \sqrt{(1 - s)}; \quad t = x - \frac{2}{2 - m} y^{(2 - m)/2}(1 - 2s);
\]
(5.17)
\[ j_\alpha(r) \] is the Bessel–Clifford function \([25, \text{ p. 731, eq. 37.8}]\) defined by
\[
j_\alpha(r) = \Gamma(\alpha + 1) \left( \frac{r}{2} \right)^\alpha J_\alpha(r) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(1 + \alpha)_n} \left( \frac{r}{2} \right)^{2n},
\]
which satisfies the following ordinary differential equation of the second order
\[
j''_\alpha(r) + \frac{1 + 2\alpha}{r} j'_\alpha(r) + j_\alpha(r) = 0.
\] (5.18)

**Theorem 1.** If \(\tau(x) \in C^3(\hat{J})\) and \(v(x) \in C^2(\hat{J})\), then the function \(U(x, y)\), defined by the formula (5.16), is a twice continuously differentiable, moreover, unique, solution of the Cauchy problem for the equation (5.1) with initial data (5.3) and (5.4) in \(D\).

**Proof.** The uniqueness of the solution of the stated problem follows from the method of obtaining the formula (5.16). To prove the remaining assertions of the theorem, we rewrite the solution \(U(x, y)\), defined by (5.16), in the form
\[
U(x, y) = \kappa_1 U_1(x, y) + \kappa_2 U_2(x, y),
\]
where
\[
U_1(x, y) = \int_0^1 s^\beta (1 - s)^\beta P(y, s; \lambda) r(t) ds + \int_0^1 s^\beta (1 - s)^\beta Q(y, s; \lambda) r'(t) ds,
\]
\[
U_2(x, y) = y \int_0^1 s^\beta (1 - s)^\beta J_{\beta}(r) r(t) ds;
\]
\[
P(y, s; \lambda) = \tilde{J}_\beta(r) + r \frac{1}{1 + 2\beta} \tilde{J}_{\beta}'(r),
\]
\[
Q(y, s; \lambda) = \frac{1}{1 - m} \sqrt{\frac{2 - m}{2}} (2s - 1) \tilde{J}_\beta(r);
\]
the variables \(r\) and \(t\) are defined in (5.17). It is easy to show that
\[
U_1(x, 0) = \tau(x), \quad U_2(x, 0) = 0, \quad \lim_{y \to +\infty} \frac{\partial U_1}{\partial y} = 0, \quad \lim_{y \to +\infty} \frac{\partial U_2}{\partial y} = v(x).
\]
This implies the satisfaction of the initial conditions (5.3) and (5.4). Let’s show that \(L(U_1) = 0\). Substituting the function \(U_1\) and its second derivatives with respect to \(x\) and \(y\) into the equation (5.1), we get
\[
L(U_1) = \sum_{i=1}^{5} R_i(x, y; \lambda),
\]
where
\[
R_1(x, y; \lambda) = \int_0^1 s^\beta (1 - s)^\beta \left( 1 - y^m t_y^3 \right) Q(y, s; \lambda) \tau'''(t) ds,
\]
\[
R_2(x, y; \lambda) = \int_0^1 s^\beta (1 - s)^\beta (1 - y^m t_y^3) P(y, s; \lambda) \tau''(t) ds,
\]
\[
R_3(x, y; \lambda) = -y^m \int_0^1 s^\beta (1 - s)^\beta (2Q_y \cdot t_y + Q \cdot t_{yy}) \tau''(t) ds,
\]
\[
R_4(x, y; \lambda) = - \int_0^1 s^\beta (1 - s)^\beta \left[ (Q_{yy} + P \cdot t_{yy} + 2P_y \cdot t_y) y^m + \lambda^2 Q \right] \tau'(t) ds,
\]
\[
R_5(x, y; \lambda) = - \int_0^1 s^\beta (1 - s)^\beta \left[ y^m P_{yy} + \lambda^2 P \right] \tau(t) ds.
\]
Since \( t_y = -y^{-\frac{3}{2}}(1 - 2s) \) and, therefore, \( 1 - y^m t_y^2 = 4x(1 - s) \), then

\[
R_1(x, y; \lambda) = -\frac{4}{1-m} y^\frac{3}{2m} \int_0^1 s^{\beta+1} (1-s)^{\beta+1} (1-2s) J_\beta(r) r''(t) \, ds.
\]

Hence, integrating by parts, we get

\[
R_1(x, y; \lambda) = \frac{4 - 3m}{2(1-m)} \int_0^1 s^{\beta}(1-s)^{\beta}(1-2s)^2 J_\beta(r) r''(t) \, ds - \frac{2(2 - m)}{1-m} \int_0^1 s^{\beta+1}(1-s)^{\beta+1} J_\beta(r) r''(t) \, ds
\]

\[+rac{2\lambda}{1-m} y^\frac{3}{2m} \int_0^1 s^{\beta+\frac{1}{2}}(1-s)^{\beta+\frac{1}{2}} (1-2s)^2 J_\beta(r) r''(t) \, ds,
\]

while we took into account that

\[
\frac{\partial t}{\partial s} = \frac{4}{2-m} y^{-\frac{3}{2m}}, \quad \frac{\partial r}{\partial s} = \frac{2\lambda}{2-m} \sqrt{1-s} y^{-\frac{3}{2m}}.
\]

Consider a sum \( R_{123} = R_1 + R_2 + R_3 \) which can be reduced to a form convenient for further research:

\[
R_{123} = -\frac{2m}{1-m} \int_0^1 s^{\beta+1}(1-s)^{\beta+1} J_\beta(r) r''(t) \, ds - \frac{2\lambda}{1-m} y^\frac{3}{2m} \int_0^1 s^{\beta+\frac{1}{2}}(1-s)^{\beta+\frac{1}{2}} (1-2s)^2 J_\beta(r) r''(t) \, ds
\]

\[+rac{8\lambda}{1-m} y^\frac{3}{2m} \int_0^1 s^{\beta+\frac{1}{2}}(1-s)^{\beta+\frac{1}{2}} J_\beta(r) r''(t) \, ds.
\]

Now integrating by parts in all three integrals and introducing the notation \( R_{1234} = R_{123} + R_4 \), we find

\[
R_{1234} = \frac{(2 - 3m)\lambda}{1-m} \int_0^1 s^{\beta+\frac{1}{2}}(1-s)^{\beta+\frac{1}{2}} (1-2s) J_\beta(r) r'(t) \, ds
\]

\[+\frac{\lambda^2}{1-m} y^{-\frac{3}{2m}} \int_0^1 s^{\beta}(1-s)^{\beta}(1-2s) J_\beta(r) r'(t) \, ds
\]

\[+\frac{\lambda^2}{1-m} y^\frac{3}{2m} \int_0^1 s^{\beta+\frac{1}{2}}(1-s)^{\beta+\frac{1}{2}} J_\beta'(r) \left[ J_\beta''(r) + \frac{1 + 2\beta}{r} J_\beta'(r) \right] r'(t) \, ds
\]

\[+\frac{8\lambda^2}{1-m} y^\frac{3}{2m} \int_0^1 s^{\beta+\frac{1}{2}}(1-s)^{\beta+\frac{1}{2}} J_\beta''(r) r'(t) \, ds.
\]

According to the relation (5.18), the integrand in square brackets equals to \(-J_\beta'(r)\), then, taking into account equality \( 1 - (1 - 2s)^2 = 4x(1 - s) \), the sum \( R_{1234} \) takes a more simplified form

\[
R_{1234} = \frac{8\lambda^2}{1-m} y^{-\frac{3}{2m}} \int_0^1 s^{\beta+1}(1-s)^{\beta+1} (1-2s) \left[ J_\beta''(r) + \frac{1 + 4\beta}{2r} J_\beta'(r) + \frac{1}{2} J_\beta(r) \right] r'(t) \, ds.
\]

The relation (5.18) helps to simplify the form of the last integral:

\[
R_{1234} = -\frac{4\lambda^2}{1-m} y^\frac{3}{2m} \int_0^1 s^{\beta+1}(1-s)^{\beta+1} (1-2s) \left[ \frac{1}{r} J_\beta'(r) + J_\beta'(r) \right] r'(t) \, ds.
\]
Now consider $R_5$:

$$R_5 = \frac{2 - m}{2(1 - m)} \lambda^2 \int_0^1 s^\beta (1 - s)^\beta \left[ r J_\beta'(r) + (1 + 2\beta) J_\beta(r) \right] \tau(t) \, ds$$

$$- \frac{2(2 - m)}{1 - m} \lambda^2 \int_0^1 s^{\beta+1} (1 - s)^{\beta+1} \left[ r J''_\beta(r) + (3 + 4\beta) J'_\beta(r) + \frac{8\beta(1 + \beta)}{r} J_\beta(r) \right] \tau(t) \, ds.$$  

Integrating by parts in $R_{1,2,3,4}$, using the relation (5.18) in $R_5$, and considering everything $R_1$–$R_5$ together, we obtain

$$L(U_1) = \lambda^2 \int_0^1 s^\beta (1 - s)^\beta \left[ J''_\beta(r) + \frac{1 + 2\beta}{r} J'_\beta(r) + J_\beta(r) \right] \tau(t) \, ds.$$  

By virtue of (5.18), we obtain $L(U_1) = 0$. Similarly, we get $L(U_2) = 0$. Theorem 1 is proved. □

**Conflicts of interest.**

The authors declare no conflicts of interest.

**REFERENCES**


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