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A HYPERVALUATION OF A HYPERFIELD ONTO A TOTALLY ORDERED CANONICAL HYPERGROUP

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Abstract

This paper attempts an exposition of the connection between valuation theory and hyperstructure theory. In this regards, by considering the notion of totally ordered canonical hypergroup we define a hypervaluation of a hyperfield onto a totally ordered canonical hypergroup and obtain some related basic results.

1. Motivation

The theory of valuations was started in 1912 by the Hungarian mathematician J. Kürschák [24]. Kürschák introduced the concept of a valuation of a field, as being real valued functions on the set of nonzero elements of the field, satisfying certain properties. Ostrowski [30], Hasse [18], Schmidt [34], and others developed this theory. These are the classical valuations. Krull [23] extended the concept of the valuation of a field, by allowing the values to be in any totally ordered abelian additive group. Such valuations are often called *Krull valuations,* the classical valuations are special Krull valuations. In fact valuation theory has its origin in Krull's paper [23]. Krull provided a basic preface to valuation theory and established the principal relations between ring theory and the theory of totally ordered groups. An abelian group $(\Gamma, +, 0)$, together with a binary relation $\leqq$ on Γ is called *totally ordered abelian group* if satisfies the following axioms: for all

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$a, b, c \in \Gamma$, (1) (reflexivity) $a \leqq a$; (2) (antisymmetry) $a \leqq b$, $b \leqq a \Longrightarrow a = b$; (3) (transitivity) $a \leqq b$, $b \leqq c \Longrightarrow a \leqq c$; (4) (totality) $a \leqq b$ or $b \leqq a$; (5) $a \leqq b \Longrightarrow a + c \leqq b + c$. The first three axioms simply say that $\leqq$ is an order relation on Γ.

Let $\Gamma = (\Gamma, +, \leqq)$ be a totally ordered abelian group and K be a field. A surjective map $v : K \to \Gamma \cup \{\infty\}$ is called a Krull valuation when the following properties are satisfied: for all $x, y \in K$, (1) $v(x) = \infty$ if and only if $x = 0$; (2) $v(xy) = v(x) + v(y)$; (3) $v(x + y) \geqq \min \left\{ v(x), v(y) \right\}$. It is understood that $\infty \notin \Gamma$, $\alpha + \infty = \infty + \alpha = \infty$ and $\alpha < \infty$ for every $\alpha \in \Gamma$.

The valuation theory and its notations can be found in many books, we refer the reader to [5], [16], [32] and [33] for general notations and definitions of valuation theory. A comprehensive review of the valuation theory can be seen in [2], [17] and [31].

Here we develop this notion to algebraic hyperstructures theory. Algebraic hyperstructures are a suitable generalization of classical structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. The hypergroup notion was introduced in 1934 by a French mathematician F. Marty [25], at the 8th congress of Scandinavian Mathematicians. He published some notes on hypergroups, using them in different contexts: algebraic functions, rational fractions, non-commutative groups. Hypergroups are studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics: geometry, topology, cryptography and code theory, graphs and hypergraphs, probability theory, binary and n-ary relations, theory of fuzzy and rough sets, automata theory, artificial intelligence, etc. The canonical hypergroups is a special class of the hypergroup. J. Mittas was the first who studied them independently from the other operations [26–28]. Some connected hyperstructures with canonical hypergroups were introduced and analyzed by P. Corsini [6, 8–10], P. Bonansinga [3, 4], K. Serafimidis [35, 36], M. Kontantinidou [20], M. De Salvo [14]. The notion of hyperring is an immediate generalization of the notion of ring. M. Krasner introduced the notions of the hyperring and the hyperfield [22]. There exist several types of hyperrings. For example, additive hyperring, multiplicative hyperring and general hyperrings. An important class of additive hyperrings is Krasner hyperrings [11, 12, 29, 37]. Surveys of the hyperstructure theory can be found in the books of P. Corsini [6], T. Vougiouklis [38], P. Corsini and V. Leoreanu [7], Davvaz and V. Leoreanu [12], B. Davvaz [13]. Alajbegovic and Mockor introduce an multirings and define the notion of an $m-$valuation on an multiring [1]. Davvaz and Salasi [15], introduced the notion of hypervaluation on a hyperring R. For this, as in the classical case we need a mapping from R onto a totally ordered group G.

In this paper, first we consider the notions of hyperstructures theory, and then we define totally ordered canonical hypergroup and by considering these notions we obtain some results. Further we define hypervaluation on a hyperfield F. For this, we need a mapping from F onto a totally ordered canonical hypergroup $(H, \circ, \leqq)$, where the element $\infty \notin H$ is adjoined in such a way that $a \circ \infty = \infty \circ a = \infty \circ \infty = \infty > a$ for all $a \in H$. Then we prove some fundamental properties of the hypervaluation.

2. Preliminaries and basic definitions

This section explains some basic definitions that have been used in the sequel. First we recall here some basic notions of hypergroup theory.

Let H be a non-empty set and $\circ : H \times H \to \wp^*(H)$ be a hyperoperation, where $\wp^*(H)$ is the family of non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid.* For any two non-empty subsets A and B of H and $x \in H$, we have $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $A \circ x = A \circ \{x\}$, $x \circ B = \{x\} \circ B$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all $a, b, c \in H$, we have $(a \circ b) \circ c = a \circ (b \circ c)$, which means that $\bigcup_{u \in a \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v$. A hypergroupoid $(H, \circ)$ is called a *quasihypergroup* if for all $a \in H$, we have $a \circ H = H \circ a = H$. This condition is also called the *reproduction axiom.* Finally a hypergroupoid $(H, \circ)$ which is both a semihypergroup and a quasihypergroup is called a *hypergroup.* We say that a semihypergroup $(H, \circ)$ is *canonical hypergroup* if: it is commutative; it has a scalar identity (also called *scalar unit*), which means that exists $e \in H$ such that $x \circ e = x$ for every $x \in R$; it is reversible, which means that if $x \in y \circ z$, then there exist the inverses y' of y and z' of z, such that $z \in y' \circ x$ and $y \in x \circ z'$. A nonempty subset K of a canonical hypergroup H is called canonical *subhypergroup* of H if K itself is a canonical hypergroup under the same hyperoperation as that of H. Equivalently, a non-empty subset K of a canonical hypergroup H is a canonical subhypergroup of H if for every $x, y \in K$, $x \circ y' \subseteqq K$ [13].

Example 1 (see [19]). Consider the hyperoperation $\circ$ on the unit interval $[0, 1]$ by

$$x \circ y = \begin{cases} \{\max\{x, y\}\}, & x \neq y, \\ [0, x], & x = y. \end{cases}$$

It can be verified that $([0, 1], \circ)$ is a canonical hypergroup. We can see that 0 is the scalar identity of $([0, 1], \circ)$ and for every x in $[0, 1]$, x is the inverse of x.

The next example constracted from Krasner's paper [21] and it is the following:

Example 2. Consider the ring $\mathbb{Z}$ and subgroup $\mathbb{N}$ of its multiplicative semigroup. The set $\mathbb{Z}/\mathbb{N} = \{a\mathbb{N} \mid a \in \mathbb{Z}\}$ with the hyperaddition given by

$$a\mathbb{N} \oplus b\mathbb{N} = \{c\mathbb{N} \mid c \in a\mathbb{N} + b\mathbb{N}\}$$

is a canonical hypergroup, where 0 is the identity of $\mathbb{Z}/\mathbb{N}$ and $-a\mathbb{N}$ is the inverse of $a\mathbb{N}$ in $\mathbb{Z}/\mathbb{N}$.

Let $(G, \circ_G)$ and $(H, \circ_H)$ be canonical hypergroups and e_G, e_H be the scalar identities of them. Let f be a mapping from G into H such that $f(e_G) = e_H$. Then, f is called: an *inclusion homomorphism* if $f(x \circ_G y) \subseteqq f(x) \circ_H f(y)$, for all $x, y \in G$; a *strong homomorphism* or for briefly a *homomorphism* if $f(x \circ_G y) = f(x) \circ_H f(y)$, for all $x, y \in G$; f is an *isomorphism* if f is an *one to one* and *onto* homomorphism. The *kernel* of f is defined by $\ker f = \{x \in G \mid f(x) = e_H\}$. A *Krasner hyperring* (in the following called just hyperring) is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms: $(R, +)$ is a canonical hypergroup; relating to the multiplication, $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x \cdot 0 = 0 \cdot x = 0$; the multiplication is distributive with respect to the hyperoperation $+$. A hyperring $(R, +, \cdot)$ is called *commutative* (with unit element) if $(R, \cdot)$ is a commutative semigroup (with unit element). A nonempty subset A of hyperring $(R, +, \cdot)$ is said to be a *subhyperring* of R if $(A, +, \cdot)$ is itself a hyperring. A hyperring $(R, +, \cdot)$ is called a: *hyperfield* if $(R \setminus \{0\}, \cdot)$ is a commutative group; *hyperdomain* if $(R, +, \cdot)$ is a commutative hyperring with unit element and $ab = 0$ implies that $a = 0$ or $b = 0$ for all $a, b \in R$.

Example 3. The set $R = \{0, 1, 2\}$ with the hyperaddition and the multiplication defined as follows is hyperfield.

$+$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{1,2\}$	R
2	$\{2\}$	R	$\{1,2\}$

$\cdot$	0	1	2
0	0	0	0
1	0	2	1
2	0	1	2

Example 4. Let $\mathbb{Q}$ be the field of rational numbers and consider subgroup $\mathbb{N}$ of its multiplicative semigroup. Then $\mathbb{Q}/\mathbb{N}$ with the hyperaddition and the multiplication given by

$$a\mathbb{N} \oplus b\mathbb{N} = \{c\mathbb{N} \mid c \in a\mathbb{N} + b\mathbb{N}\},\ \ a\mathbb{N} \cdot b\mathbb{N} = ab\mathbb{N}$$

is a hyperfield.

A nonempty subset I of a hyperring $(R, +, \cdot)$ is called a *hyperideal* if and only if $a, b \in I$ implies $a - b \subseteqq I$; $a \in I, r \in R$ imply $r \cdot a \in I$; and it is called a *prime* hyperideal if $a \cdot b \in I$ implies $a \in I$ or $b \in I$, for each $a, b \in I$. A proper hyperideal M of hyperring R is a *maximal* hyperideal of R if the only hyperideals of R that contain M are M itself and R. Let R_1 and R_2 be hyperrings. A map $f : R_1 \to R_2$ is called a *homomorphism* if the following conditions are satisfied: $f(a.b) = f(a).f(b)$ for all $a, b \in R_1$; $f(a + b) \subseteqq f(a) + f(b)$ for all $a, b \in R_1$. A map f is called an epimorphism if f is a surjective homomorphism and also if for every $a_2, b_2 \in R_2$ the following holds: $(\forall\, y \in a_2 + b_2)$ $(\exists a_1, b_1 \in R_1)$ $(\exists x \in a_1 + b_1)$, $f(a_1) = a_2$, $f(b_1) = b_2$, $f(x) = y$.

Finally, a map f is said to be an isomorphism if it is a bijective homomorphism satisfying: $f(a + b) = f(a) + f(b)$ for all $a, b \in R_1$.

3. Totally ordered canonical hypergroup

In this section, we define totally ordered canonical hypergroup and investigate some of their properties.

Definition 3.1. A canonical hypergroup $(H, \circ)$ is called *ordered* if there exists an order relation $\leqq$ on H such that $a \leqq b$ implies $a \circ c \leqq b \circ c$ for all $a, b, c \in H$. Where for $A, B \subseteqq H$, $A \leqq B$ means that for all $a \in A$ there exists $b \in B$ and for all $b \in B$ there exists $a \in A$ such that $a \leqq b$. An ordered canonical hypergroup $(H, \circ)$ is called *totally ordered* if for each $a, b \in H$ either $a \leqq b$ or $b \leqq a$.

Definition 3.2. For an ordered canonical hypergroup $(H, \circ, \leqq)$ the set $P = H^+ := \{h \in H \mid h \geqq e\}$ is called the *set of positive elements* of H (or the positive cone of H) and it has the following properties:

1. $e \in P$;
2. $P \cap P^{-1} = \{e\}$, where $P^{-1} = \{p^{-1} \mid p \in P\}$;
3. $P \circ P = P$.

Theorem 3.3. *Let $(H, \circ)$ be a canonical hypergroup and $K \subseteqq H$ be a subset such that:*

(1) $e \in K$*;*

(2) $K \cap K^{-1} = \{e\}$*;*

(3) $K \circ K = K$*.*

Then there exists an order relation $\leqq$ on H such that $(H, \circ, \leqq)$ is an ordered canonical hypergroup and $K = P$. Moreover, $(H, \circ, \leqq)$ is totally ordered if and only if $K \cup K^{-1} = H$.

Proof. We define the relation $\leqq$ as follows:

$$(\forall\, x, y \in H)\ x \leqq y \Leftrightarrow y \circ x^{-1} \cap K \neq \emptyset.$$

Since $e \in K$ and $e \in x \circ x^{-1}$ for all $x \in K$, hence $x \circ x^{-1} \cap K \neq \emptyset$; then $x \leqq x$, i.e., $\leqq$ is reflexive. Suppose that $x \leqq y$ and $y \leqq x$ for $x, y \in H$. Then $y \circ x^{-1} \cap K \neq \emptyset$ and $x \circ y^{-1} \cap K \neq \emptyset$. So $x \circ y^{-1} \cap K^{-1} \neq \emptyset$. Whence $e \in x \circ y^{-1}$, i.e., $x = y$ proving $\leqq$ is antisymmetric. Now, let $x \leqq y$ and $y \leqq z$

where $x, y, z \in H$. Then, there exist $a \in y \circ x^{-1} \cap K$ and $b \in z \circ y^{-1} \cap K$. So $y \in a \circ x$ and $y^{-1} \in z^{-1} \circ b$, then $e \in y^{-1} \circ y \subseteqq z^{-1} \circ (a \circ b) \circ x$. Hence there exist $c \in a \circ b$ and $d \in c \circ x$ such that $e \in z^{-1} \circ d$. This implies that $z = d \in c \circ x$. Hence, $c \in z \circ x^{-1}$, i.e., $a \circ b \cap z \circ x^{-1} \neq \emptyset$, therefore $z \circ x^{-1} \cap K \neq \emptyset$, which satisfies the condition for $x \leqq z$, and so $\leqq$ is transitive. Thus $\leqq$ is an order relation. Now, assume that $x \leqq y$ and $z \in H$. Since $x \leqq y$ so exists $m \in y \circ x^{-1} \cap K$ and since $\leqq$ is reflexive exists $n \in z \circ z^{-1} \cap K$. whence $m \circ n \subseteqq (y \circ z) \circ (x \circ z)^{-1}$ and $m \circ n \subseteqq P \circ P = P$. On the other hand $(y \circ z) \circ (x \circ z)^{-1} = y \circ z \circ z^{-1} \circ x^{-1} = y \circ x^{-1} \circ z \circ z^{-1}$. So $(y \circ z) \circ (x \circ z)^{-1} \cap K \neq \emptyset$. Hence $x \circ z \leqq y \circ z$. Thus $(H, \circ, \leqq)$ is an ordered canonical hypergroup. By the definition of $\leqq$ we get $x \in K$ if and only if $e \leqq x$ and so $K = P$.

Now let H be a totally ordered, then $x \leqq e$ or $e \leqq x$, for all $x \in H$. So $e \in x \circ x^{-1} \leqq e \circ x^{-1} = x^{-1}$ and so $x \in K$ or $x \in K^{-1}$, observe that $x = (x^{-1})^{-1}$. Thus $H = K \cup K^{-1}$.

Conversely, suppose that $H = K \cup K^{-1}$. If $x \nleqq y$, for $x, y \in H$, then $y \circ x^{-1} \cap K = \emptyset$. Hence $y \circ x^{-1} \cap K^{-1} \neq \emptyset$, assume that $z \in y \circ x^{-1} \cap K^{-1}$. So $z \circ z^{-1} \leqq z^{-1} \circ e = z^{-1}$, therefore $z^{-1} \in K$, proving $(H, \circ, \leqq)$ is totally ordered. □

Example 5. Considering the canonical hypergroup $\mathbb{Z}/\mathbb{N}$ in Example 2, it can be verified that $\mathbb{Z}/\mathbb{N}$ is a totally ordered canonical hypergroup with the relation $a\mathbb{N} \leqq b\mathbb{N}$ if and only if $\big(b\mathbb{N} \oplus (-a\mathbb{N})\big) \cap \big(\mathbb{N} \cup \{0\}\big) \neq \emptyset$.

Let $(G, \circ_G, \leqq_G)$ and $(H, \circ_H, \leqq_H)$ be two ordered canonical hypergroups and let $f : G \to H$ be a map. Then f is called an *order homomorphism* if it is a homomorphism and

$$(\forall\, x, y \in G) x \leqq_G y \Longrightarrow f(x) \leqq_H f(y).$$

A subhypergroup K of totally ordered canonical hypergroup H is called the *isolated* subhypergroup if for every $k \in K$, we have $\{h \in H \mid e \leqq h \leqq k\} \subseteqq K$.

We denote the set of all isolated subhypergroups of H by I_H.

Theorem 3.4. *Let H_1 and H_2 be totally ordered canonical hypergroups and K be an isolated subhypergroup of H_1.*

(1) *H_1/K is totally ordered by the relation:*

$$(h_1 \in H_1)\ h_1 \circ K \geqq e \circ K \Leftrightarrow \exists k \in K, h_1 \geqq k.$$

The mapping $\rho : H_1 \to H_1/K$ given by $\rho(h_1) = h_1 \circ K$ is an order homomorphism.

(2) *If $\phi : H_1 \to H_2$ is an order homomorphism, then* $\ker \phi$ *is an isolated subhypergroup of H_1.*

(3) *If* $\phi : H_1 \to H_2$ *is an order homomorphism, then* $H_1/K \cong Im\phi$.

Proof. In order to prove (1) define $\rho : H_1 \to H_1/K$ by $\rho(h_1) = h_1 \circ K$ and let $S = \{h \in H_1 \mid h \geqq e\}$, $\bar{S} = \rho(S)$. Observe that $\bar{S} \cap \bar{S}^{-1} = \{K\}$. Indeed, fix $h_1 \circ K, h_2 \circ K \in \bar{S}$ and let $h_1 \circ K = (h_2 \circ K)^{-1}$. Thus $h_1, h_2 \in S$ and $\rho(h_1) = \rho(h_2)^{-1}$, hence $\rho(h_1 \circ h_2) = e \circ K = K$, so $h_1 \circ h_2 \subseteqq K$. Next, since $h_1, h_2 \geqq e$, we have that $e \leqq h_1 \leqq h_1 \circ h_2$ and since K is isolated, then $h_1 \in K$. It follows that $h_1 \circ K = \rho(h_1) = e \circ K = K$. It is easy to verify that $\bar{S} \circ \bar{S} = \bar{S}$. Since $e \circ K = K \in \bar{S}$, $\bar{S} = \bar{S} \circ K \subseteqq \bar{S} \circ \bar{S}$. Now let $h_1 \circ K, h_2 \circ K \in \bar{S}$. Then $e \leqq h_1$ and $e \leqq h_2$ and so $e \leqq h_1 \circ h_2$ which implies that $h_1 \circ h_2 \subseteqq S$, hence $(h_1 \circ h_2) \circ K \subseteqq \bar{S}$. Therefore, $\bar{S} \circ \bar{S} \subseteqq \bar{S}$. In order to check that $\bar{S} \cup \bar{S}^{-1} = H_1/K$ fix $h \circ K \in H_1/K$. Then $h \in S$ or $h^{-1} \in S$, so $h \circ K \in \bar{S}$ or $(h \circ K)^{-1} \in \bar{S}$. It now follows by Theorem 3.3 that $\bar{S}$ is the positive cone of H_1/K and H_1/K is totally ordered canonical hypergroup and:

$$\begin{aligned} h \circ K \geqq e \circ K &\Leftrightarrow \rho(h) = h \circ K \subseteqq \bar{S} \\ &\Leftrightarrow (\exists a \in S)\ \rho(h) = \rho(a) \\ &\Leftrightarrow (\exists a \in S)\ (\exists k \in K)\ h \in a \circ k \\ &\Leftrightarrow (\exists k \in K)\ h \geqq k. \end{aligned}$$

Obviously ρ preserves ordering.

To prove (2) fix an arbitrary $k \in \ker\phi$ and let $h \in H_1$ be such that $e \leqq h \leqq k$. Then $e = \phi(e) \leqq \phi(h) \leqq \phi(k) = e$, so $h \in \ker\phi$ and hence $\{h \in H \mid e \leqq h \leqq k\} \subseteqq \ker\phi$.

To finish the proof of the theorem observe that by the first isomorphism theorem of canonical hypergroup there exists an isomorphism $\psi : H_1/\ker\phi \to Im\phi$ such that $\psi \circ \rho = \phi$. We shall show that ψ is order-preserving. Fix $k \in \ker\phi$ and let $k \circ \ker\phi \geqq e \circ \ker\phi$, then there exists $h \in \ker\phi$ such that $k \geqq h$. Therefore $\psi(k \circ \ker\phi) = \phi(k) \geqq \phi(h) = e$. □

4. Hypervaluation

In this section, we define the *hypervaluation* on a hyperfield and drive some basic properties of hypervaluation. Also, we show the existence of a hypervaluation for any canonical hypergroup.

Let $(F, +, \cdot)$ be a hyperfield and $(H, \circ, \leqq)$ be a totally ordered canonical hypergroup.

Definition 4.1. We define *hypervaluation* ν on a hyperfield F to be a surjective map $\nu : F \to H \cup \{\infty\}$ satisfying the following axioms: for all $x, y \in F$

1. $\nu(x) = \infty$ if and only if $x = 0$;
2. $\nu(-x) = \nu(x)$;
3. $\nu(x \cdot y) \in \nu(x) \circ \nu(y)$;
4. $z \in x + y \Longrightarrow \nu(z) \geqq \min\big\{\nu(x), \nu(y)\big\}$.

If $H = 0$ or if H is a totally ordered group, and F is a field, ν is a classical Krull valuation. Hence all Krull valuations are trivial hypervaluation.

Example 6. 1. Let p be a fixed prime number. If x is any rational number other than 0, we can write x in the form $x = p^\alpha \frac{a}{b}$, where $a, b \in \mathbb{Z}$, $p \nmid a$, and $p \nmid b$, and $\alpha \in \mathbb{Z}$.

2. The map $\mu_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$, given by $\mu_p(x) = \alpha$ for all $x \in \mathbb{Q} \setminus \{0\}$ and $\mu_p(0) = \infty$ is a trivial hypervaluation. The valuation μ_p is called p-adic valuation.

Example 7. Consider the hyperfield $\mathbb{Q}/\mathbb{N}$ in Example 4 and totally ordered canonical hypergroup $\mathbb{Z}/\mathbb{N}$ in Example 5. From the notations of Example 6(1), if $x\mathbb{N} \in \mathbb{Q}/\mathbb{N}$ other than 0, we can write $x\mathbb{N}$ in the form $x\mathbb{N} = p^\alpha \frac{a}{b}\mathbb{N}$, where $a, b \in \mathbb{Z}$, $p \nmid a$, and $p \nmid b$, and $\alpha \in \mathbb{Z}$. . We define the map $\nu_p : \mathbb{Q}/\mathbb{N} \longrightarrow \mathbb{Z}/\mathbb{N} \cup \{\infty\}$ given by $\nu_p(x\mathbb{N}) = \alpha\mathbb{N}$ for all $x\mathbb{N} \in \mathbb{Q}/\mathbb{N} \setminus \{0\}$ and $\nu_p(0) = \infty$. Now, we check that ν_p satisfies the conditions of Definition 4.1. The first two conditions follow immediately from the definition of ν_p. Furthermore, if $y\mathbb{N} = p^\beta \frac{a'}{b'}\mathbb{N}$ where $a', b' \in \mathbb{Z}$, $p \nmid a'$, $p \nmid b'$, and $\beta \in \mathbb{Z}$, then $x\mathbb{N} \cdot y\mathbb{N} = p^{\alpha+\beta}\frac{aa'}{bb'}\mathbb{N}$ where $p \nmid aa'$, $p \nmid bb'$. Therefore $\nu_p(x\mathbb{N} \cdot y\mathbb{N}) = (\alpha + \beta)\mathbb{N}$. Hence $\nu_p(x\mathbb{N} \cdot y\mathbb{N}) \in \nu_p(x\mathbb{N}) \oplus \nu_p(y\mathbb{N})$. We can easily show that the last condition is satisfied.

Lemma 4.2. *Let $\nu : F \to H \cup \{\infty\}$ be a hypervaluation on a hyperfield F. We have the following properties:*

(1) $\nu(1) = e$.

(2) $\nu(x^{-1}) = -\nu(x)$, *where for any $\alpha \in H$, $-\alpha$ is denote the inverse of α.*

(3) $\nu(x \circ y^{-1}) \in \nu(x) \circ \big(-\nu(y)\big)$.

(4) $\nu(x) \neq \nu(y) \Longrightarrow (\forall\, z \in x + y)\ \nu(z) = \min\big\{\nu(x), \nu(y)\big\}$.

Proof. (1) We have that $e \in H$, hence there exists $x \in F$ such that $e = \nu(x)$, which implies $e = \nu(x) = \nu(x.1) \in \nu(x) \circ \nu(1)$ and thus, $\nu(1) = e$.

(2) By using the identity $x.x^{-1} = 1$, we have $e = \nu(1) = \nu(x.x^{-1}) \in \nu(x) \circ \nu(x^{-1})$, hence $\nu(x^{-1}) \in e \circ \big(-\nu(x)\big) = -\nu(x)$. Therefore $\nu(x^{-1}) = -\nu(x)$.

(3) From the part (2) and Definition 4.1, we have

$$\nu(x/y) = \nu(xy^{-1}) \in \nu(x) \circ \big(-\nu(y)\big).$$

(4) Suppose $z \in x+y$ and $\nu(z) \neq \min\{\nu(x), \nu(y)\}$, we may assume that $\nu(x) < \nu(y)$. This implies $\nu(z) > \min\{\nu(x), \nu(y)\}$, in particular $\nu(z) > \nu(x)$. Since $z \in x+y$, we have $x \in z-y$ and so:

$$\nu(x) \geqq \min\{\nu(z), \nu(-y)\} = \min\{\nu(z), \nu(y)\} > \nu(x),$$

which is a contradiction. □

Definition 4.3. Let F be a hyperfield. A hyperring $R \subseteqq F$ is called the *hypervaluation hyperring,* if for any $a \in F$ we have $a \in R$ or $a^{-1} \in R$.

Theorem 4.4. *Let $\nu : F \to H \cup \{\infty\}$ be a hypervaluation on a hyperfield F.*

(1) *The set $R_\nu = \{x \in F \mid \nu(x) \geqq e\}$ is a canonical hypervaluation hyperring. We shall call it the hypervaluation hyperring associated with ν.*

(2) *The set $M_\nu = \{x \in F \mid \nu(x) > e\}$ is the only maximal hyperideal of R_ν.*

(3) *The set $U_\nu = \{x \in F \mid \nu(x) = e\}$ is a group consistent of all units of the hyperring R_ν.*

Proof. (1) First we show that R_ν is a hyperring. Let $x, y \in R_\nu$, From the Lemma 4.2, for every $z \in x-y$, we have $\nu(z) \geqq \min\{\nu(x), \nu(y)\} \geqq e$. Hence $x - y \subseteqq R_\nu$. Also we have $\nu(xy) \in \nu(x) \circ \nu(y) \geqq e$, thus $xy \in R_\nu$. To show that this is a hypervaluation hyperring fix $a \in F$ and suppose that $a \notin R_\nu$. Then $\nu(a) < e$. This implies that $\nu(a^{-1}) = -\nu(a) > e$, so $a^{-1} \in R_\nu$.

(2) Clearly, M_ν is a hyperideal of R_ν. If I is an hyperideal of R_ν and if I is not contained in M_ν, then I contains a unit of R_ν; whence $I = R_\nu$. Thus, M_ν is the unique maximal hyperideal of R_ν.

(3) It is enough to show that for every $x, y \in U_\nu$, $xy^{-1} \in U_\nu$. We have $\nu(xy^{-1}) \in \nu(x) \circ \nu(y^{-1}) = e$, hence $xy^{-1} \in U_\nu$. □

Example 8. Consider the hypervaluation ν_p of the hyperfield $\mathbb{Q}/\mathbb{N}$ in Example 7. We easily see that the hypervaluation hyperring R_{ν_p} and the maximal hyperideal M_{ν_p} of this as follows:

$$R_{\nu_p} = \left\{\frac{a}{b}\mathbb{N} \mid a, b \in \mathbb{Z},\ p \nmid b\right\} \quad M_{\nu_p} = \left\{p\frac{a}{b}\mathbb{N} \mid a, b \in \mathbb{Z},\ p \nmid b\right\}.$$

We denote by the V_F the set of all hypervaluations on the hyperfield F and denote by $[\nu]$ the element of V_F containing the hypervaluation ν.

Proposition 4.5. *Let F be a hyperfield and let $\Re_F$ be the set of all subhyperrings R of F such that*

$$\big(\forall\, x \in F^* = F \setminus \{0\}\big)\ x \notin R \Longrightarrow x^{-1} \in R.$$

Then there exists bijection $\psi : V_F \to \Re_F$ *such that*

$$\psi\big([\nu]\big) = R_\nu = \big\{ x \in F \mid \nu(x) \geqq e \big\}.$$

PROOF. We show how the inverse map $\psi^{-1} : \Re_F \to V_F$ can be constructed.

Let $R \in \Re_F$ and let U be the group of units of R. Define a system $F^*/U = \Big(\big\{ (xU) \mid x \in F^* \big\}, \circ, U, ^{-I}\Big)$, where $(xU)^{-I} = x^{-1}U$ and

$$(xU) \circ (yU) = \{zU \mid z \in xU + yU\}.$$

Then $(F^*/U, \circ)$ is a canonical hypergroup. This canonical hypergroup may be totally ordered by the Theorem 3.3, relation $\leqq$ is defined as follows

$$xU \leqq yU \Leftrightarrow (yU) \circ (x^{-1}U) \cap R/U \neq \emptyset.$$

We define a map $\nu : F^* \to F^*/U$ by $\nu(x) = xU$, and $\nu(0) = \infty$. Then ν is a hypervaluation, and $R_\nu = R$. Then we set $\psi^{-1}(R) := [\nu] \in V_F$. $\square$

PROPOSITION 4.6. *Let F be a hyperfield, H a totally ordered canonical hypergroup and $\nu : F \to H \cup \{\infty\}$ a hypervaluation. Let R be the hypervaluation hyperring associated with ν and let O_R be the set of all subhyperrings of hyperfield F containing R. Let $\nu_B : F \to H_B \cup \{\infty\}$ be a hypervaluation associated with hyperring $B \in O_R$. Then there exists exactly one inclusion homomorphism $\lambda : H \to H_B$ such that $\nu_B = \lambda \circ \nu$, λ is an order-preserving surjection and* $\ker \lambda = \nu\big(U(B)\big)$.

PROOF. Fix $B \in O_R$ and observe that $U(R) \subseteqq U(B)$. The hypervaluations ν and ν_B determine the inclusion homomorphisms $\nu : U(F) \longrightarrow H$ and $\nu_B : U(F) \longrightarrow H_B$ such that $U(R) = \ker \nu$, $U(B) = \ker \nu_B$. Hence $H \cong U(F)/U(R)$ and $H_B \cong U(F)/U(B)$. If there exists a mapping λ such that $\nu_B = \lambda \circ \nu$, then for $a \in U(F)$ must the relation $\lambda\big(aU(R)\big) = \nu_B(a)$, to be true. Therefore there exists at most one mapping such as λ. Now we show that λ is well-defined. In fact, if $aU(R) = bU(R)$ then $ab^{-1} \in U(R) \subseteqq U(B) = \ker \nu_B$. So λ is welldefined. Clearly λ is an inclusion homomorphism. Moreover, $im\lambda = im\nu_B$ and $\ker \lambda = \nu(\ker \nu_B) = \nu\big(U(B)\big)$. Since ν_B is surjective it follows that also λ is a surjection. It remains to show that λ is order-preserving. Fix $h \in H$ and let $h \geqq e_H$. Since ν is a surjection, there exists $x \in F$ such that $h = \nu(x)$. Thus since $\nu(x) = h \geqq e$, we have that $x \in R_\nu \subseteqq B$. Suppose that $\lambda(h) < e_{H_B}$. Then $e_{H_B} > \lambda(h) = \lambda\big(\nu(x)\big) = \nu_B(x)$, so $x \notin B$ is a contradiction. $\square$

If $\nu : F \to H \cup \{\infty\}$ be a hypervaluation on a hyperfield F, for $K \in I_H$, we put:

$$R_\nu(K) := \{x \in K \mid \nu(x) \in K^+\}.$$

PROPOSITION 4.7. *Let H_1 and H_2 be totally ordered canonical hypergroups and let $f : H_1 \cup \{\infty\} \longrightarrow H_2 \cup \{\infty\}$, where $f(\infty) = \infty$ be an ordered homomorphism. Further, let $\nu : F \to H_1 \cup \{\infty\}$ be a hypervaluation. Then $f\nu := f \circ \nu$ is a hypervaluation. If we suppose that $K \in I_{H_1}$ and f is an ordered epimorphism then $f(K) \in I_{H_2}$ and $R_\nu(K) = R_{f\nu}\big(f(K)\big)$.*

PROOF. Since ν is a hypervaluation and f is an ordered homomorphism the conditions of Definition 4.1 for $f\nu$ are satisfied and $f\nu$ is a hypervaluation. Suppose that f is an ordered epimorphism. Then $f(K)$ is an isolated subhypergroup in H_2. In fact let $a, b \in f(K)$ then exists $k_1, k_2 \in K$ such that $a = f(k_1), b = f(k_2)$. Hence $a \circ b^{-1} = f(k_1) \circ f(k_2)^{-1} = f(k_1 \circ k_2^{-1}) \subseteq f(K)$ so $f(K)$ is a subhypergroup of H_2. Suppose now that $a = f(k_1) \leqq c \leqq b = f(k_2)$. Then exist $k \in H_1$ such that $c = f(k)$ and hence we have $f(k_1) \leqq f(k) \leqq f(k_2)$ and since f is order epimorphism so $k \in K$. Hence $z = f(k) \in f(K)$, proving $f(K)$ is isolated. Let $x \in R_\nu(K)$. Then $\nu(x) \in K^+$ and so exist $k \in K^+$ such that $\nu(x) = k$. Then $f\nu(x) = f(k) \in f(K)^+$. Hence $R_\nu(K) \subseteq R_{f\nu}\big(f(K)\big)$. Now if $x \in R_{f\nu}\big(f(K)\big)$, then $f\nu(x) = f\big(\nu(x)\big) \in f(K)^+$ which implies that $\nu(x) \in K^+$. Hence $x \in R_\nu(K)$ and so $R_{f\nu}\big(f(K)\big) \subseteq R_\nu(K)$. □

Consider the totally ordered canonical hypergroup $\mathbb{Z}/\mathbb{N}$ in Example 5 and map μ_p in Example 6. According to the above theorem, we express the following example.

EXAMPLE 9. Let us define a map ρ from $\mathbb{Z}$ onto $\mathbb{Z}/\mathbb{N}$ by setting $\rho(a) = a\mathbb{N}$. It is clear that ρ is a inclusion homomorphism and onto. We consider the following composition of maps:

$$\mathbb{Q}^* \xrightarrow{\mu_p} \mathbb{Z} \xrightarrow{\rho} \mathbb{Z}/\mathbb{N}$$

The map $\nu = \rho \circ \mu_p$ with $\nu(0) = \infty$ is a hypervaluation on $\mathbb{Q}$.

THEOREM 4.8. *Let $\nu : F \to H \cup \{\infty\}$ be a hypervaluation on hyperfield F and let R be the corresponding hypervaluation hyperring. Then there exists a bijective order-preserving correspondence between the set I_H of isolated subhypergroups of H and the set O_R of all subhyperrings of hyperfield F containing R.*

PROOF. Let $B \in O_R$. Then $U(B) \supseteqq U(R)$ and there exists an order-epimorphism $\lambda : F^*/U(R) \to F^*/U(B)$, where for any $\bar{x} \in F^*/U(R)$, $\lambda(\bar{x}) = \bar{x}U(B)$, and ordering of these canonical hypergroups as in the proof of Proposition 4.5. Since the maps $\nu_R : F^* \to F^*/U(R)$ and $\nu_B : F^* \to F^*/U(B)$ are hypervaluation, we may identify H with $F^*/U(R)$ and $F^*/U(B)$ with H/H_B, where H_B is the kernel of λ. Hence, we define a map $\gamma : O_R \to I_H$ such that $\gamma(B) = H_B$.

Conversely, if $E \in I_H$, we consider the following composition of maps:

$$F^* \xrightarrow{\nu} H \xrightarrow{\rho} H/E$$

where for any $x \in H$, $\rho(x) = xE$, it is clear that $\rho \circ \nu := \nu'$ with $\nu'(0) = \infty$ is a hypervaluation on F and we may define a map $w : I_H \to O_R$ such that $w(E) = R_{\nu'} \supseteqq R$. It is easy to see that these maps are order-preserving and mutually inverse. □

Definition 4.9. Two hypervaluations ν_1, ν_2 on a hyperfield F are said to be equivalent, notationally $\nu_1 \equiv \nu_2$, if and only if $(R_{\nu_1}, M_{\nu_1}) = (R_{\nu_2}, M_{\nu_2})$.

Theorem 4.10. *Let F be a hyperfield and ν_1 and ν_2 hypervaluations on F. Then the following statements hold:*

(1) $R_{\nu_1} = R_{\nu_2} \subsetneqq F \Rightarrow M_{\nu_1} = M_{\nu_2}$*;*

(2) $M_{\nu_1} = M_{\nu_2} \Rightarrow R_{\nu_1} = R_{\nu_2}$*;*

(3) $\nu_1 \equiv \nu_2$ *if and only if there exists an order-preserving isomorphism f such that*

$$f : H_{\nu_1} \cup \{\infty\} \longrightarrow H_{\nu_2} \cup \{\infty\}, \quad f(\infty) = \infty, \quad \nu_2 = f \circ \nu_1.$$

Proof. The proof is similar to the proof of lemma 2.1.7 in [2], by considering the suitable modification with using the Definition 4.1 and 4.9 and Proposition 4.4. □

Definition 4.11. Let $(H, \circ, \leqq)$ be a totally ordered canonical hypergroup. We say that a subset U of the set H is an *upper class* if:

1. $\emptyset \neq U \neq H$,
2. $\alpha \in U$, $\beta \in H$ with $\alpha < \beta$, implies $\beta \in U$.

Example 10. For any totally ordered canonical hypergroup H, the set of positive element of H or the positive cone of H is an upper class of H.

Definition 4.12. An upper class U of $(H, \circ, \leqq)$ is called a *prime upper class* if $U \subset H' = \{\gamma \in H \mid e < \gamma\}$ and if $\alpha, \beta \notin U$, then $\alpha \circ \beta \nsubseteq U$.

Theorem 4.13. *Let $\nu : F \to H \cup \{\infty\}$ be a hypervaluation on hyperfield F. If I is a nonzero hyperideal of R_ν, then $\nu(I) \setminus \infty$ is a prime upper class of H if and only if I is a prime hyperideal.*

Proof. First, suppose that $\nu(I) \setminus \infty$ is a prime upper class of H. If $a \notin I, b \notin I$, then $\nu(a) \notin \nu(I), \nu(b) \notin \nu(I)$, so $\nu(a) \circ \nu(b) \nsubseteq \nu(I)$. Hence $\nu(ab) \notin \nu(I)$, therefore $ab \notin I$. Thus, I is a prime hyperideal. For the converse, assume that I is a prime hyperideal. If $\alpha, \beta \notin \nu(I)$, then exist

$a, b \in K \setminus I$ such that $\alpha = \nu(a), \beta = \nu(b)$. Thus from the definition of hypervaluation ν we have $\nu(ab) \in \nu(a) \circ \nu(b)$. But since I is a prime hyperideal $ab \notin I$. Therefore $\nu(ab) \notin \nu(I)$. Hence $\alpha \circ \beta \nsubseteq \nu(I)$. □

EXAMPLE 11. Consider the maximal hyperideal M_{ν_p} in Example 8. Then according to the above theorem $\nu(M_{\nu_p}) \setminus \infty$ is a prime upper class of $\mathbb{Z}/\mathbb{N}$.

We recall that a hyperfield $(F, +, \cdot)$ with a total order $\leqq$ on F is an ordered hyperfield if satisfies the following properties: (1) if $a \leqq b$ then $a + c \leqq b + c$; (2) if $0 \leqq a$ and $0 \leqq b$ then $0 \leqq a \cdot b$.

THEOREM 4.14. *Let $(H, \circ, \leqq)$ is any totally ordered canonical hypergroup, then there exists a hyperfield F and a hypervaluation ν on F with $\nu(F^*) = H$.*

PROOF. Let K be an arbitrary ordered hyperfield, and let R be the set of order-preserving mappings $f : H \to K$ such that the set

$$\sigma(f) = \big\{ \gamma \in H \mid f(\gamma) \neq 0 \big\}$$

is finite. We define the hyperoperations and multiplication on R as follows:

$$(f + g)(\gamma) = f(\gamma) + g(\gamma),$$

$$(fg)(\gamma) = \sum_{\gamma \in \gamma_1 \circ \gamma_2} f(\gamma_1) g(\gamma_2).$$

Then R becomes a commutative hyperring with unit element e, where $e(0) = 1$ and $e(\gamma) = 0$ for all $\gamma \neq 0$. Moreover, R is a hyperdomain, because if $f, g \neq 0$, if γ_1 is the smallest element of $\sigma(f)$, and if γ_2 is the smallest element of $\sigma(g)$, then

$$(fg)(\gamma_1 \circ \gamma_2) = \sum_{\gamma_1 \circ \gamma_2 \subseteqq \gamma_3 \circ \gamma_4} f(\gamma_3) g(\gamma_4) \neq 0,$$

so $fg \neq 0$. In fact the smallest element of $\sigma(fg)$ is a member of $\gamma_1 \circ \gamma_2$.

Let $\nu : R \to H$ be defined by $\nu(f)$= the smallest element of $\sigma(f)$, when $f \neq 0$ and $\nu(0) = \infty$. Then $\nu(fg) \in \nu(f) \circ \nu(g)$ as seen above, for all $h \in (f + g)$, $\nu(h) \geqq \min \big\{ \nu(f), \nu(g) \big\}$ as in easily checked. Let F be the hyperfield of fractions of R. We extend ν to F by letting $\nu\big(\frac{f}{g}\big) \in \nu(f) \circ \big(-\nu(g) \big)$. Then ν is a hypervaluation on F. □

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