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A CLASSIFIER FOR SIMPLE ISOLATED COMPLETE INTERSECTION SINGULARITIES

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Abstract

M. Giusti's classification of the simple complete intersection singularities is characterized in terms of invariants. This is a basis for the implementation of a classifier in the computer algebra system SINGULAR.

1. Introduction

We report about a classifier for simple isolated complete intersection singularities in the computer algebra system SINGULAR [2], [5]. In [1] V. Arnold classified the simple hypersurface singularities, the famous A-D-E-singularities. M. Giusti gave a classification of simple complete intersection singularities which are not hypersurfaces [4]. The singularities in Giusti's classification are given by normal forms.

The aim of this paper is to describe Giusti's classification in terms of certain invariants. Based on this description we are not forced to compute the normal form for finding the type of the singularity. This is usually more complicated and may be space and time consuming. For the classification of hypersurface singularities we refer to the SINGULAR library classify.lib [2].

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2. Simple complete intersection singularities in dimension 0

Let $\big(V\big(\langle f,g\rangle\big),0\big) \subseteqq (\mathbb{C}^2,0)$ be the germ of a complete intersection singularity. M. Giusti proved in [4] that $\big(V\big(\langle f,g\rangle\big),0\big)$ is simple iff it is isomorphic to a complete intersection in the following list.

Type	Normal Form	Milnor Number
$F^{n,p}_{n+p-1}$, $n,p \geqq 2$	$(xy, x^n + y^p)$	$n+p-1$
G_5	(x^2, y^3)	5
G_7	(x^2, y^4)	7
$H_{n+3}, n \geqq 4$	$(x^2 + y^n, xy^2)$	$n+3$
$I_{2p-1}, p \geqq 4$	$(x^2 + y^3, y^p)$	$2p-1$
$I_{2q+2}, q \geqq 2$	$(x^2 + y^3, xy^q)$	$2q+2$

We want to give a description of the type of the singularity without producing the normal form. Given $\big(V\big(\langle f,g\rangle\big),0\big) \subseteqq (\mathbb{C}^2,0)$ we consider the ideal $I = \langle f,g\rangle \subseteqq \mathbb{C}[[x,y]]$. We fix the local degree reverse lexicographical ordering $>$ on $\mathbb{C}[[x,y]]$ with $y > x$. We will denote by $L(I)$ the leading ideal of I with respect to this ordering and by $LM(f)$ the leading monomial of f, $f \in \mathbb{C}[[x,y]]$.

Proposition 2.1. *Let $I = \langle f,g\rangle \subseteqq \mathbb{C}[[x,y]]$ be an $\mathfrak{m}$-primary ideal and $d = \dim_{\mathbb{C}}\big(\mathbb{C}[[x,y]]/I\big)$, $\mathfrak{m} = \langle x,y\rangle$.*

(1) *If $\dim_{\mathbb{C}}(I+\mathfrak{m}^3/\mathfrak{m}^3) = 2$ then $\big(V(I),0\big)$ is of type $F^{2,d-1}_{d-1}$.*

(2) *If $\dim_{\mathbb{C}}(I+\mathfrak{m}^3/\mathfrak{m}^3) = 1$ and a generator of $I+\mathfrak{m}^3/\mathfrak{m}^3$ is reduced in $\mathbb{C}[[x,y]]/\mathfrak{m}^3$, let $\phi : \mathbb{C}[[x,y]] \to \mathbb{C}[[x,y]]$ be an automorphism such that $\phi(I) = \langle xy+a, b\rangle, a,b \in \mathfrak{m}^3$. If $L\big(\phi(I)\big) = \langle xy, x^p, y^q\rangle$ then $\big(V(I),0\big)$ is of type $F^{p,q-1}_{d-1}$.*

(3) *If $\dim_{\mathbb{C}}(I+\mathfrak{m}^3/\mathfrak{m}^3) = 1$ and a generator of $I+\mathfrak{m}^3/\mathfrak{m}^3$ is a square let $\psi : \mathbb{C}[[x,y]] \to \mathbb{C}[[x,y]]$ be an automorphism such that $\psi(I) = \langle x^2+g, h\rangle$, $g,h \in \mathfrak{m}^3$ and assume that x^2 does not divide the monomials of h of degree $\leqq d$.*
Let c be the Milnor number of $(V(x^2+g),0) \subseteqq (\mathbb{C}^2,0)$.
If $d = 6$ then $\big(V(I),0\big)$ is of type G_5.
If $d \geqq 7$ and $c = 2$ then $\big(V(I),0\big)$ is of type I_{d-1}.
If $LM(h) = y^4$, $c \geqq 3$ then $\big(V(I),0\big)$ is of type G_7.
If $LM(h) = xy^2$, $c \geqq 3$ then $\big(V(I),0\big)$ is of type H_{d-1}.

(4) *In all other cases $\big(V(I),0\big)$ is not simple.*

Proof. Since a minimal standard basis of $\langle xy+a, b\rangle$, $a, b \in \mathfrak{m}^3$ is $\{xy+a, x^p+f, y^q+g\}$ for suitable $f, g \in \mathbb{C}[[x,y]]$, $f \in \mathfrak{m}^{p+1}$, $g \in \mathfrak{m}^{q+1}$. (1) and (2) follow directly from the classification of Giusti. To prove (3) note that a minimal standard bases of $\langle x^2+g, h\rangle$, $g, h \in \mathfrak{m}^3$, is $\{x^2+g, h\}$, monomials of h of degree $\leqq d$ are not divisible by x^2. If $LM(h) = y^p$ for some p (in this case $d = 2p$) then obviously $\{x^2+g, h\}$ is a minimal standard basis. If $LM(h) = xy^q$ for some q then a minimal standard basis is $\{x^2+g, h, y^p+e\}$ for some $p \geqq q+3$ and a suitable $e \in \mathfrak{m}^{q+3}$ (in this case $d = p+q$). This is a consequence of the fact that $y > x$, the monomials of h up to degree d are not divisible by x^2 and therefore $spoly(x^2+g, h) \in \mathfrak{m}^{q+3}$. If $d = 6$ then $LM(h) = y^3$ since in the other case $d \geqq 7$. It follows from Giusti's classification that $\langle x^2+g, y^3+h\rangle$, $g, h \in \mathfrak{m}^3$, $LM(h) < y^3$ is of Type G_5.

If $c = 2$ and $d \geqq 7$ then we may assume that $g = y^3$. If $L(I) = \langle x^2, y^p\rangle$, $p \geqq 4$ (in case $p = 3$ we have $d = 6$) then we obtain from Giusti's classification that $\big(V(I), 0\big)$ is of type I_{2p-1}.

If $L(I) = \langle x^2, xy^q, y^p\rangle$, for some $p \geqq q+3$ we obtain $p = q+3$ since $LM\big(\text{spoly}(x^2+y^3, h)\big) = y^{q+3}$. Again Giusti's classification gives I_{2q+2}. Now we may assume $c \geqq 3$ (the case $c = \infty$ included). If $LM(h) = y^p$ and $d = 8$ we obtain $p = 4$. We may assume $g = y^{c+1}$, since $c \geqq 3$ we can change x^2+g adding a suitable multiple of h to increase c. If $LM(h) = xy^2$ then $\big(V(I), 0\big)$ is of type H_{d-1}. This we obtain analyzing the proof of Giusti's classification. □

We summarize this case in Algorithm 1.[1]

[1] The corresponding procedures are implemented in Singular in the library `classifyci.lib`.

Algorithm 1 0-dimensional simple complete intersections

Input: $I = \langle f, g\rangle \in \langle x, y\rangle^2\mathbb{C}[[x, y]]$
Output: the type of the singularity $\big(V(I), 0\big)$

compute $d = \dim_{\mathbb{C}} \big(\mathbb{C}[[x, y]]/I\big)$;
compute $s = \dim_{\mathbb{C}} \big(I + \langle x, y\rangle^3/\langle x, y\rangle^3\big)$;
if $s = 2$ **then**
  return $(F^{2,d-1}_{d-1})$;
if $s = 1$ **then**
  choose a homogenous generator h of $I + \langle x, y\rangle^3/\langle x, y\rangle^3$;
  if h splits into two factors **then**
    choose an automorphism ϕ such that $\phi(h) = xy$. Compute generators of the leading ideal $L\big(\phi(I)\big) = \langle xy, x^p, y^q\rangle$
    return $(F^{p,q-1}_{d-1})$;
  else
    if $d = 6$ **then**
      return (G_5);
    choose an automorphism ϕ such that $\phi(h) = x^2$ choose $g, h \in \langle x, y\rangle^3$ such that x^2 does not divide the monomials of h of degree $\leqq d$ and $\phi(I) = \langle x^2 + g, h\rangle$.
    Compute c the Milnor number of $x^2 + g$.
    if $c = 2$ **then**
      return (I_{d-1});
    else
      if $LM(h) = y^4$ **then**
        return (G_7);
      if $LM(h) = xy^2$ **then**
        return (H_{d-1});
return $(\textit{not simple})$;

3. Simple complete intersection singularities in dimension 1

Let $\big(V\big(\langle f, g\rangle\big), 0\big) \subseteqq (\mathbb{C}^3, 0)$ be the germ of a complete intersection singularity. Assume it is not a hypersurface singularity. M. Giusti proved in [4] that $\big(V\big(\langle f, g\rangle\big), 0\big)$ is simple if and only if it is isomorphic to a complete intersection in the following list.

Type	Normal form	Milnor Number
$S_{n+3},\ n \geqq 2$	$(x^2+y^2+z^n, yz)$	$n+3$
T_7	$(x^2+y^3+z^3, yz)$	7
T_8	$(x^2+y^3+z^4, yz)$	8
T_9	$(x^2+y^3+z^5, yz)$	9
U_7	$(x^2+yz, xy+z^3)$	7
U_8	(x^2+yz+z^3, xy)	8
U_9	$(x^2+yz, xy+z^4)$	9
W_8	(x^2+z^3, y^2+xz)	8
W_9	(x^2+yz^2, y^2+xz)	9
Z_9	(x^2+z^3, y^2+z^3)	9
Z_{10}	(x^2+yz^2, y^2+z^3)	10

Similarly to section 2 we want to give a description of the type of a singularity without producing the normal form. Giusti's classification is based on the classification of the 2-*jet* I_2 of $\langle f, g\rangle$. The 2-*jet* is a homogenous ideal generated by 2 polynomials of degree 2. Let $\bigcap_{i=1}^s Q_i$ be the irredundant primary decomposition in $\mathbb{C}[x,y,z]$. According to Giusti's classification we obtain simple singularities only in the following cases.

Type	Characterization	Normal form of I_2
S_5	$s=4$, $Q_1,\dots,Q_4$ prime	$(x^2+y^2+z^2, yz)$
S_n	$s=3$, $\text{mult}(Q_1)=\text{mult}(Q_2)=1$, $\text{mult}(Q_3)=2$	(x^2+y^2, yz)
T	$s=2$, $\text{mult}(Q_1)=\text{mult}(Q_2)$	(x^2, yz)
U	$s=2$, $\text{mult}(Q_1)=1$, $\text{mult}(Q_2)=3$	(x^2+yz, xy)
W	$s=1$ and $\sqrt{I_2}^3 \nsubseteq I_2$	(x^2, y^2+xz)
Z	$s=1$ and $\sqrt{I_2}^3 \subseteqq I_2$	(x^2, y^2)

Here the multiplicity is given by the Hilbert polynomial of the corresponding homogeneous ideal.

PROPOSITION 3.1. *Let $I=\langle f,g\rangle \subseteqq \langle x,y,z\rangle^2\mathbb{C}[[x,y,z]]$ defines a complete intersection singularity and I_2 is* 2-jet. *Let $I_2=\bigcap_{i=1}^s Q_i$ be the irredundant primary decomposition. Let μ be the Milnor number of $\mathbb{C}[[x,y,z]]/I$.*

(1) *if $s=4$ then $\big(V(I),0\big)$ is of type S_5.*

(2) *if $s=3$ then $\big(V(I),0\big)$ is of type S_μ.*

(3) *if $s=2$ and* $\text{mult}(Q_1)=\text{mult}(Q_2)=2$ *and*

  (a) *$7 \leqq \mu \leqq 8$ then $\big(V(I),0\big)$ is of type T_μ.*
  or

  (b) *$\mu=9$ and $\big(V(I),0\big)$ has two branches then $\big(V(I),0\big)$ is of type T_9.*

(4) *if* $s = 2$, $7 \leqq \mu \leqq 9$ *and* $\operatorname{mult}(Q_1) \neq \operatorname{mult}(Q_2)$ *then* $\big(V(I), 0\big)$ *is of type* U_μ.

(5) *if* $s = 1$, $8 \leqq \mu \leqq 9$ *and* $\sqrt{I_2}^3 \nsubseteq I_2$ *then* $\big(V(I), 0\big)$ *is of type* W_μ.

(6) *if* $s = 1$, $8 \leqq \mu \leqq 9$ *and* $\sqrt{I_2}^3 \subseteqq I_2$ *then* $\big(V(I), 0\big)$ *is of type* Z_μ.

(7) *In all other cases* $\big(V(I), 0\big)$ *is not simple.*

The following two lemmas are the basis for the proposition.

Lemma 3.2. *With the notations of Proposition* 3.1 *assume that* $s = 2$ *and* $\operatorname{mult}(Q_1) = \operatorname{mult}(Q_2) = 2$. *There is an automorphism* $\phi \in \operatorname{Aut}_{\mathbb{C}}\big(\mathbb{C}[[x, y, z]]\big)$ *such that* $\phi(I) = \langle x^2 + y^a + z^b + g, yz + h\rangle$, $3 \leqq a \leqq b \leqq \infty$, $g \in \langle x, y, z\rangle^{b+1}$, $h \in \langle x, y, z\rangle^3$.

(1) *If* $a = 3$ *and* $b \geqq 6$ *or* $a \geqq 4$ *and* $b \geqq 5$ *then* $\mu\big(\mathbb{C}[[x, y, z]]/I\big) \geqq 10$.

(2) *If* $a = 3$ *and* $b = 5$ *or* $a = b = 4$ *then* $\mu\big(\mathbb{C}[[x, y, z]]/I\big) \geqq 9$.

Proof. We may assume that $I = \langle x^2 + g, yz + h\rangle$ with $g, h \in \langle x, y, z\rangle^3$. Let $g = xg_1 + g_2$, $g_2 \in \mathbb{C}[[y, z]]$, $g_1 \in \langle x, y, z\rangle^2$. Consider

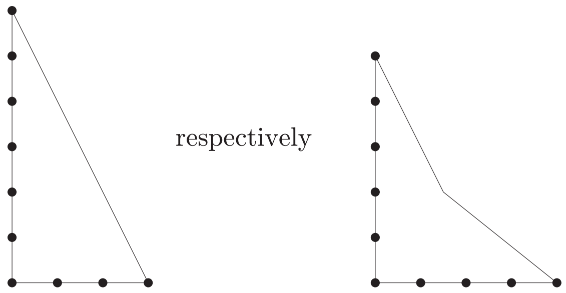
$$\phi \in \operatorname{Aut}_{\mathbb{C}}\big(\mathbb{C}[[x, y, z]]\big)$$

defined by $\phi(x) = x - \frac{1}{2}g_1$, $\phi(y) = y$, $\phi(z) = z$ then $\phi(x^2 + g) = x^2 + g_2 + \frac{1}{4}g_1^2$. Since $g_1 \in \langle x, y, z\rangle^4$ we may iterate this process and assume that $I = \langle x^2 + g, yz + h\rangle$ with $g \in \mathbb{C}[[y, z]]$. Let $a = \operatorname{ord}(g)$. If yz does divide the a-*jet* of g we subtract a suitable multiple of $yz + h$ from $x^2 + g$ and obtain $I = \langle x^2 + \tilde{g}, yz + h\rangle$, $\operatorname{ord}(\tilde{g}) > a$. Using an automorphism as in the beginning we may assume that $\tilde{g} \in \mathbb{C}[[y, z]]$. Repeating this we obtain $I = \langle x^2 + g, yz + h\rangle$, $g \in \mathbb{C}[[y, z]]$. If $g \neq 0$ and $\operatorname{ord}(g) = a$ then we may assume (if necessary exchanging y and z) that $g = y^a + \alpha z^a + g_1$, $g_1 \in \langle y, z\rangle^{a+1}\mathbb{C}[[y, z]]$. If $\alpha = 0$ we continue like this. We obtain finally $I = \langle x^2 + y^a + z^b + g, yz + h\rangle$, $3 \leqq a \leqq b \leqq \infty$ ($a = \infty$, $b = \infty$ included), $\operatorname{ord}(g) > b$, $h \in \langle x, y, z\rangle^3$.

To estimate the Milnor number in case (1) consider the following deformation $I_t = \langle x^2 + y^a + z^b + g, tx + yz + h\rangle$ for small $|t|$ we have

$$\mu\big(\mathbb{C}[[x, y, z]]/I\big) \geqq \mu\big(\mathbb{C}[[x, y, z]]/I_t\big).$$

If $t \neq 0$ I_t defines a plane curve singularity. It is enough to consider the cases $a = 3$, $b = 6$ and $a = 4$, $b = 5$. We obtain as Newton polygon



with corresponding Milnor numbers 10. Now we assume that $a = b = 4$ (case (2)) then the singularity is semi-quasihomogeneous with weights $(w_1, w_2, w_3) = (2, 1, 1)$ and degrees $(d_1, d_2) = (4, 2)$. The corresponding formulae for the Milnor number is (cf. [3])

$$\mu = 1 + \frac{d_1 d_2}{w_1 w_2 w_3}(d_1 + d_2 - w_1 - w_2 - w_3)$$

and we obtain 9.

Similarly the other case ($a = 3$ and $b = 5$) can be settled. This proves the Lemma. □

LEMMA 3.3. *With the notations of 3.1 assume that* $s = 1$ *and* $\sqrt{I_2}^3 \subsetneqq I_2$. *There exists automorphism* $\phi \in \mathrm{Aut}_{\mathbb{C}}\big(\mathbb{C}[[x, y, z]]\big)$ *such that* $\phi(I) = \langle x^2 + \alpha z^a + \beta y z^{a-1} + g, y^2 + \gamma z^b + \delta x z^{b-1} + h\rangle,\ 3 \leqq a \leqq b < \infty,\ g \in \langle x, y, z\rangle^{a+1}$, $h \in \langle x, y, z\rangle^{b+1}$.

(1) *If* $a = b = 3$ *and* $\alpha = \gamma = 0$ *then* $\mu\big(\mathbb{C}[[x, y, z]]/I\big) \geqq 11$.

(2) *If* $b \geqq 4$ *then* $\mu\big(\mathbb{C}[[x, y, z]]/I\big) \geqq 11$.

PROOF. Assume that $\mathrm{ord}(g) = a$ and $g = xg_1 + g_2$, $g_2 \in \mathbb{C}[[y, z]]$. Using the transformation $x \to x - \frac{1}{2}g_1$ and reducing with $y^2 + h$ we may assume that $I = \langle x^2 + \alpha z^a + \beta y z^{a-1} + g, y^2 + h\rangle$, $g \in \langle x, y, z\rangle^{a+1}$. Similarly we can adjust h. It remains to prove the estimation of the Milnor number. It is enough to prove this for the case $a = b = 3$ and $\alpha = \gamma = 0$, $\beta \neq 0$, $\delta \neq 0$, since we can always find a deformation to this case. We have $I = \langle x^2 + yz^2 + g$, $y^2 + xz^2 + h\rangle,\ g, h \in \langle x, y, z\rangle^4$. The Milnor number of $\mu\big(\mathbb{C}[[x, y, z]]/I\big)$ is given by the following formula

$$\mu\big(\mathbb{C}[[x, y, z]]/I\big)$$

$$= \dim_{\mathbb{C}}\big(\mathbb{C}[[x,y,z]]/\langle f, M\rangle\big) - \dim_{\mathbb{C}}\big(\mathbb{C}[[x,y,z]]/\langle \partial f/\partial x, \partial f/\partial y, \partial f/\partial z\rangle\big)$$

with $f = x^2 + yz^2 + y^2 + xz^2 + g + h$ and M the ideal of the 2-*minors* of $\partial(x^2 + yz^2 + g, y^2 + xz^2 + h)/\partial(x,y,z)$.

Analyzing the standard basis computations to compute the corresponding dimensions we see that their leading terms do not depend on g and h. We obtain 16 resp. 3 and therefore the Milnor number is 13. □

Proof of Proposition 3.1. (1) and (2) are direct consequences of the proof of Giusti's classification.

Using lemma 3.2 we obtain (3) as follows. According to Giusti's classification a simple singularity of type T must have Milnor number 7, 8 or 9. We obtain 7 in case $a = b = 3$, 8 in case $a = 3$, $b = 4$ and 9 in case $a = 3$, $b = 5$ or $a = b = 4$. If $a = 3$ and $b = 5$ then we obtain the type T_9 according to Giusti's classification. The corresponding curve has two irreducible branches. It can be distinguished from the curve defined by $a = b = 4$ since the curve has 4 smooth branches. This proves (3).

To prove (4) we may assume that $I = \langle x^2 + yz + g, xy + h\rangle$, $g, h \in \langle x,y,z\rangle^3$. Using a suitable automorphism we may assume that $I = \langle x^2 + yz + g, xy + \beta z^3 + h\rangle$, $g \in \langle x,y,z\rangle^3$ and $h \in \langle x,y,z\rangle^4$. According to the proof of Giusti's classification we obtain the type U_7 if $\beta \neq 0$. In case that $\beta = 0$ and z^3 is a monomial in g we obtain U_8. If z^3 is not a monomial in g and z^4 is a monomial in h we obtain U_9. In all other cases the singularity is not simple. It remains to prove that $\mu\big(\mathbb{C}[[x,y,z]]/I\big) \geqq 10$, in case that $\beta = 0$ and z^3 is not a monomial of g, z^4 is not a monomial of h. Using a suitable deformation of I we may assume that z^4 is a monomial of g. In this case the singularity is semi-quasihomogeneous with weights $\left(\frac{1}{2}, \frac{3}{4}, \frac{1}{4}\right)$ of degree $\left(1, \frac{5}{4}\right)$ as one can easily check. The corresponding Milnor number is 11. This implies $\mu\big(\mathbb{C}[[x,y,z]]/I\big) \geqq 11$.

To prove (5) we may assume that $I = \langle x^2 + g, y^2 + xz + h\rangle$, $h, g \in \langle x,y,z\rangle^3$. Using a suitable automorphism we may assume that $I = \langle x^2 + \alpha z^3 + \beta yz^2 + g, y^2 + xz + h\rangle$, $h \in \langle x,y,z\rangle^3$, $g \in \langle x,y,z\rangle^4$. According to the proof of Giusti's classification we obtain the type W_8 if $\alpha \neq 0$ and W_9 if $\alpha = 0$ and $\beta \neq 0$. It remains to prove that $\mu\big(\mathbb{C}[[x,y,z]]/I\big) \geqq 10$, if $\alpha = \beta = 0$. Using a suitable deformation we may assume that z^4 is a monomial in g. Then $\langle x^2 + g, y^2 + xz + h\rangle$ is semi-quasihomogeneous with weights $\left(\frac{1}{2}, \frac{3}{8}, \frac{1}{4}\right)$ and degree $\left(1, \frac{3}{4}\right)$ as one can easily check the corresponding Milnor number is 11. This implies $\mu\big(\mathbb{C}[[x,y,z]]/I\big) \geqq 11$.

To prove (6) we may assume that $I = \langle x^2 + g, yz + h\rangle$, $g, h \in \langle x,y,z\rangle^3$. Using Lemma 3.3 we obtain $I = \langle x^2 + \alpha z^a + \beta yz^{a-1} + g,\ y^2 + \gamma z^b + \delta xz^{b-1} + h\rangle$, $3 \leqq a \leqq b$, $g \in \langle x,y,z\rangle^{a+1}$, $h \in \langle x,y,z\rangle^{b+1}$.

According to the proof in Giusti's classification we obtain for $a = b = 3$ and $\alpha\gamma \neq 0$ the type Z_9 and $\alpha\gamma = 0$, $\alpha + \gamma \neq 0$ the type Z_{10}. If $\alpha = \gamma = 0$ or $b \geqq 4$ then because of the lemma 3.3 the Milnor number is greater than 10. This proves (6). □

We summarize this case in Algorithm 2.[2]

[2]The corresponding procedures are implemented in Singular in the library `classifyci.lib`.

Algorithm 2 1-dimensional complete intersections

Input: $I = \langle f, g \rangle \subseteqq \langle x, y, z \rangle^2 \mathbb{C}[[x, y, z]]$ isolated complete intersection curve singularity.

Output: The type of the singularity $\big(V(I), 0\big)$.

compute I_2 the 2-*jet* of I;
compute $I_2 = \bigcap_{i=1}^{s} Q_i$ the irredundant primary decomposition over $\mathbb{C}$;
compute μ the Milnor number of $\big(\mathbb{C}[[x, y, z]]/I\big)$;
if $s = 4$ **then**
  return (S_5);
if $s = 3$ **then**
  return (S_μ);
if $s = 2$ **then**
  compute $m_1 = \mathrm{mult}(Q_1)$, $m_2 = \mathrm{mult}(Q_2)$
  if $m_1 = m_2$ **then**
    if $7 \leqq \mu \leqq 8$ **then**
      return (T_μ);
    if $\mu = 9$ **then**
      compute the number b of branches of the curve using a resolution of the singularity
      if $b = 2$ **then**
        return (T_9);
      else
      return $(\textit{not simple})$;
  else
  if $7 \leqq \mu \leqq 9$ **then**
    return (U_μ);
if $s = 1$ **then**
  compute R the radical of I_2
  if $R^3 \nsubseteq I_2$ **then**
    if $8 \leqq \mu \leqq 9$ **then**
      return (W_μ);
    else
    return *not simple*;
else
  if $8 \leqq \mu \leqq 9$ **then**
    return (Z_μ);
  else
    return $(\textit{not simple})$;
return $(\textit{not simple})$;

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