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THE PRINCIPAL FIBRE BUNDLE ON LORENTZIAN ALMOST R-PARA CONTACT STRUCTURE

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ABSTRACT

The purpose of this paper is to study the principal fibre bundle (P, M, G, π_p) with Lie group G, where M admits Lorentzian almost paracontact structure (ϕ, ξ_p, η_p, g) satisfying certain condtions on $(1, 1)$ tensor field J, indeed possesses an almost product structure on the principal fibre bundle. In the later sections, we have defined trilinear frame bundle and have proved that the trilinear frame bundle is the principal bundle and have proved in Theorem 5.1 that the Jacobian map π^* is the isomorphism.

KEYWORDS

principal fibre bundle, Lie group, trilinear frame, Cartesian product manifold, linear connections, almost product structure, projection map

MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 53D15; Secondary 57R25

1. INTRODUCTION

The theory of principal fibre bundles was developed by S. Kobayashi [7], S. Kobayashi and K. Nomizu [6], Bishop and Critterdon [1] and many others. Principal fibre bundles have played an important role in differential geometry and mathematical physics. Motivated by the works of Boothby-Wong [2] and K. Taleb [9], the authors have defined and studied Lorentzian almost r-paracontact structure [5]. In Section 2, we recall basic definitions of principal fibre bundles, Lie algebra and general linear group. Section 3 deals with Lorentzian almost r-para-contact structure and almost product structure. In Section 4, we have proved that the trilinear frame bundle is the principal fibre bundle while the last Section is dedicated to proving that the horizontal subspace is isomorphic to the tangent space of the product manifold by the Jacobian map of the projection map.

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2. PRELIMINARIES

A set (P, M, G, π) is called the principal fibre bundle if P is a differentiable manifold, G a Lie group and

(i) G acts on P differentiably to the right i.e. there exists a differentiable map $P \times G \longrightarrow P$ such that $(u, g) \longrightarrow ug$ where $u \in P, g \epsilon G$ and $ug \in P$ where $(ug)h = u(gh)$ for all $g, h \in G$.

(ii) M is the quotient manifold $P = G$ and the projection map: P/M is differentiable.

(iii) For each $x \in M$ and for every neighborhood U of x, the set $\pi^{-1}(U)$ is isomorphic to $U \times G$ [4].

Let G be a C^∞ manifold. If G is a group and the maps

(i) $G \times G \longrightarrow G$ such that $(g_1, g_2) \longrightarrow g_1 g_2$ and

(ii) $G \longrightarrow G$ such that g, g^{-1} are differentiable, then G is called the Lie group. Here g_1, g_2 and g are arbitrary elements of G.

If $Gl(n, R)$ be the set of all $n \times n$ non-singular matrices over R, then $Gl(n, R)$ is a group under matrix multiplication. If $g \in Gl(n, R), g$ can be expressed as $(g^a_b), g^a_b \in R$. g^a_b can be treated as coordinates and induce the manifold structure in $Gl(n, R)$ and $Gl(n, R)$ is a Lie group. It is called general linear group.

Let G be a Lie group and let $g \longrightarrow G$. Then the map $L_g : \ G \longrightarrow G$ such that $L_g(h) = gh, h \in G$ is an automorphism of G called left translation. If $X$is a vector field on G such that $(Lg)_* X = X, X$ is called left invariant vector field where $(Lg)_*$ denotes Jacobian map of Lg.

Let G be a Lie group and S be the set of vector fields over G. If for all $X, Y \in S, [X; Y] \in S$ such that

$$\begin{aligned} &\text{(i)} && [X, Y] = -[Y, X] \\ &\text{(ii)} && [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \end{aligned} \tag{2.1}$$

for all $X, Y, Z \in S$, then S is called Lie algebra over G. It is easy to show that the set of all left invariant vector fields form the Lie Algebra over G [10].

3. LORENTZIAN ALMOST R-PARA-CONTACT STRUCTURE

Let M be a differentiable manifold of C^∞ class and $T(M)$ denotes the tangent bundle of M. Suppose that there are given a tensor field ϕ of type (1, 1), a vector field ξ_p and a 1-form $\eta_p, p = 1, 2, \dots, r$ satisfying [3]

$$\begin{aligned} &\text{(i)} && \phi^2 = I - \sum_{p=1}^{r} \xi_p \otimes \eta_p \\ &\text{(ii)} && \phi \xi_p = 0 \\ &\text{(iii)} && \eta_p \circ \phi = 0 \\ &\text{(iv)} && \eta_p(\xi_q) = \delta_{pq} \end{aligned} \tag{3.1}$$

where $p = 1, 2, \dots, r$ and δ_{pq} denote the Kronecker delta. Thus the manifold M satisfying conditions (3.1) will be said to possess Lorentzian almost r-para-contact structure [5, 8].

THEOREM 3.1. Let (P, M, G, π_p) be the principal fibre bundle with Lie group G. The $(1, 1)$ tensor field J satisfying

$$\text{(i)} \qquad \pi(JX) = \phi\pi X - \left\{ a\bar{\eta_p}(\omega X) + b\eta_p(\pi X) \right\} \xi_p \tag{3.2}$$

and

$$\text{(ii)} \qquad \omega(JX) = \phi\omega X - \left\{ a^{-1}(1 - b^2)\eta_p(\pi X) - b\bar{\eta_p}(\omega X) \right\} \xi_p \tag{3.3}$$

gives an almost product structure on P.

Proof. Let (P, M, G, π_p) be the principal fibre bundle with Lie group G and the projection map π_p and let ω be the connection 1-form in P. Let u^h and $A*$ be lift of $u \in \chi(M)$ and the fundamental vector field with respect to $A \in \wp$ where $\wp$ is the Lie algebra of the Lie group G. Suppose further that M admits Lorentzian almost r-paracontact structure (ϕ, ξ_p, η_p, g). Let $(\bar{\phi}, \bar{\xi_p}, \bar{\eta_p})$ be the left invariant

Lorentzian almost r-paracontact structure over the Lie group G. Let ω be connection 1-form on P. For a tensor field J of type $(1,1)$ on P, we define the structure on M as

$$\text{(i)} \qquad \pi(JX) = \phi\pi X - \left\{ a\bar{\eta}_p(\omega X) + b\eta_p(\pi X) \right\} \xi_p \tag{3.4}$$

and

$$\text{(ii)} \qquad \omega(JX) = \phi\omega X - \left\{ a^{-1}(1-b^2)\eta_p(\pi X) - b\bar{\eta}_p(\omega X) \right\} \xi_p \tag{3.5}$$

where a and b are real numbers. Then we can easily verify that

$$\pi(J^2X) = \pi X \tag{3.6}$$

and

$$\omega(J^2X) = \omega X \tag{3.7}$$

hence J gives an almost product structure on P. □

THEOREM 3.2. Let (P, M, G, π_p) be the principal fibre bundle with Lie group G. The $(1,1)$ tensor field J satisfying

$$\text{(i)} \qquad JX^* = (\bar{\phi}A)^* - a\eta_p(A)\xi_p^h - b\bar{\eta}_p(\bar{A})\xi_p^* \tag{3.8}$$

$$\text{(ii)} \qquad Ju^* = (\phi u)^h + b\eta_p(u)\xi_p^h - a^{-1}(1-b^2)\eta_p(u)\xi_p^* \tag{3.9}$$

gives an almost product structure on P.

Proof. For $A, B \in S$, S the Lie algebra over the Lie group G and for $u, v \in \chi(M)$, we have [3]

$$\begin{aligned} &\text{(i)} && [A^* u^h] = 0 \\ &\text{(ii)} && [A^*, B^*] = [A, B]^* \\ &\text{(iii)} && [u^h, v^h] = -\Omega(u^h, v^h) \end{aligned} \tag{3.10}$$

where A^* is the fundamental vector field with respect to A, u^h the horizontal lift and Ω the curvature form of the connection.

For the fundamental vector field A^* and the horizontal lift u^h, let us define

$$\begin{aligned} &\text{(i)} && JX^* = (\bar{\phi}A)^* - a\eta_p(A)\xi_p^h - b\bar{\eta}_p(\bar{A}\xi_p^* \\ &\text{(ii)} && Ju^* = (\phi u)^h + b\eta_p(u)\xi_p^h - a^{-1}(1-b^2)\eta_p(u)\xi_p^* \end{aligned} \tag{3.11}$$

then we can easily to verify that

$$\begin{aligned} &\text{(i)} && J^2A^* = A^* \\ &\text{(ii)} && J^2\xi_p^h = \xi_p^h \end{aligned} \tag{3.12}$$

Hence J again gives an almost product structure on P. □

4. TRILINEAR FRAME BUNDLE

Let M_1, M_2, M_3 be three C^∞ manifolds each of dimension n. If $x \in M_1, y \in M_2$ and $z \in M_3$ then (x, y, z) is the point of cartesian product manifold $M_1 \times M_2 \times M_3$. Let $(x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n)$ and $(z^1, z^2, \ldots, z^n)$ be local coordinate systems about x, y, z in the manifolds M_1, M_2 and M_3 respectively. Then $\left\{(x^i, y^j, z^k), 1 \le i, j, k \le n\right\}$ of n^3 triplets forms the local coordinate system about (x, y, z) in the product manifold. The canonical basis vectors about (x, y, z) in $M_1 \times M_2 \times M_3$ are $\left\{ \left(\frac{\partial}{\partial x^i}\right), \left(\frac{\partial}{\partial y^j}\right), \left(\frac{\partial}{\partial z^k}\right), 1 \le i, j, k \le n \right\}$.

If U_a, V_b and W_c be tangent vectors to M_1, M_2, M_3 at points x, y, z respectively, we can write

$$U_\alpha = U_a^i \frac{\partial}{\partial x^i}, \quad V_\alpha = V_b^j \frac{\partial}{\partial y^j}, \quad W_\alpha = W_a^k \frac{\partial}{\partial z^k} \tag{4.1}$$

It is easy to show that the set $\{U_a, V_b, W_c, 1 \le a, b, c \le n\}$ is the set of n^3 vectors forming the basis of tangent space of the product manifold $M_1 \times M_2 \times M_3$.

Let us call the set $\left\{x^i, y^j, z^k, U^i_a, V^j_b, W^k_c\right\}$ as the trilinear frame at (x, y, z) of $M_1 \times M_2 \times M_3$. We denote by TL the set of the trilinear frames at different points of the product manifold $M_1 \times M_2 \times M_3$. It can be shown that TL is also a differentiable manifold.

Let $u = (x^i, y^j, z^k, U^i_a, V^j_b, W^k_c) \in TL$ and $g, h, p \in Gl(n, R)$ where $g = (g^a_b), h = (h^b_c), p = (p^c_d)$. We define action of $Gl(n, R) \times Gl(n, R) \times Gl(n, R)$ on TL as follows

$$((x^i, y^j, z^k, U^i_a, V^j_b, W^k_c), ((g^a_b), (h^b_c), (p^c_d)) \longrightarrow (x^i, y^j, z^k, U^i_a g^a_d, V^j_b h^b_c, W^k_c p^c_f) \tag{4.2}$$

It is mapping

$$TL \times Gl(n, R) \times Gl(n, R) \times Gl(n, R) \longrightarrow TL.$$

Let us call the set

$$\{TL, M_1 \times M_2 \times M_3, \pi, Gl(n, R) \times Gl(n, R) \times Gl(n, R)\}$$

the trilinear frame bundle over the product manifold where π is projection map.

THEOREM 4.1. The trilinear frame bundle is the principal fibre bundle.

Proof. To show that $\{TL, M_1 \times M_2 \times M_3, \pi, Gl(n, R) \times Gl(n, R) \times Gl(n, R)\}$ is the principal fibre bundle, we have to show

(i) $Gl(n, R) \times Gl(n, R) \times Gl(n, R)$ acts on TL differentiably to the right. It is obvious by the equation (2.1). We can also define

$$(x^i, y^j, z^k, U^i_a g^a_d, V^j_b h^b_c, W^k_c p^c_f)\left((q^a_b), (r^b_c), s^c_d\right) = (x^i, y^j, z^k, U^i_a, V^j_b, W^k_c)\left(g^a_d q^d_c, h^b_f r^f_l, p^c_m s^m_n\right).$$

Hence $Gl(n, R) \times Gl(n, R) \times Gl(n, R)$ acts on TL differentiably to the right.

(ii) $M_1 \times M_2 \times M_3$ can be treated as the quotient manifold $TL/Gl(n, R) \times Gl(n, R) \times Gl(n, R)$ and the projection map $\pi : TL \longrightarrow M_1 \times M_2 \times M_3$ is differentiable.

(iii) Let $(x, y, z) \in M_1 \times M_2 \times M_3$ and U be coordinate neighbourhood of (x, y, z) in the product manifold. $\left\{(x^i, y^j, z^k), 1 \leq i, j, k \leq n\right\}$ is the local coordinate system in U. As $\pi^{-1}(U) \subset TL$, we can take

$$\pi^{-1}(U) = \left\{(x^i, y^j, z^k, g^a_b, h^c_d, p^c_f)\right\}.$$

As $(x^i, y^j, z^k) \in U$ and $(g^a_b, h^c_d, p^c_f) \in Gl(n, R) \times Gl(n, R) \times Gl(n, R)$ so

$$(x^i, y^j, z^k, g^a_b, h^c_d, p^c_f) \in U \times Gl(n, R) \times Gl(n, R) \times Gl(n, R).$$

We can take the identity map

$$I : \pi^{-1}(U) \longrightarrow U \times Gl(n, R) \times Gl(n, R) \times Gl(n, R)$$

such that $(x^i, y^j, z^k, g^a_b, h^c_d, p^c_f) \longrightarrow (x^i, y^j, z^k, g^a_b, h^c_d, p^c_f)$. Since identity map is always the isomorphism so $\pi^{-1}(U)$ is isomorphic to $U \times Gl(n, R) \times Gl(n, R) \times Gl(n, R)$. Thus the trilinear frame bundle becomes the principal fibre bundle. □

5. LINEAR CONNECTIONS

Let $\{TL, M_1 \times M_2 \times M_3, \pi, Gl(n, R) \times Gl(n, R) \times Gl(n, R)\}$ be the trilinear frame bundle and for $u \in TL$, Tu be tangent space of TL at u. Let L^v_u be the subspace of Tu such that

$$L^v_u = \{X \in T|\pi^*(X) = 0\},$$

where π^* is the jacobian map of π.

A connection Γ on TL is an assignment of the subspace Γ^h_u of Tu such that

$$Tu = L^v_u + \Gamma^h_u \tag{5.1}$$

Further

$$T_u g^h = (R_g) * \Gamma^h_u \tag{5.2}$$

where R_g is the right translation. Let us call L^v_u and Γ^h_u as vertical and horizontal subspaces of Tu. We can express arbitrary $X \in Tu$ as

$$X = Y + Z, \tag{5.3}$$

where $Y \in L^v_u$ and $Z \in \Gamma^h_u$.

If $Z = 0, X$ is called vertical vector field and if $Y = 0, X$ is horizontal vector field.

THEOREM 5.1. The horizontal subspace Γ_u^h is isomorphic to the tangent space T_p of the product manifold at $p = \pi(u)$ by the Jacobian map π^* of the projection map.

Proof. Since π is the map $TL \longrightarrow M_1 \times M_2 \times M_3$ so π_* is the linear transformation $Tu \longrightarrow T_p$. But $\pi^* T_u = \pi^* \Gamma_u^h$ as $\pi^* L_u^v = 0$. Thus $\pi^* \Gamma_u^h \subset T_p$ or $\pi^* Tu \subset T_p$.

Now for arbitrary $X \in Tp$, we can construct a tangent vector $(X; 0; 0; 0)$ in $U \times Gl(n, R) \times Gl(n, R) \times Gl(n, R)$, 0 null matrix of $Gl(n; R)$ and U the coordinate neighborhood of p in $M_1 \times M_2 \times M_3$. Since trilinear frame bundle is the principal fibre bundle, $\pi^{-1}(U)$ is isomorphic to $U \times Gl(n, R) \times Gl(n, R)$ and isomorphism is the identity map. So we can assume that $(X, 0, 0, 0) \in Tu$ and

$$\pi^*(X, 0, 0, 0) = X.$$

Thus $Tp \subset \pi^* Tu$ and therefore

$$Tp = \pi^* Tu = \pi^* \Gamma_u^h.$$

π^* will be isomorphism if we show that it is one to one. Let $X, Y \in \Gamma_u^h$ all arbitrary and

$$\pi * X = \pi * Y \Longrightarrow \pi * (X - Y) = 0 \Longrightarrow X - Y \in L_u^v.$$

Since $X, Y \in \Gamma_u^h \Rightarrow X - Y \in \Gamma_u^h$ and consequently

$$X - Y = 0 \Longrightarrow X = Y.$$

Hence π^* is the isomorphism and the theorem is proved. □

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