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ON LOWER AND UPPER BOUNDS FOR PROBABILITIES OF UNIONS AND THE BOREL–CANTELLI LEMMA

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Abstract

We obtain new lower and upper bounds for probabilities of unions of events. These bounds are sharp. They are stronger than earlier ones. General bounds may be applied in arbitrary measurable spaces. We have improved the method that has been introduced in previous papers. We derive new generalizations of the first and second parts of the Borel–Cantelli lemma.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $\{A_n\}$ be a sequence of events. Put $U_n = \bigcup_{i=1}^n A_i$ and $\xi_n = I_{A_1} + I_{A_2} + \cdots + I_{A_n}$ for $n = 1, 2, \ldots$, where I_B denotes the indicator of event B.

Bounds for $\mathbf{P}(U_n)$ play an important role in probability and statistics. Many of them are based on moments $\alpha_k(n) = \mathbf{E}\xi_n^k$, $k = 1, 2, \ldots$, of random variable ξ_n. For example, Bonferroni type inequalities use binomial moments of ξ_n, but one can write them in terms of $\alpha_k(n)$ as well.

Chung and Erdős (1952) have derived the most simple and applicable lower bound for $\mathbf{P}(U_n)$ of this type that is

$$\mathbf{P}(U_n) \geqq \frac{\alpha_1^2(n)}{\alpha_2(n)}.$$

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It follows from the Cauchy–Buniakowski inequality. It is very convenient because of

$$\alpha_1(n) = \sum_{i=1}^{n} \mathbf{P}(A_i), \quad \alpha_2(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{P}(A_i A_j).$$

Various generalizations of the Chung–Erdős inequality were obtained in Dawson and Sankoff (1967), Gallot (1966), Kounias (1968), Kwerel (1975), Boros and Prékopa (1989), Galambos and Simonelli (1996), de Caen (1997), Kuai, Alajaji and Takahara (2000), Prékopa (2009), Frolov (2012) and references therein.

For example, in Kwerel (1975) and Boros and Prékopa (1989), one can find upper and lower bounds for $\mathbf{P}(U_n)$ that are based on $\alpha_k(n)$ for $1 \leqq k \leqq 3$ and $\alpha_k(n)$ for $1 \leqq k \leqq 4$. The lower bounds are stronger than the Chung–Erdős inequality. They are simple enough in applications as well. Indeed, for every $k \leqq n$, moments $\alpha_k(n)$ are sums of probabilities of intersections of k events from $A_1, A_2, \ldots, A_n$. Note that a precision of bounds increases when numbers of used moments enlarge.

From the other hand, making use of the Hölder inequality, one can conclude that

$$\mathbf{P}(U_n) \geqq \lim_{p \to 1+} \left(\frac{(\mathbf{E}\xi_n)^p}{\mathbf{E}\xi_n^p} \right)^{1/(p-1)} \geqq \left(\frac{(\mathbf{E}\xi_n)^p}{\mathbf{E}\xi_n^p} \right)^{q/p},$$

where $p > 1$ and $1/p + 1/q = 1$. It follows that applications of L_p-norms of $\xi_n / \mathbf{E}\xi_n$ with $p > 2$ do not give lower bounds stronger than the Chung–Erdős inequality. In particular, one can not derive better bounds, using $\alpha_2(n)$ and $\alpha_3(n)$ instead of $\alpha_1(n)$ and $\alpha_2(n)$, for instance. Stronger bounds have to involve moments of smaller orders and then we have to use moments of non-integer orders. Of course, these bounds are more complicated in calculations. One can find such lower bounds in Frolov (2012) and upper bounds in Frolov (2014). Note that in Frolov (2012), one part of bounds is obtained by applications of the Cauchy–Buniakowski and Hölder inequalities and another one is proved by a method which we improve in this paper. One can find a discussion on relationship of these bounds.

In this paper, we improve the method from Frolov (2012) and derive new upper and lower bound for $\mathbf{P}(U_n)$. Some of them include quantities similar to $\alpha_k(n)$, and may be calculated relatively simply.

Every new bound for $\mathbf{P}(U_n)$ may be used to obtain generalizations of the Borel–Cantelli lemma, the classical variant of which is as follows.

The Borel–Cantelli lemma. 1) *If the series* $\sum_{n=1}^{\infty} \mathbf{P}(A_n)$ *converges, then* $\mathbf{P}(A_n \text{ i.o.}) = 0$, *where*

$$\{A_n \text{ i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

2) *If* $\{A_n\}$ *are independent and the series* $\sum_{n=1}^{\infty} \mathbf{P}(A_n)$ *diverges, then* $\mathbf{P}(A_n \text{ i.o.}) = 1$.

The first part of the Borel–Cantelli lemma works in many situations, but the assumption of independence in the second one is restrictive. Therefore, main attention was paid to generalization of the second part.

The most important generalization of the second part of the Borel–Cantelli lemma has been obtained by Erdős and Rényi (1959). They proved that $\mathbf{P}(A_n \text{ i.o.}) = 1$ provided $L = 1$, where

$$L = \liminf_{n\to\infty} \frac{\alpha_2(n)}{\alpha_1^2(n)}.$$

This result implies in particular that the independence may be relaxed to pairwise independence in the second part. Of course, the proof is based on the Chung–Erdős inequality. Kochen and Stone (1964) and Spitzer (1964) have proved that $\mathbf{P}(A_n \text{ i.o.}) \geqq 1/L$. Further generalizations of the second part of the Borel–Cantelli lemma have been obtained in Kounias (1968), Móri and Székely (1983), Anděl and Dupaš (1989), Martikainen and Petrov (1990), Petrov (2002), Xie (2008), Feng, Li and Shen (2009), Frolov (2012) and references therein. Note that Móri and Székely (1983) obtained the lower bound for $\mathbf{P}(A_n \text{ i.o.})$ in terms of non-integer moments of $\xi_n/\mathbf{E}\xi_n$. Generalizations of the first part of the Borel–Cantelli lemma may be found in Frolov (2014) and references therein. One can find results under additional assumptions on dependence of events, conditional Borel–Cantelli lemma and further references in Chandra (2012).

The rest of the paper is organized as follows. We present our method and general inequalities in Section 2. Section 3 contains some new bounds for probabilities of unions. Note that these results may be also applied to measures of unions in arbitrary measurable spaces. In Section 4, new generalizations of the Borel–Cantelli lemma are proved.

2. Method and general results

Our method is based on the following result which is a generalization of Theorem 1 in Frolov (2012).

Theorem 1. *Let ℓ and N be natural numbers such that $2 \leqq \ell \leqq N$. Let $\{r_i,\ 1 \leqq i \leqq N\}$ and $\{f_{ki},\ 1 \leqq k \leqq \ell,\ 1 \leqq i \leqq N\}$ be arrays of non-negative real numbers. For $1 \leqq k \leqq \ell$, put*

$$\bar{s}_k = \sum_{i=1}^{N} f_{ki} r_i. \tag{1}$$

Assume that there exist real numbers $c_1, c_2, \ldots, c_N$ and $a_1, a_2, \ldots, a_\ell$ such that

$$\sum_{i=1}^{N} (1 - c_i) r_i = \sum_{i=1}^{\ell} a_i \bar{s}_i. \tag{2}$$

If $c_i \geqq 0$ for all $i = 1, 2, \ldots, N$, then

$$R = \sum_{i=1}^{N} r_i \geqq \sum_{i=1}^{\ell} a_i \bar{s}_i. \tag{3}$$

If $c_i \leqq 0$ for all $i = 1, 2, \ldots, N$, then

$$R \leqq \sum_{i=1}^{\ell} a_i \bar{s}_i. \tag{4}$$

Inequalities (3) *and* (4) *turn to equalities if, for some $1 \leqq i_1 < i_2 < \cdots < i_\ell \leqq N$, the numbers $r_{i_1}, r_{i_2}, \ldots, r_{i_\ell}$ are solutions of the linear system*

$$\sum_{j=1}^{\ell} f_{k i_j} r_{i_j} = \bar{s}_k, \quad 1 \leqq k \leqq \ell, \tag{5}$$

$r_i = 0$ for all $i \neq i_k$ and $c_{i_k} = 0$ for all $1 \leqq k \leqq \ell$. In this case, $R = r_{i_1} + r_{i_2} + \cdots + r_{i_\ell}$.

Proof. By (2), we have

$$R - \sum_{i=1}^{\ell} a_i \bar{s}_i = \sum_{i=1}^{N} c_i r_i,$$

that yields the assertion of Theorem 1. □

Now we describe the method which gives sharp lower and upper bounds for R.

We first choose number ℓ and array $\{f_{ki}\}$. The next step is to take c_i. To satisfy relation (2), the simplest choice of c_i is

$$c_i = 1 - \sum_{j=1}^{\ell} a_j f_{ji}$$

for all $1 \leqq i \leqq N$. Since the bounds for R have to be sharp (i.e. they have to turn to equalities for some set of numbers $r_1, r_2, \dots, r_N$), we will take coefficients a_j such that $c_{i_k} = 0$ for some $1 \leqq i_1 < i_2 < \cdots < i_\ell \leqq N$. To this end, we need a way to determine $i_1, i_2, \dots, i_\ell$.

Note that if $f_{ki} = i^k$, $1 \leqq k \leqq \ell$, then putting

$$c_i = \prod_{k=1}^{\ell} \left(1 - \frac{i}{i_k}\right)$$

for all $1 \leqq i \leqq N$ gives a simple way to find $i_1, i_2, \dots, i_\ell$ such that $c_i \geqq 0$ or $c_i \leqq 0$ for every i. Indeed, assume first that $\ell = 2$. If we put $i_1 = m - 1$ and $i_2 = m$, where $2 \leqq m \leqq N$, then $c_i \geqq 0$ for all i. If we take $i_1 = 1$ and $i_2 = N$, then $c_i \leqq 0$ for every i. Here and in the sequel, natural number m is a parameter which will be specified in proofs below. Further, suppose that $\ell = 3$. If we choose $i_1 = m - 1$, $i_2 = m$ and $i_3 = N$ with $2 \leqq m \leqq N - 1$, then $c_i \geqq 0$ for all i. If we take $i_1 = 1$, $i_2 = m - 1$ and $i_3 = m$, where $3 \leqq m \leqq N$, then $c_i \leqq 0$ for every i. For $\ell = 4$, putting $i_1 = m - 1$, $i_2 = m$, $i_3 = N - 1$ and $i_4 = N$ yields that $c_i \geqq 0$ for all i. If $\ell = 4$ and $i_1 = 1$, $i_2 = m - 1$, $i_3 = m$ and $i_4 = N$, $3 \leqq m \leqq N - 1$, then $c_i \leqq 0$ for every i. And so on. We will only use this way in the sequel to choose $i_1, i_2, \dots, i_\ell$.

When we know $i_1, i_2, \dots, i_\ell$, coefficients a_j may be found as solutions of the following linear system:

$$\sum_{j=1}^{\ell} a_j f_{j i_k} = 1, \quad k = 1, 2, \dots, \ell.$$

Taking into account that f_{ki} may differ from i^k, we have to make certain that the choice of $i_1, i_2, \dots, i_\ell$ yields desired inequalities for c_i. If we construct a lower bound, then we have to check that $c_i \geqq 0$ for all i. If we deal with an upper bound, then we have to verify that $c_i \leqq 0$ for all i. By Theorem 1, we get either inequality (3), or inequality (4). Since indices i_k depend on m, we make an optimization over m.

Note that in the case $f_{ki} = i^k$, we get

$$c_i = \prod_{k=1}^{\ell} \left(1 - \frac{i}{i_k}\right) = \sum_{k=1}^{\ell} a_k i^k$$

for all i. It means that a_j are coefficients in the decomposition of c_i over degrees of i and all $i, i^2, \dots, i^\ell$ are in this decomposition. Unfortunately, we can not follow this pattern to find a_j in general case. Indeed, if $f_{ki} = i^{\gamma_k}$, $\gamma_k > 0$, for example, we could put

$$c_i = \prod_{k=1}^{\ell} \left(1 - \left(\frac{i}{i_k}\right)^{\gamma_k - \gamma_{k-1}}\right).$$

In this case we will not obtain a desired decomposition with all $i^{\gamma_1}, i^{\gamma_2}, \dots, i^{\gamma_\ell}$. A variant with γ_k instead of $\gamma_k - \gamma_{k-1}$ yields the same problem as well.

The above reasons lead us to the following result.

Corollary 1. *Assume that for some* $1 \leqq i_1 < i_2 < \dots < i_\ell \leqq N$, *coefficients* a_j *are solutions of the following linear system:*

$$\sum_{j=1}^{\ell} a_j f_{j i_k} = 1, \quad k = 1, 2, \dots, \ell. \tag{6}$$

Put

$$c_i = 1 - \sum_{j=1}^{\ell} a_j f_{ji} \tag{7}$$

for all $1 \leqq i \leqq N$. *Assume that the numbers* $r^*_{i_1}, r^*_{i_2}, \dots, r^*_{i_\ell}$ *are solutions of the linear system* (5) *and* $r^*_i = 0$ *for all* $i \neq i_k$, $1 \leqq i \leqq N$.

If $c_i \geqq 0$ *for all* $i = 1, 2, \dots, N$, *then* $R \geqq R^* = r^*_{i_1} + r^*_{i_2} + \dots + r^*_{i_\ell}$.

If $c_i \leqq 0$ *for all* $i = 1, 2, \dots, N$, *then* $R \leqq R^*$.

Proof. By (7), we have (2). If $c_i \geqq 0$ for all $i = 1, 2, \dots, N$, then inequality (3) holds. By definition of $r^*_1, r^*_2, \dots, r^*_N$, we get

$$\bar{s}^*_k = \sum_{i=1}^{N} f_{ki} r^*_i = \bar{s}_k, \quad k = 1, 2, \dots, \ell.$$

The last equality follows from (5). It yields that

$$R \geqq \sum_{i=1}^{\ell} a_i \bar{s}_i^*.$$

By (6), we have $c_{i_k} = 0$ for all $k = 1, 2, \dots, \ell$. Applying Theorem 1 to $r_1^*, r_2^*, \dots, r_N^*$, we conclude that

$$R^* = \sum_{i=1}^{\ell} a_i \bar{s}_i^* \leqq R.$$

The case $c_i \leqq 0$ for all $i = 1, 2, \dots, N$ may be considered in the same way. □

We choose $\{f_{ik}\}$ such that results will be simpler. To this end, in the sequel, we put $f_{ik} = i^{a+(k-1)\varrho}$ for all $1 \leqq i \leqq N$ and $1 \leqq k \leqq \ell$, where $a > 0$ and $\varrho > 0$. Then relation (1) turns to

$$\bar{s}_k = \sum_{i=1}^{N} i^{a+(k-1)\varrho} r_i, \quad 1 \leqq k \leqq \ell. \tag{8}$$

For the case $a = \varrho = 1$, we will use a special notation

$$s_k = \sum_{i=1}^{N} i^k r_i, \quad 1 \leqq k \leqq \ell. \tag{9}$$

We start with the case $\ell = 2$. Our first result is a lower bound for R.

THEOREM 2. *Define $\bar{s}_1$ and $\bar{s}_2$ by* (8). *Put $\bar{\delta} = (\bar{s}_2/\bar{s}_1)^{1/\varrho}$, $\theta = \bar{\delta} - [\bar{\delta}]$ and $\bar{\theta} = \big(\bar{\delta}^{\rho} - (\bar{\delta} - \theta)^{\varrho}\big) / \big((\bar{\delta} + 1 - \theta)^{\varrho} - (\bar{\delta} - \theta)^{\varrho}\big) \in [0, 1)$, where $[\cdot]$ denotes the integer part of the number in brackets. Here and in the sequel, we suggest that $0/0 = 0$.*

Then

$$R \geqq \frac{\bar{\theta} \bar{s}_1^{(a+\varrho)/\varrho}}{\big(\bar{s}_2^{1/\varrho} + (1-\theta)\bar{s}_1^{1/\varrho}\big)^a} + \frac{(1-\bar{\theta})\bar{s}_1^{(a+\varrho)/\varrho}}{\big(\bar{s}_2^{1/\varrho} - \theta \bar{s}_1^{1/\varrho}\big)^a}. \tag{10}$$

Note that if $\bar{s}_1 = 0$, then $r_i = 0$ for all i, $\bar{s}_2 = \bar{\delta} = \theta = \bar{\theta} = 0$ and (10) is trivial. If $\bar{s}_1 > 0$, then $\bar{\delta} \geqq 1$. Moreover, if $\bar{\delta} = 1$, then $r_i = 0$ for all $i \geqq 2$, $\bar{s}_2 = \bar{s}_1 = r_1$, $\theta = \bar{\theta} = 0$ and (10) turns to $R \geqq r_1$.

PROOF. For every natural m, $2 \leqq m \leqq N$, put $i_1 = m-1$ and $i_2 = m$. By (6) and (7), we have $c_i = 1 - a_1 i^a - a_2 i^{a+\varrho}$ for $i = 1, 2, \ldots, N$, where a_1 and a_2 satisfy the following linear system:

$$(m-1)^a a_1 + (m-1)^{a+\varrho} a_2 = 1,$$

$$m^a a_1 + m^{a+\varrho} a_2 = 1.$$

It follows that

$$a_1 = \frac{m^{a+\varrho} - (m-1)^{a+\varrho}}{m^{a+\varrho}(m-1)^a - m^a(m-1)^{a+\varrho}},$$

$$a_2 = -\frac{m^a - (m-1)^a}{m^{a+\varrho}(m-1)^a - m^a(m-1)^{a+\varrho}},$$

and

$$c_i = 1 - \frac{m^{a+\varrho} - (m-1)^{a+\varrho}}{m^{a+\varrho}(m-1)^a - m^a(m-1)^{a+\varrho}} i^a + \frac{m^a - (m-1)^a}{m^{a+\varrho}(m-1)^a - m^a(m-1)^{a+\varrho}} i^{a+\varrho}$$

for all $i = 1, 2, \ldots, N$.

Now we check that $c_i \geqq 0$ for all i. Consider function $f(x) = 1 - a_1 x^a - a_2 x^{a+\varrho}$ for real $x \geqq 0$. We have $f(0) = 1$, $f(m-1) = f(m) = 0$, $f(+\infty) = +\infty$ and $f'(x) = -x^{a-1}(aa_1 + (a+\varrho)a_2 x^{\varrho})$. It is clear that there exists a unique solution of equation $f'(x) = 0$. It follows that function $f(x)$ takes its minimum at $x_0 \in (m-1, m)$ and $f(x) \leqq 0$ for $x \in (m-1, m)$ and $f(x) \geqq 0$ otherwise. Hence, $c_i = f(i) \geqq 0$ for all $i = 1, 2, \ldots, N$.

Linear system (5) is

$$(m-1)^a r_{m-1} + m^a r_m = \bar{s}_1,$$

$$(m-1)^{a+\varrho} r_{m-1} + m^{a+\varrho} r_m = \bar{s}_2.$$

Solving this system, we get

$$r^*_{m-1} = \frac{\bar{s}_1 m^{a+\varrho} - \bar{s}_2 m^a}{m^{a+\varrho}(m-1)^a - m^a(m-1)^{a+\varrho}},$$

$$r^*_m = \frac{\bar{s}_2 (m-1)^a - \bar{s}_1 (m-1)^{a+\varrho}}{m^{a+\varrho}(m-1)^a - m^a(m-1)^{a+\varrho}}.$$

Inequalities $r^*_{m-1} \geqq 0$ and $r^*_m \geqq 0$ imply that $(\bar{s}_2/\bar{s}_1)^{1/\varrho} \leqq m \leqq 1 + (\bar{s}_2/\bar{s}_1)^{1/\varrho}$, which coincides with $\bar{\delta} \leqq m \leqq 1 + \bar{\delta}$. This inequality defines m uniquely for non-integer $\bar{\delta}$. If $\bar{\delta}$ is integer then there are two variants. Anyone of them may be excluded. Hence, without loss of generality, we assume in the sequel that $\bar{\delta} < m \leqq 1 + \bar{\delta}$. Then

$$m = 1 + [\bar{\delta}] = 1 + \bar{\delta} - \theta,$$

provided $\bar{\delta} < N$. It follows in this case that

$$\bar{\theta} = \frac{\bar{\delta}^\varrho - (m-1)^\varrho}{m^\varrho - (m-1)^\varrho} \in [0, 1).$$

Taking into account that $\bar{s}_2 \leqq N^\varrho \bar{s}_1$, we see that $\bar{\delta} \leqq N$. Hence, we finally put $m = \min\{1 + [\bar{\delta}], N\} \leqq N$.

Assume that $\bar{\delta} < N$. Then $m = 1 + [\bar{\delta}]$ and

$$(m-1)^a = \bar{s}_1^{-a/\varrho}\big(\bar{s}_2^{1/\varrho} - \theta \bar{s}_1^{1/\varrho}\big)^a.$$

It follows that

$$r^*_{m-1} = \frac{\bar{s}_1(m^\varrho - \bar{\delta}^\varrho)}{(m-1)^a(m^\varrho - (m-1)^\varrho)} = \frac{(1-\bar{\theta})\bar{s}_1^{(a+\varrho)/\varrho}}{\big(\bar{s}_2^{1/\varrho} - \theta \bar{s}_1^{1/\varrho}\big)^a}.$$

Similarly,

$$m^a = \bar{s}_1^{-a/\varrho}\big(\bar{s}_2^{1/\varrho} + (1-\theta)\bar{s}_1^{1/\varrho}\big)^a.$$

This implies that

$$r^*_m = \frac{\bar{s}_1\big(\bar{\delta}^\varrho - (m-1)^\varrho\big)}{m^a\big(m^\rho - (m-1)^\rho\big)} = \frac{\bar{\theta}\bar{s}_1^{(a+\varrho)/\varrho}}{\big(\bar{s}_2^{1/\varrho} + (1-\theta)\bar{s}_1^{1/\varrho}\big)^a}.$$

By Corollary 1, we have $R \geqq r^*_{m-1} + r^*_m$. Substituting of r^*_{m-1} and r^*_m in the last inequality yields inequality (10).

Assume that $\bar{\delta} = N$. Then

$$0 = \bar{s}_2 - N^\varrho \bar{s}_1 = \sum_{i=1}^{N-1} (i^\varrho - N^\varrho) i^a r_i,$$

where $(i^\varrho - N^\varrho)i^a < 0$ for $i \leqq N-1$. The latter implies that $r_1 = r_2 = \cdots = r_{N-1} = 0$. Moreover, $\bar{\delta} = N$ yields that $\theta = \bar{\theta} = 0$. Hence inequality (10) turns to inequality $R \geqq r_N$ which holds obviously. □

Theorem 2 yields simpler bounds as follows.

Corollary 2. *Define* $\bar{s}_1$ *and* $\bar{s}_2$ *by* (8). *If* $\varrho \geqq 1$, *then*

$$R \geqq \frac{\bar{s}_1^{(a+\varrho)/\varrho}}{\bar{s}_2^{a/\varrho}}. \tag{11}$$

If $\varrho < 1$, *then*

$$R \geqq \frac{1-\bar{\theta}}{1-\theta}\,\frac{\bar{s}_1^{(a+\varrho)/\varrho}}{\bar{s}_2^{a/\varrho}}. \tag{12}$$

Proof. Put

$$g(\theta,\bar{\theta}) = \frac{\bar{\theta}\bar{s}_1^{(a+\varrho)/\varrho}}{\left(\bar{s}_2^{1/\varrho} + (1-\theta)\bar{s}_1^{1/\varrho}\right)^a} + \frac{(1-\bar{\theta})\bar{s}_1^{(a+\varrho)/\varrho}}{\left(\bar{s}_2^{1/\varrho} - \theta\bar{s}_1^{1/\varrho}\right)^a}.$$

We have

$$g(\theta,\bar{\theta}) = g(\theta,\theta) + (\bar{\theta}-\theta)\left(\frac{\bar{s}_1^{(a+\varrho)/\varrho}}{\left(\bar{s}_2^{1/\varrho} + (1-\theta)\bar{s}_1^{1/\varrho}\right)^a} - \frac{\bar{s}_1^{(a+\varrho)/\varrho}}{\left(\bar{s}_2^{1/\varrho} - \theta\bar{s}_1^{1/\varrho}\right)^a}\right).$$

If $\bar{\theta} \leqq \theta$, then $g(\theta,\bar{\theta}) \geqq g(\theta,\theta)$. It follows from inequality (10) that

$$R \geqq \frac{\theta\bar{s}_1^{(a+\varrho)/\varrho}}{\left(\bar{s}_2^{1/\varrho} + (1-\theta)\bar{s}_1^{1/\varrho}\right)^a} + \frac{(1-\theta)\bar{s}_1^{(a+\varrho)/\varrho}}{\left(\bar{s}_2^{1/\varrho} - \theta\bar{s}_1^{1/\varrho}\right)^a}.$$

The right-hand side of the last inequality takes its minimum over θ for $\theta = 0$. Hence, inequality (10) implies inequality (11) provided $\bar{\theta} \leqq \theta$.

Inequality $\bar{\theta} \leqq \theta$ is equivalent to $\bar{\delta}^{\varrho} \leqq (1-\theta)(\bar{\delta}-\theta)^{\varrho} + \theta(\bar{\delta}-\theta+1)^{\varrho}$. The latter inequality holds when x^{ϱ} is a convex function of x, i.e. for $\varrho \geqq 1$. For $\varrho < 1$, x^{ϱ} is a concave function of x and, therefore, $\bar{\theta} > \theta$.

So, we have proved inequality (11). To check inequality (12), we note that for $\bar{\theta} > \theta$,

$$g(\theta,\bar{\theta}) \geqq \min\left\{\frac{\bar{\theta}}{\theta}, \frac{1-\bar{\theta}}{1-\theta}\right\} g(\theta,\theta) = \frac{1-\bar{\theta}}{1-\theta} g(\theta,\theta).$$

Inequality (12) now follows from (10). □

Theorem 2 and Corollary 2 yield the following result in the case $a = \varrho = 1$.

Corollary 3. *Define* s_1 *and* s_2 *by* (9). *Put* $\delta = s_2/s_1$ *and* $\theta = \delta - [\delta]$. *The following inequality holds:*

$$R \geqq \frac{\theta s_1^2}{s_2 + (1-\theta)s_1} + \frac{(1-\theta)s_1^2}{s_2 - \theta s_1} \geqq \frac{s_1^2}{s_2}.$$

Note that the middle part of the last inequality takes its minimum over θ for $\theta = 0$.

Now we turn to upper bounds for $\ell = 2$. Our next result is as follows.

Theorem 3. *Define* $\bar{s}_1$ *and* $\bar{s}_2$ *by* (8). *The following inequality holds:*

$$R \leqq \frac{N^{a+\varrho} - 1}{N^{a+\varrho} - N^a}\bar{s}_1 - \frac{N^a - 1}{N^{a+\varrho} - N^a}\bar{s}_2. \tag{13}$$

Proof. Take $i_1 = 1$ and $i_2 = N$. By (6) and (7), we have $c_i = 1 - a_1 i^a - a_2 i^{a+\varrho}$ for all $i = 1, 2, \dots, N$, where a_1 and a_2 are such that

$$a_1 + a_2 = 1,$$
$$N^a a_1 + N^{a+\varrho} a_2 = 1.$$

Hence

$$a_1 = \frac{N^{a+\varrho} - 1}{N^{a+\varrho} - N^a}, \quad a_2 = -\frac{N^a - 1}{N^{a+\varrho} - N^a},$$

and

$$c_i = 1 - \frac{N^{a+\varrho} - 1}{N^{a+\varrho} - N^a} i^a + \frac{N^a - 1}{N^{a+\varrho} - N^a} i^{a+\varrho}$$

for all $i = 1, 2, \dots, N$.

Let us check that $c_i \leqq 0$ for all i. Consider again function $f(x) = 1 - a_1 x^a - a_2 x^{a+\varrho}$ for real $x \geqq 0$. We have $f(0) = 1$, $f(1) = f(N) = 0$, $f(+\infty) = +\infty$ and $f'(x) = -x^{a-1}(aa_1 + (a+\varrho)a_2 x^{\varrho})$. We see that there exists a unique solution x_0 of equation $f'(x) = 0$. It follows that function $f(x)$ takes its minimum at $x_0 \in (1, N)$, $f(x) \leqq 0$ for $x \in (1, N)$ and $f(x) \geqq 0$ otherwise. Hence, $c_i = f(i) \leqq 0$ for all $i = 1, 2, \dots, N$.

Therefore Theorem 1 yields inequality (13). □

Theorem 3 implies the following result for $a = \varrho = 1$.

Corollary 4. *Define* s_1 *and* s_2 *by* (9). *The following inequality holds:*

$$R \leqq \frac{N+1}{N} s_1 - \frac{1}{N} s_2.$$

Comparing Theorems 2 and 3, we see that lower bounds seem more interesting for $\ell = 2$. The situation will change in the case $\ell = 3$, to which we turn now. We start again with a lower bound for R.

THEOREM 4. *Define* $\bar{s}_1$, $\bar{s}_2$ *and* $\bar{s}_3$ *by* (8). *Put* $\bar{\delta}_1 = N^{\varrho}\bar{s}_1 - \bar{s}_2$, $\bar{\delta}_2 = N^{\varrho}\bar{s}_2 - \bar{s}_3$, $\bar{\delta} = (\bar{\delta}_2/\bar{\delta}_1)^{1/\varrho}$, $\theta = \bar{\delta} - [\bar{\delta}]$ *and* $\bar{\theta} = (\bar{\delta}^{\varrho} - (\bar{\delta}-\theta)^{\varrho})/\big((\bar{\delta}+1-\theta)^{\varrho} - (\bar{\delta}-\theta)^{\varrho}\big) \in [0,1)$.

The following inequality holds:

$$R \geqq \frac{\bar{\delta}_1(1-\bar{\theta})\big(N^a - (\bar{\delta}-\theta)^a\big)}{N^a(\bar{\delta}-\theta)^a\big(N^{\varrho} - (\bar{\delta}-\theta)^{\varrho}\big)} + \frac{\bar{\delta}_1\bar{\theta}\big(N^a - (\bar{\delta}-\theta+1)^a\big)}{N^a(\bar{\delta}-\theta+1)^a\big(N^{\varrho} - (\bar{\delta}-\theta+1)^{\varrho}\big)} + \frac{\bar{s}_1}{N^a}. \tag{14}$$

Note that if $\bar{\delta}_1 = 0$, then $r_1 = r_2 = \cdots = r_{N-1} = 0$, $\bar{\delta} = \bar{\delta}_2 = \theta = \bar{\theta} = 0$ and (14) turns to $R \geqq r_N$ that holds obviously. If $\bar{\delta}_1 > 0$, then $\bar{\delta} \geqq 1$ in view of

$$\bar{\delta}_1 = \sum_{i=1}^{N-1}(N^{\varrho} - i^{\rho})i^a r_i \leqq \sum_{i=1}^{N-1}(N^{\varrho} - i^{\rho})i^{a+\varrho} r_i = \bar{\delta}_2.$$

The latter also yields that $\bar{\delta} = 1$ only if $r_2 = \cdots = r_{N-1} = 0$. In the last case, $\bar{\delta}_1 = \bar{\delta}_2 = (N^{\varrho} - 1)r_1$, $\theta = \bar{\theta} = 0$ and (14) turns to $R \geqq r_1 + r_N$.

It follows that we may assume that $\bar{\delta} > 1$ in the sequel.

PROOF. Take natural m, $2 \leqq m \leqq N-1$, and put $i_1 = m-1$, $i_2 = m$, $i_3 = N$. By (6) and (7), we get $c_i = 1 - a_1 i^a - a_2 i^{a+\varrho} - a_3 i^{a+2\varrho}$, where a_1, a_2 and a_3 are determined by the following system of linear equations

$$(m-1)^a a_1 + (m-1)^{a+\varrho} a_2 + (m-1)^{a+2\varrho} a_3 = 1,$$

$$m^a a_1 + m^{a+\varrho} a_2 + m^{a+2\varrho} a_3 = 1,$$

$$N^a a_1 + N^{a+\varrho} a_2 + N^{a+2\varrho} a_3 = 1.$$

Then we have

$$a_1 = \frac{m^{a+\varrho}N^{a+\varrho}(N^{\varrho} - m^{\varrho}) - (m-1)^{a+\varrho}N^{a+\varrho}\big(N^{\varrho} - (m-1)^{\varrho}\big)}{\Delta_m} + \frac{m^{a+\varrho}(m-1)^{a+\varrho}\big(m^{\varrho} - (m-1)^{\varrho}\big)}{\Delta_m},$$

$$a_2 = -\frac{m^a N^a (N^{2\varrho} - m^{2\varrho}) - (m-1)^a N^a \big(N^{2\varrho} - (m-1)^{2\varrho}\big)}{\Delta_m}$$
$$+ \frac{m^a (m-1)^a \big(m^{2\varrho} - (m-1)^{2\varrho}\big)}{\Delta_m},$$
$$a_3 = \frac{m^a N^a (N^{\varrho} - m^{\varrho}) - (m-1)^a N^a \big(N^{\varrho} - (m-1)^{\varrho}\big)}{\Delta_m}$$
$$+ \frac{m^a (m-1)^a \big(m^{\varrho} - (m-1)^{\varrho}\big)}{\Delta_m},$$

where $\Delta_m = N^a m^a (m-1)^a (m^{\varrho} - (m-1)^{\varrho})\big(N^{\varrho} - (m-1)^{\varrho}\big)\,(N^{\varrho} - m^{\varrho})$. Considering function $f(x) = 1 - a_1 x^a - a_2 x^{a+\varrho} - a_3 x^{a+2\varrho}$, one can check that $c_i \geqq 0$ for all $i = 1, 2, \ldots, N$.

Solving linear system (5), that is

$$(m-1)^a r_{m-1} + m^a r_m + N^a r_N = \bar{s}_1,$$
$$(m-1)^{a+\varrho} r_{m-1} + m^{a+\varrho} r_m + N^{a+\varrho} r_N = \bar{s}_2,$$
$$(m-1)^{a+2\varrho} r_{m-1} + m^{a+2\varrho} r_m + N^{a+2\varrho} r_N = \bar{s}_3,$$

we get

$$r^*_{m-1} = m^a N^a (N^{\varrho} - m^{\varrho}) \frac{m^{\varrho} N^{\varrho} \bar{s}_1 - (N^{\varrho} + m^{\varrho}) \bar{s}_2 + \bar{s}_3}{\Delta_m},$$
$$r^*_m = -(m-1)^a N^a \big(N^{\varrho} - (m-1)^{\varrho}\big)$$
$$\times \frac{(m-1)^{\varrho} N^{\varrho} \bar{s}_1 - \big(N^{\varrho} + (m-1)^{\varrho}\big)\,\bar{s}_2 + \bar{s}_3}{\Delta_m},$$
$$r^*_N = (m-1)^a m^a \big(m^{\varrho} - (m-1)^{\varrho}\big)\,\frac{(m-1)^{\varrho} m^{\varrho} \bar{s}_1 - \big(m^{\varrho} + (m-1)^{\varrho}\big)\,\bar{s}_2 + \bar{s}_3}{\Delta_m}.$$

Making use of inequalities $r^*_{m-1} \geqq 0$ and $r^*_m \geqq 0$, we conclude that

$$\left(\frac{\bar{\delta}_2}{\bar{\delta}_1}\right)^{1/\varrho} \leqq m \leqq 1 + \left(\frac{\bar{\delta}_2}{\bar{\delta}_1}\right)^{1/\varrho}.$$

The latter is equivalent to $\bar{\delta} \leqq m \leqq 1 + \bar{\delta}$. By the same reason as in the proof of Theorem 2, we assume that $\bar{\delta} < m \leqq 1 + \bar{\delta}$. Taking into account that $\bar{\delta}_2 \leqq (N-1)^{\varrho} \bar{\delta}_1$, we obtain $\bar{\delta} \leqq N - 1$. Hence, we put $m = \min\{1 + [\bar{\delta}], N - 1\}$.

It is not difficult to check that

$$r^*_{m-1} = \frac{\bar\delta_1(1-\bar\theta)}{(\bar\delta-\theta)^a\big(N^\varrho-(\bar\delta-\theta)^\varrho\big)},$$

$$r^*_m = \frac{\bar\delta_1\bar\theta}{(\bar\delta-\theta+1)^a\big(N^\varrho-(\bar\delta-\theta+1)^\varrho\big)},$$

$$r^*_N = \frac{\bar s_1}{N^a} - \frac{\bar\delta_1}{N^a}\left(\frac{\bar\theta}{\big(N^\varrho-(\bar\delta-\theta+1)^\varrho\big)} + \frac{1-\bar\theta}{\big(N^\varrho-(\bar\delta-\theta)^\varrho\big)}\right).$$

Corollary 1 implies that $R \geqq r^*_{m-1}+r^*_m+r^*_N$. Substituting the formulae for r^*_{m-1}, r^*_m and r^*_N in the last inequality, we arrive at (14). □

Theorem 4 implies the next result.

Corollary 5. *Under notations of Theorem* 4, *if* $a \leqq \varrho$, *then*

$$(15)\quad R \geqq \frac{\bar\delta_1(1-\bar\theta)(N^a-\bar\delta^a)}{N^a(\bar\delta-\theta)^a(N^\varrho-\bar\delta^\varrho)} + \frac{\bar\delta_1\bar\theta\big(N^a-(\bar\delta+1)^a\big)}{N^a(\bar\delta-\theta+1)^a\big(N^\varrho-(\bar\delta+1)^\varrho\big)} + \frac{\bar s_1}{N^a}.$$

If $a \geqq \varrho$, *then*

$$(16)\quad R \geqq \frac{\bar\delta_1(1-\bar\theta)\big(N^a-(\bar\delta-1)^a\big)}{N^a(\bar\delta-\theta)^a\big(N^\varrho-(\bar\delta-1)^\varrho\big)} + \frac{\bar\delta_1\bar\theta(N^a-\bar\delta^a)}{N^a(\bar\delta-\theta+1)^a(N^\varrho-\bar\delta^\varrho)} + \frac{\bar s_1}{N^a}.$$

Proof. We need the following technical result.

Lemma 1. *If either* $0<u<v<1$, *or* $1<u<v$, *then* $f(x)=\frac{1-u^x}{1-v^x}$, $x>0$, *is a decreasing function.*

We omit the proof of Lemma 1.

Put $u=(\bar\delta-\theta)/N$ and $v=\bar\delta/N$. Since $\bar\delta_2 \leqq (N-1)^\varrho\bar\delta_1$, we have $v<1$. If $a<\varrho$, then by Lemma 1,

$$\frac{1-u^a}{1-v^a} > \frac{1-u^\varrho}{1-v^\varrho},$$

which is equivalent to

$$\frac{N^a-(\bar\delta-\theta)^a}{N^a-\bar\delta^a} > \frac{N^\varrho-(\bar\delta-\theta)^\varrho}{N^\varrho-\bar\delta^\varrho}.$$

We will have an opposite inequality for $a>\varrho$.

It follows that

$$\frac{N^a - (\bar{\delta} - \theta)^a}{N^\varrho - (\bar{\delta} - \theta)^\varrho}$$

take its minimum over θ for $\theta = 0$, if $a \leqq \varrho$, and for $\theta = 1$, if $a \geqq \varrho$.

Making use of Lemma 1, one can check that the same holds true for

$$\frac{N^a - (\bar{\delta} - \theta + 1)^a}{N^\varrho - (\bar{\delta} - \theta + 1)^\varrho}.$$

Now Corollary 5 follows from the latter and Theorem 4. □

For $\varrho \geqq 1$, inequalities (15) and (16) imply simpler bounds.

Corollary 6. *Assume that $\varrho \geqq 1$. Then $\bar{\theta} \leqq \theta$ and one can replace $\bar{\theta}$ by θ in* (15) *and* (16)*. Moreover, if $a \leqq \varrho$ in addition, then*

$$R \geqq \frac{\bar{\delta}_1 (N^a - \bar{\delta}^a)}{N^a \bar{\delta}^a (N^\varrho - \bar{\delta}^\varrho)} + \frac{\bar{s}_1}{N^a}.$$

If $a \geqq \varrho$ in addition, then

$$R \geqq \frac{\bar{\delta}_1 \big(N^a - (\bar{\delta} - 1)^a\big)}{N^a \bar{\delta}^a \big(N^\varrho - (\bar{\delta} - 1)^\varrho\big)} + \frac{\bar{s}_1}{N^a}.$$

Proof. One can check that $\bar{\theta} \leqq \theta$ and one can put $\bar{\theta} = \theta = 0$ in (15) and (16) in the same way as in the proof of Corollary 2. □

The next result is an upper bound for R.

Theorem 5. *Define $\bar{s}_1$, $\bar{s}_2$ and $\bar{s}_3$ by* (8)*. Put $\hat{\delta}_1 = \bar{s}_2 - \bar{s}_1$, $\hat{\delta}_2 = \bar{s}_3 - \bar{s}_2$, $\hat{\delta} = (\hat{\delta}_2/\hat{\delta}_1)^{1/\varrho}$, $\theta = \hat{\delta} - [\hat{\delta}]$ and $\hat{\theta} = \big(\hat{\delta}^\varrho - (\hat{\delta} - \theta)^\varrho\big)/\big((\hat{\delta} + 1 - \theta)^\varrho - (\hat{\delta} - \theta)^\varrho\big) \in [0, 1)$.*

The following inequality holds:

$$R \leqq \bar{s}_1 - \frac{\hat{\delta}_1 (1 - \hat{\theta}) \big((\hat{\delta} - \theta)^a - 1\big)}{(\hat{\delta} - \theta)^a \big((\hat{\delta} - \theta)^\varrho - 1\big)} - \frac{\hat{\delta}_1 \hat{\theta} \big((\hat{\delta} - \theta + 1)^a - 1\big)}{(\hat{\delta} - \theta + 1)^a \big((\hat{\delta} - \theta + 1)^\varrho - 1\big)}. \tag{17}$$

Note that if $\hat{\delta}_1 = 0$, then then $r_i = 0$ for all $i \geqq 2$, $\hat{\delta}_2 = \hat{\delta} = \theta = \hat{\theta} = 0$, $\bar{s}_1 = r_1$ and (17) holds. If $\hat{\delta}_1 > 0$, then taking into account that

$$\hat{\delta}_2 = s_3 - s_2 = \sum_{i=2}^{N} i^{a+\varrho} (i^\varrho - 1) r_i \geqq 2^\varrho \sum_{i=2}^{N} i^a (i^\varrho - 1) r_i = 2^\varrho (s_2 - s_1) = 2^\varrho \hat{\delta}_1,$$

we arrive at $\hat{\delta} \geqq 2$.

PROOF. Take natural m, $3 \leqq m \leqq N$, and put $i_1 = 1$, $i_2 = m-1$, $i_3 = m$. By (6) and (7), $c_i = 1 - a_1 i^a - a_2 i^{a+\varrho} - a_3 i^{a+2\varrho}$, where a_1, a_2 and a_3 are solutions of linear system

$$a_1 + a_2 + a_3 = 1,$$

$$(m-1)^a a_1 + (m-1)^{a+\varrho} a_2 + (m-1)^{a+2\varrho} a_3 = 1,$$

$$m^a a_1 + m^{a+\varrho} a_2 + m^{a+2\varrho} a_3 = 1.$$

Then we have

$$a_1 = \frac{(m^{a+2\varrho}-1)\big((m-1)^{a+\varrho}-1\big) - (m^{a+\varrho}-1)\big((m-1)^{a+2\varrho}-1\big)}{\Delta_m},$$

$$a_2 = -\frac{(m^{a+2\varrho}-1)\big((m-1)^{a}-1\big) - (m^{a}-1)\big((m-1)^{a+2\varrho}-1\big)}{\Delta_m},$$

$$a_3 = \frac{(m^{a+\varrho}-1)\big((m-1)^{a}-1\big) - (m^{a}-1)\big((m-1)^{a+\varrho}-1\big)}{\Delta_m},$$

where $\Delta_m = m^a(m-1)^a\big(m^\varrho - (m-1)^\varrho\big)\big((m-1)^\varrho - 1\big)(m^\varrho - 1)$. Considering function $f(x) = 1 - a_1 x^a - a_2 x^{a+\varrho} - a_3 x^{a+2\varrho}$, one can check that $c_i \geqq 0$ for all $i = 1, 2, \ldots, N$.

Linear system (5) is as follows.

$$r_1 + (m-1)^a r_{m-1} + m^a r_m = \bar{s}_1,$$

$$r_1 + (m-1)^{a+\varrho} r_{m-1} + m^{a+\varrho} r_m = \bar{s}_2,$$

$$r_1 + (m-1)^{a+2\varrho} r_{m-1} + m^{a+2\varrho} r_m = \bar{s}_3.$$

We have

$$r_1^* = m^a(m-1)^a\big(m^\varrho - (m-1)^\varrho\big) \times \frac{m^\varrho(m-1)^\varrho \bar{s}_1 - \big(m^\varrho + (m-1)^\varrho\big)\bar{s}_2 + \bar{s}_3}{\Delta_m},$$

$$r_{m-1}^* = -m^a(m^\varrho - 1)\frac{m^\varrho \bar{s}_1 - (m^\varrho + 1)\bar{s}_2 + \bar{s}_3}{\Delta_m},$$

$$r_m^* = (m-1)^a\big((m-1)^\varrho - 1\big)\,\frac{(m-1)^\varrho \bar{s}_1 - \big((m-1)^\varrho + 1\big)\,\bar{s}_2 + \bar{s}_3}{\Delta_m}.$$

Again making use of $r_{m-1}^* \geqq 0$ and $r_m^* \geqq 0$, we get

$$\left(\frac{\hat{\delta}_2}{\hat{\delta}_1}\right)^{1/\varrho} \leqq m \leqq 1 + \left(\frac{\hat{\delta}_2}{\hat{\delta}_1}\right)^{1/\varrho}.$$

This inequality is equivalent to $\hat{\delta} \leqq m \leqq \hat{\delta} + 1$. By the same reason as in the proof of Theorem 2, we may assume that $\hat{\delta} < m \leqq \hat{\delta} + 1$. Remember that $\hat{\delta} \geqq 2$. It follows that we can put $m = \min\{1 + [\hat{\delta}], N\}$.

It is not difficult to check that

$$r_1^* = \bar{s}_1 - \hat{\delta}_1 \left(\frac{1-\hat{\theta}}{(\hat{\delta}-\theta)^\varrho - 1} + \frac{\hat{\theta}}{(\hat{\delta}+1-\theta)^\varrho - 1}\right),$$

$$r_{m-1}^* = \frac{\hat{\delta}_1(1-\hat{\theta})}{(\hat{\delta}-\theta)^a\big((\hat{\delta}-\theta)^\varrho - 1\big)},$$

$$r_m^* = \frac{\hat{\delta}_1\hat{\theta}}{(\hat{\delta}+1-\theta)^a\big((\hat{\delta}+1-\theta)^\varrho - 1\big)}.$$

It follows from Corollary 1 that $R \leqq r_1^* + r_{m-1}^* + r_m^*$. Substituting of r_1^*, r_{m-1}^* and r_m^* in the latter inequality yields (17). □

Theorem 5 gives simpler bounds as well.

Corollary 7. *Under notations of Theorem* 5, *if* $a \leqq \varrho$, *then*

$$R \leqq \bar{s}_1 - \frac{\hat{\delta}_1(1-\hat{\theta})(\hat{\delta}^a - 1)}{(\hat{\delta}-\theta)^a(\hat{\delta}^\varrho - 1)} - \frac{\hat{\delta}_1\hat{\theta}\big((\hat{\delta}+1)^a - 1\big)}{(\hat{\delta}-\theta+1)^a\big((\hat{\delta}+1)^\varrho - 1\big)}.$$

If $a \geqq \varrho$, *then*

$$R \leqq \bar{s}_1 - \frac{\hat{\delta}_1(1-\hat{\theta})\big((\hat{\delta}-1)^a - 1\big)}{(\hat{\delta}-\theta)^a\big((\hat{\delta}-1)^\varrho - 1\big)} - \frac{\hat{\delta}_1\hat{\theta}(\hat{\delta}^a - 1)}{(\hat{\delta}-\theta+1)^a(\hat{\delta}^\varrho - 1)}.$$

For $\varrho \geqq 1$, Corollary 7 implies the next result.

Corollary 8. *Assume that* $\varrho \geqq 1$. *If* $a \leqq \varrho$, *then*

$$R \leqq \bar{s}_1 - \frac{\hat{\delta}_1(\hat{\delta}^a - 1)}{\hat{\delta}^a(\hat{\delta}^\varrho - 1)}.$$

If $a \geqq \varrho$, *then*

$$R \leqq \bar{s}_1 - \frac{\hat{\delta}_1\big((\hat{\delta} - 1)^a - 1\big)}{\hat{\delta}^a\big((\hat{\delta} - 1)^\varrho - 1\big)}.$$

Proofs of Corollaries 7 and 8 follow the same pattern as those of Corollaries 5 and 6. We omit details.

Remark 1. The right-hand side of (17) may be obtained from the right-hand side of (14) by a formal replacement $N = 1$, $\bar{\delta}_1 = -\hat{\delta}_1$, $\bar{\delta}_2 = -\hat{\delta}_2$, $\bar{\delta} = \hat{\delta}$ and $\bar{\theta} = \hat{\theta}$.

Remark 2. All inequalities of Theorems 2–5 are sharp. For each of these inequalities, there exists a set of numbers $r_1, r_2, \ldots, r_N$ such that the inequality turns to equality.

3. Bounds for probabilities of unions

In this section, we discuss bounds for probabilities of unions of events which follow from the results of Section 2. Note that these bounds may be applied to measures of unions of sets in arbitrary measurable spaces.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. For events $A_1, A_2, \ldots, A_N$, put $U = \bigcup_{i=1}^N A_i$. Denote $B_i = \{\omega \in \Omega : \omega$ belongs to exactly i events of $A_1, A_2, \ldots, A_N\}$ and $p_i = \mathbf{P}(B_i)$, $i = 0, 1, \ldots, N$. Then

$$\mathbf{P}(U) = \sum_{i=1}^N p_i.$$

The simplest application of the above method is to put $r_i = p_i$ for $i = 1, 2, \ldots, N$. Then the general results of the previous sections yield Theorems 2–5 and Corollaries 1 and 2 in Frolov (2012) and Theorem 3 and 4 and Corollaries 1–5 in Frolov (2014) that are generalizations of earlier results.

Note that one may also consider more general events that may be represented by sums $p_{i_1} + p_{i_2} + \cdots + p_{i_M}$ where $i_1 < i_2 < \cdots < i_M$, $M \leqq N$. For example, sum $p_t + p_{t+1} + \cdots + p_N$ equals to probability that at least t events from $A_1, A_2, \dots, A_N$ occur. This requires a modification of the above method and will be done elsewhere.

We now turn to another representations of $\mathbf{P}(U)$ which is a starting point of our method as well. By Lemma 1 in Kuai, Alajaji and Takahara (2000), we have

$$\mathbf{P}(U) = \sum_{k=1}^{N} \sum_{i=1}^{N} \frac{1}{i} p_{ik},$$

where $p_{ik} = \mathbf{P}(B_i A_k)$.

We also give a simple proof of the last equality. Putting $\xi_N = I_{A_1} + I_{A_2} + \cdots + I_{A_N}$, we get $\xi_N I_{B_i} = i I_{B_i}$ and

$$\sum_{k=1}^{N} \sum_{i=1}^{N} \frac{1}{i} p_{ik} = \mathbf{E}\biggl(\sum_{k=1}^{N} \sum_{i=1}^{N} \frac{1}{i} I_{B_i} I_{A_k} \biggr) = \mathbf{E}\biggl(\sum_{i=1}^{N} \frac{1}{i} I_{B_i} \xi_N \biggr)$$

$$= \mathbf{E}\biggl(\sum_{i=1}^{N} I_{B_i} \biggr) = \mathbf{P}(U).$$

For every fixed k, putting $r_{ik} = p_{ik}/i$ and

$$R_k = \sum_{i=1}^{N} r_{ik},$$

we can take bounds for R_k from our general results.

Denote

$$\bar{s}_j(k) = \sum_{i=1}^{N} i^{a+(j-1)\varrho} r_{ik}, \tag{18}$$

$$s_j(k) = \sum_{i=1}^{N} i^{j} r_{ik}, \tag{19}$$

for $1 \leqq j \leqq \ell$ and $k = 1, 2, \dots, N$.

An application of Theorem 2 yields the next result.

Theorem 6. *Define $\bar{s}_1(k)$ and $\bar{s}_2(k)$ by* (18), *$k = 1, 2, \dots, N$. Put $\bar{\delta}_k = \bigl(\bar{s}_2(k)/\bar{s}_1(k)\bigr)^{1/\varrho}$, $\theta_k = \bar{\delta}_k - [\bar{\delta}_k]$ and*

$$\bar{\theta}_k = \bigl(\bar{\delta}_k^{\varrho} - (\bar{\delta}_k - \theta_k)^{\varrho}\bigr) / \bigl((\bar{\delta}_k + 1 - \theta_k)^{\varrho} - (\bar{\delta}_k - \theta_k)^{\varrho}\bigr) \in [0, 1),$$

where $k = 1, 2, \dots, N$.

Then

(20)

$$\mathbf{P}(U) \geqq \sum_{k=1}^{N} \left\{ \frac{\bar{\theta}_k \bar{s}_1^{(a+\varrho)/\varrho}(k)}{\left(\bar{s}_2^{1/\varrho}(k) + (1-\theta_k)\bar{s}_1^{1/\varrho}(k)\right)^a} + \frac{(1-\bar{\theta}_k)\bar{s}_1^{(a+\varrho)/\varrho}(k)}{\left(\bar{s}_2^{1/\varrho}(k) - \theta_k \bar{s}_1^{1/\varrho}(k)\right)^a} \right\}.$$

For $a = \varrho = 1$, Theorem 6 implies Theorem 1 in Kuai, Alajaji and Takahara (2000). By Corollary 3, we may put $\bar{\theta}_k = \theta_k = 0$ in (20) and obtain a result in de Caen (1997).

It is clear that one can use all results from Section 2 to derive upper and lower bounds similar to that of Theorem 6. Theorem 4 implies the following result.

THEOREM 7. *Define* $\bar{s}_1(k)$, $\bar{s}_2(k)$ *and* $\bar{s}_3(k)$ *by* (18), $k = 1, 2, \dots, N$. *Put* $\bar{\delta}_{1k} = N^{\varrho}\bar{s}_1(k) - \bar{s}_2(k)$, $\bar{\delta}_{2k} = N^{\varrho}\bar{s}_2(k) - \bar{s}_3(k)$, $\bar{\delta}_k = (\bar{\delta}_{2k}/\bar{\delta}_{1k})^{1/\varrho}$, $\theta_k = \bar{\delta}_k - [\bar{\delta}_k]$ *and* $\bar{\theta}_k = (\bar{\delta}_k^{\varrho} - (\bar{\delta}_k - \theta_k)^{\varrho})/\left((\bar{\delta}_k + 1 - \theta_k)^{\varrho} - (\bar{\delta}_k - \theta_k)^{\varrho}\right) \in [0, 1)$, $k = 1, 2, \dots, N$.

The following inequality holds:

$$\mathbf{P}(U) \geqq \sum_{k=1}^{N} \left\{ \frac{\bar{\delta}_{1k}(1-\bar{\theta}_k)\left(N^a - (\bar{\delta}_k - \theta_k)^a\right)}{N^a(\bar{\delta}_k - \theta_k)^a\left(N^{\varrho} - (\bar{\delta}_k - \theta_k)^{\varrho}\right)} + \frac{\bar{\delta}_{1k}\bar{\theta}_k\left(N^a - (\bar{\delta}_k - \theta_k + 1)^a\right)}{N^a(\bar{\delta}_k - \theta_k + 1)^a\left(N^{\varrho} - (\bar{\delta}_k - \theta_k + 1)^{\varrho}\right)} + \frac{\bar{s}_1(k)}{N^a} \right\}.$$

For $a = \varrho = 1$, we obtain the next result from Corollary 6.

COROLLARY 9. *Define* $s_1(k)$, $s_2(k)$ *and* $s_3(k)$ *by* (19) *and put* $\bar{\delta}_{1k} = Ns_1(k) - s_2(k)$, $\bar{\delta}_{2k} = Ns_2(k) - s_3(k)$ *for* $k = 1, 2, \dots, N$.

The following inequality holds:

$$\mathbf{P}(U) \geqq \frac{1}{N} \sum_{k=1}^{N} \left\{ \frac{\bar{\delta}_{1k}^2}{\bar{\delta}_{2k}} + s_1(k) \right\}. \tag{21}$$

Note that for all $k = 1, 2, \dots, N$, we have

$$s_1(k) = \sum_{i=1}^{N} p_{ik} = \mathbf{E}\left(\sum_{i=1}^{N} I_{B_i} I_{A_k} \right) = \mathbf{E}\left(I_U I_{A_k}\right) = \mathbf{P}(A_k),$$

$$s_2(k) = \sum_{i=1}^N i p_{ik} = \mathbf{E}\Big(\sum_{i=1}^N i I_{B_i} I_{A_k}\Big) = \mathbf{E}\,(\xi_N I_{A_k}) = \sum_{i=1}^N \mathbf{P}(A_i A_k),$$

$$s_3(k) = \sum_{i=1}^N i^2 p_{ik} = \mathbf{E}\Big(\sum_{i=1}^N i^2 I_{B_i} I_{A_k}\Big) = \mathbf{E}\left(\xi_N^2 I_{A_k}\right) = \sum_{i=1}^N \sum_{j=1}^N \mathbf{P}(A_i A_j A_k).$$

It follows that

$$\begin{aligned} \bar{\delta}_{1k} &= \sum_{i=1}^N \mathbf{P}(\overline{A_i} A_k) = \mathbf{E}\big((N-\xi_N) I_{A_k}\big)\,, \\ \bar{\delta}_{2k} &= \sum_{i=1}^N \sum_{j=1}^N \mathbf{P}(A_i \overline{A_j} A_k) = \mathbf{E}\big(\xi_N (N-\xi_N) I_{A_k}\big)\,, \end{aligned} \tag{22}$$

for all $k = 1, 2, \ldots, N$.

Now we turn to upper bounds. The next result follows from Theorem 5.

Theorem 8. *Define* $\bar{s}_1(k)$, $\bar{s}_2(k)$ *and* $\bar{s}_3(k)$ *by* (18), $k = 1, 2, \ldots, N$. *Put* $\hat{\delta}_{1k} = \bar{s}_2(k) - \bar{s}_1(k)$, $\hat{\delta}_{2k} = \bar{s}_3(k) - \bar{s}_2(k)$, $\hat{\delta}_k = (\hat{\delta}_{2k}/\hat{\delta}_{1k})^{1/\varrho}$, $\theta_k = \hat{\delta}_k - [\hat{\delta}_k]$ *and* $\hat{\theta}_k = \big(\hat{\delta}_k^\varrho - (\hat{\delta}_k - \theta_k)^\varrho\big) / \big((\hat{\delta}_k + 1 - \theta_k)^\varrho - (\hat{\delta}_k - \theta_k)^\varrho\big) \in [0, 1)$, $k = 1, 2, \ldots, N$.

The following inequality holds:

$$\mathbf{P}(U) \leqq \sum_{k=1}^N \left\{ \bar{s}_1(k) - \frac{\hat{\delta}_{1k}(1-\hat{\theta}_k)\big((\hat{\delta}_k - \theta_k)^a - 1\big)}{(\hat{\delta}_k - \theta_k)^a \big((\hat{\delta}_k - \theta_k)^\varrho - 1\big)} - \frac{\hat{\delta}_{1k}\hat{\theta}_k\big((\hat{\delta}_k - \theta_k + 1)^a - 1\big)}{(\hat{\delta}_k - \theta_k + 1)^a \big((\hat{\delta}_k - \theta_k + 1)^\varrho - 1\big)} \right\}.$$

For $a = \varrho = 1$, we obtain the next result from Corollary 8.

Corollary 10. *Define* $s_1(k)$, $s_2(k)$ *and* $s_3(k)$ *by* (19) *and put* $\hat{\delta}_{1k} = s_2(k) - s_1(k)$, $\hat{\delta}_{2k} = s_3(k) - s_2(k)$ *for* $k = 1, 2, \ldots, N$.

The following inequality holds:

$$\mathbf{P}(U) \leqq \sum_{k=1}^N \left\{ s_1(k) - \frac{\hat{\delta}_{1k}^2}{\hat{\delta}_{2k}} \right\}. \tag{23}$$

Note that

$$\hat{\delta}_{1k} = \sum_{i=1}^N \mathbf{P}(A_i A_k) - \mathbf{P}(A_k) = \mathbf{E}(\xi_N - 1) I_{A_k}, \tag{24}$$

$$(25) \qquad \hat{\delta}_{2k} = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{P}(A_i A_j A_k) - \sum_{i=1}^{N} \mathbf{P}(A_i A_k) = \mathbf{E}\xi_N(\xi_N - 1) I_{A_k}.$$

We finally mention that Theorems 6–8 and Corollaries 9 and 10 are new result.

4. Borel–Cantelli lemmas

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $\{A_n\}$ be a sequence of events. Denote

$$\{A_n \text{ i.o.}\} = \limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

For $m \leqq n$, put $U_{mn} = \bigcup_{k=m}^{n} A_k$. Since

$$\mathbf{P}(A_n \text{ i.o.}) = \lim_{m\to\infty} \lim_{n\to\infty} \mathbf{P}(U_{mn}),$$

every new upper or lower bound allows us to derive new variant of first or second part of the Borel–Cantelli Lemma. Our results of the previous section imply that

$$Q(m,n) \leqq \mathbf{P}(U_{mn}) \leqq Q'(m,n),$$

where $Q(m,n)$ and $Q'(m,n)$ are the right-hand sides of the applied lower and upper bound, correspondingly. It is clear that

$$\limsup_{m\to\infty} \limsup_{n\to\infty} Q(m,n) \leqq \mathbf{P}(A_n \text{ i.o.}) \leqq \liminf_{m\to\infty} \liminf_{n\to\infty} Q'(m,n).$$

It may happen that we cannot find these double limits. But, if for every fixed m the inequality

$$\limsup_{n\to\infty} Q(m,n) \geqq \limsup_{n\to\infty} Q(1,n)$$

holds, then we have

$$\mathbf{P}(A_n \text{ i.o.}) \geqq \limsup_{n\to\infty} Q(1,n).$$

Similarly, if for every fixed m the inequality

$$\liminf_{n\to\infty} Q'(m,n) \leqq \liminf_{n\to\infty} Q'(1,n)$$

holds, then we get

$$\mathbf{P}\left(A_n \text{ i.o.}\right) \leqq \liminf_{n\to\infty} Q'(1,n).$$

The most applicable variants of the Borel–Cantelli lemma are proved by this way. In the proof of the first part of the classical Borel–Cantelli lemma, the upper bound for $\mathbf{P}(U_{mn})$ by s_1 is used. In the Erdős–Rényi generalization of the second part of the Borel–Cantelli lemma, the inequality with s_1 and s_2 is applied. Frolov (2012) has applied the lower bound for $\mathbf{P}(U_{mn})$, based on s_1, s_2 and s_3. This yielded a generalization of the second part of the Borel–Cantelli lemma in Theorem 9 of the last paper. Note that bounds mentioned here are constructed for probabilities $r_i = p_i$.

In this section, we present new variants of the Borel–Cantelli lemma based on inequalities (21) and (23). Note that the last inequalities are constructed from bounds for numbers $r_{ik} = p_{ik}/i$.

We start with the second part of the Borel–Cantelli lemma.

THEOREM 9. *Denote* $\xi_n = I_{A_1} + I_{A_2} + \cdots + I_{A_n}$ *and* $\eta_n = n - \xi_n$ *for all natural* n*.*

Assume that

$$\frac{1}{n}\sum_{k=1}^{n} \frac{\mathbf{E}\eta_n I_{A_k}}{\mathbf{E}\eta_n\xi_n I_{A_k}} \to 0 \quad as \quad n\to\infty.$$

Then

$$\mathbf{P}\left(A_n \ i.o.\right) \geqq \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\left\{\mathbf{P}(A_k) + \frac{\left(\mathbf{E}\eta_n I_{A_k}\right)^2}{\mathbf{E}\eta_n\xi_n I_{A_k}}\right\}.$$

It follows from (22) that

$$\mathbf{E}\eta_n I_{A_k} = \sum_{i=1}^{n} \mathbf{P}\left(\overline{A_i} A_k\right), \quad \mathbf{E}\eta_n\xi_n I_{A_k} = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{P}\left(\overline{A_i} A_j A_k\right).$$

PROOF. Inequality (21) and relation (22) yield that

$$\mathbf{P}\left(\bigcup_{k=m}^{n} A_k\right) \geqq \frac{1}{n-m+1}\sum_{k=m}^{n}\left\{\mathbf{P}(A_k) + T_k(m,n)\right\},$$

where

$$T_k(m,n) = \frac{\left(\mathbf{E}(\eta_n - \eta_{m-1}) I_{A_k}\right)^2}{\mathbf{E}(\eta_n - \eta_{m-1})(\xi_n - \xi_{m-1}) I_{A_k}}.$$

We have

$$T_k(m,n) \geqq \frac{(\mathbf{E}(\eta_n - \eta_{m-1})I_{A_k})^2}{\mathbf{E}\eta_n\xi_n I_{A_k}}$$
$$= \frac{(\mathbf{E}\eta_n I_{A_k})^2 - 2\mathbf{E}\eta_n\eta_{m-1}I_{A_k} + (\mathbf{E}\eta_{m-1}I_{A_k})^2}{\mathbf{E}\eta_n\xi_n I_{A_k}}$$
$$\geqq \frac{(\mathbf{E}\eta_n I_{A_k})^2 - 2(m-1)\mathbf{E}\eta_n I_{A_k}}{\mathbf{E}\eta_n\xi_n I_{A_k}}.$$

By (21), the inequality

$$1 \geqq \frac{1}{n}\sum_{k=1}^{n} T_k(1,n) = \frac{1}{n}\sum_{k=1}^{n} \frac{(\mathbf{E}\eta_n I_{A_k})^2}{\mathbf{E}\eta_n\xi_n I_{A_k}}$$

holds for all natural n. It implies that

$$\sum_{k=1}^{m} \frac{(\mathbf{E}\eta_n I_{A_k})^2}{\mathbf{E}\eta_n\xi_n I_{A_k}} \leqq m.$$

Hence

$$\mathbf{P}\left(\bigcup_{k=m}^{n} A_k\right) \geqq \frac{1}{n}\sum_{k=m}^{n}\left\{\mathbf{P}(A_k) + \frac{(\mathbf{E}\eta_n I_{A_k})^2}{\mathbf{E}\eta_n\xi_n I_{A_k}} - \frac{2(m-1)\mathbf{E}\eta_n I_{A_k}}{\mathbf{E}\eta_n\xi_n I_{A_k}}\right\}$$
$$\geqq \frac{1}{n}\sum_{k=1}^{n}\left\{\mathbf{P}(A_k) + \frac{(\mathbf{E}\eta_n I_{A_k})^2}{\mathbf{E}\eta_n\xi_n I_{A_k}}\right\} - \frac{2m}{n} - \frac{2(m-1)}{n}\sum_{k=1}^{n}\frac{\mathbf{E}\eta_n I_{A_k}}{\mathbf{E}\eta_n\xi_n I_{A_k}}.$$

This yields that for every fixed m,

$$\mathbf{P}\left(\bigcup_{k=m}^{\infty} A_k\right) \geqq \limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left\{\mathbf{P}(A_k) + \frac{(\mathbf{E}\eta_n I_{A_k})^2}{\mathbf{E}\eta_n\xi_n I_{A_k}}\right\}.$$

The last inequality implies the desired assertion. □

Theorem 9 in Frolov (2012) contains a lower bound for $\mathbf{P}\,(A_n \text{ i.o.})$ constructed from p_i. There is an example in Frolov (2012) which shows that this lower bound is better than previous ones. One can check that for this example, the lower bounds of Theorem 9 in Frolov (2012) and Theorem 9 of this section coincide.

Now we turn to the first part of the Borel–Cantelli lemma.

THEOREM 10. *Denote* $\xi_n = I_{A_1} + I_{A_2} + \dots + I_{A_n}$ *for all natural* n *and* $\xi_0 = 0$. *If*

$$\sum_{k=m}^{n} \frac{\mathbf{E}(\xi_n - \xi_{m-1})I_{A_k}}{\mathbf{E}(\xi_n - \xi_{m-1})^2 I_{A_k}} \to 0 \quad as \quad n \to \infty \tag{26}$$

for all sufficiently large m, *then*

$$\mathbf{P}\,(A_n \ i.o.) \leqq \limsup_{m\to\infty} \limsup_{n\to\infty} \sum_{k=m}^{n} \left\{ \mathbf{P}(A_k) - \frac{\bigl(\mathbf{E}(\xi_n - \xi_{m-1})I_{A_k}\bigr)^2}{\mathbf{E}(\xi_n - \xi_{m-1})^2 I_{A_k}} \right\}.$$

If condition (26) *holds for* $m = 1$, *then*

$$\mathbf{P}\,(A_n \ i.o.) \leqq \limsup_{m\to\infty} \limsup_{n\to\infty} \sum_{k=m}^{n} \left\{ \mathbf{P}(A_k) - \frac{(\mathbf{E}\xi_n I_{A_k})^2}{\mathbf{E}\xi_n^2 I_{A_k}} \right\}.$$

It follows from (24) and (25) that

$$\mathbf{E}\xi_n I_{A_k} = \sum_{i=1}^{n} \mathbf{P}\,(A_i A_k)\,, \quad \mathbf{E}\xi_n^2 I_{A_k} = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{P}\,(A_i A_j A_k)\,.$$

PROOF. Inequality (23) and assertions (24) and (25) imply that

$$\mathbf{P}\biggl(\bigcup_{k=m}^{n} A_k\biggr) \leqq \sum_{k=m}^{n} \bigl\{\mathbf{P}(A_k) - T'_k(m,n)\bigr\}\,,$$

where

$$T'_k(m,n) = \frac{\bigl(\mathbf{E}(\xi_n - \xi_{m-1} - 1)I_{A_k}\bigr)^2}{\mathbf{E}(\xi_n - \xi_{m-1})(\xi_n - \xi_{m-1} - 1)I_{A_k}}.$$

We have

$$T'_k(m,n) \geqq \frac{\bigl(\mathbf{E}(\xi_n - \xi_{m-1} - 1)I_{A_k}\bigr)^2}{\mathbf{E}(\xi_n - \xi_{m-1})^2 I_{A_k}}$$

$$\geqq \frac{\bigl(\mathbf{E}(\xi_n - \xi_{m-1})I_{A_k}\bigr)^2 - 2\mathbf{E}(\xi_n - \xi_{m-1})I_{A_k}}{\mathbf{E}(\xi_n - \xi_{m-1})^2 I_{A_k}}$$

$$\geqq \frac{(\mathbf{E}\xi_n I_{A_k})^2 - 2\mathbf{E}\xi_n \xi_{m-1} I_{A_k} - 2\mathbf{E}\xi_n I_{A_k}}{\mathbf{E}\xi_n^2 I_{A_k}} \geqq \frac{(\mathbf{E}\xi_n I_{A_k})^2 - 2m\mathbf{E}\xi_n I_{A_k}}{\mathbf{E}\xi_n^2 I_{A_k}}.$$

It yields that

$$\mathbf{P}\Big(\bigcup_{k=m}^{n} A_k\Big)$$
$$\leqq \sum_{k=m}^{n}\left\{\mathbf{P}(A_k) - \frac{\big(\mathbf{E}(\xi_n-\xi_{m-1})I_{A_k}\big)^2}{\mathbf{E}(\xi_n-\xi_{m-1})^2 I_{A_k}}\right\} + 2\sum_{k=m}^{n}\frac{\mathbf{E}(\xi_n-\xi_{m-1})I_{A_k}}{\mathbf{E}(\xi_n-\xi_{m-1})^2 I_{A_k}}$$
$$\leqq \sum_{k=m}^{n}\left\{\mathbf{P}(A_k) - \frac{(\mathbf{E}\xi_n I_{A_k})^2}{\mathbf{E}\xi_n^2 I_{A_k}}\right\} + 2m\sum_{k=1}^{n}\frac{\mathbf{E}\xi_n I_{A_k}}{\mathbf{E}\xi_n^2 I_{A_k}}.$$

Theorem 10 follows from the latter. □

Theorem 10 generalizes the first part of the classic Borel–Cantelli lemma. If $\{A_n\}$ are independent and series $\sum_{n=1}^{\infty}\mathbf{P}(A_n)$ diverges, then by Theorem 10, the upper bound is 1. So, the bound is sharp in this case.

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