

## PASCAL'S HEXAGRAM AND DESARGUES CONFIGURATIONS

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Communicated by László Tóth

Original Research Article

Received: Feb 21, 2020 · Accepted: Nov 23, 2020

First published online: Mar 24, 2021

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### ABSTRACT

This paper solves an enumerative problem which arises naturally in the context of Pascal's hexagram. We prove that a general Desargues configuration in the plane is associated to six conical sextuples via the theorems of Pascal and Kirkman. Moreover, the Galois group associated to this problem is isomorphic to the symmetric group on six letters.

### KEYWORDS

Pascal's hexagram, Desargues configuration

### MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 14N05; Secondary 51N35

## 1. INTRODUCTION

This paper solves an enumerative problem which arises naturally in the context of Pascal's hexagram. We begin with a short statement of the problem; however, all of the underlying concepts will be explained later in substantially more detail.

### 1.1.

Let  $\mathbb{P}^2$  denote the projective plane over the complex numbers. Consider the following two *a priori* distinct geometric facts:

- (1) A Desargues configuration in the plane depends on 11 parameters.
- (2) Suppose that we have a nonsingular conic  $\mathcal{K}$  in the plane, together with six distinct points  $A, B, C, D, E, F$  on  $\mathcal{K}$ . This datum also depends on 11 parameters (five for the conic, and one each for the six points). By applying Pascal's theorem to all possible arrangements of these points, we get altogether sixty Pascal lines. Now it is a consequence of Kirkman's theorem that these lines naturally separate themselves into six Desargues configurations  $\Delta_1, \Delta_2, \dots, \Delta_6$ .

This raises the following natural question. Given an arbitrary Desargues configuration  $\Gamma$  in the plane, can we find a conic  $\mathcal{K}$  together with six points  $A, \dots, F \in \mathcal{K}$  such that  $\Gamma$  coincides with one of the  $\Delta_i$ ? Since the number of *required* parameters is equal to the number of *available* parameters,

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our intuition suggests that this should be achievable in *finitely many* ways. The main result of the paper determines this number.

**THE MAIN THEOREM (Preliminary Version).** For a general choice of  $\Gamma$ , there are **six** essentially distinct solutions to this problem.

Permuting the points  $A, \dots, F$  permutes the  $\Delta_i$ , hence the phrase ‘essentially distinct’ needs to be clarified. This will be done in Sections 3.1–3.4, to be followed by the final version of the main theorem in Section 3.8. Moreover, we will also prove that the Galois group associated to this enumerative problem is isomorphic to the symmetric group  $\mathfrak{S}_6$ .

The calculations necessary for proving the main theorem are too cumbersome to do by hand, since they involve colon ideals and Gröbner bases. All of these were done in MAPLE and confirmed in MACAULAY-2. In several places, we will have to argue that a certain property holds for a ‘general’ object of some kind. Since the property in question will be a topologically open condition, it will suffice to check the claim on a specific example.

## 1.2. Collateral reading

The literature on Pascal’s hexagram is very large. Two of the most comprehensive classical accounts are given in ‘Note II: On the *hexagrammum mysticum* of Pascal’ appended to Baker [3], and the ‘Notes’ at the end of Salmon’s treatise [16]. An excellent modern introduction to this subject may be found in the article by Conway and Ryba [5]. We refer the reader to the standard works by Coxeter [8] and Seidenberg [17] for foundational notions in projective geometry.

## 2. THE VERONESE DECOMPOSITION

We continue with a more detailed discussion of Desargues configurations and how they arise in the context of Pascal’s hexagram.<sup>1</sup>

### 2.1. The Desargues Configuration

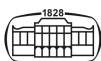
Desargues theorem says that two triangles which are in perspective from a point are also in perspective from a line (see [17, Ch. I]). This leads to the Desargues configuration consisting of ten points and ten lines, such that each point lies on exactly three lines and each line contains exactly three points. One can systematically label its points and lines in such a way that the incidence relations between them become transparent (see Diagram 1 on page 23).

Consider the set  $\text{FIVE} = \{1, 2, 3, 4, 5\}$ . The points are labelled by 3-element subsets<sup>2</sup> of FIVE, as in  $M[123], M[245]$  etc. The lines are labelled by 2-element subsets of FIVE, as in  $\ell[12], \ell[45]$  etc. The line  $\ell[\alpha]$  contains the point  $M[\beta]$  iff  $\alpha \subset \beta$ . The triangles  $M[124] - M[234] - M[134]$  and  $M[125] - M[235] - M[135]$  are in perspective from the point  $M[123]$ , and the line of perspectivity (on the far right) is  $\ell[45]$ . But it is well-known that any of the ten points  $M[\beta]$  can be chosen as the centre of perspectivity, which then uniquely determines the line of perspectivity as  $\ell[\alpha]$ , where  $\alpha = \text{FIVE} \setminus \beta$ . This labelling will reappear in a slightly different form in the context of Pascal’s hexagram.

In order to construct this configuration *ab initio*, we can start with the point  $M[123]$  which depends on two parameters, and then each of the lines  $\ell[12], \ell[13], \ell[23]$  containing it depends on one parameter each. Now choose  $M[124]$  and  $M[125]$  on the first line (depending on one parameter each) and similarly for the other two lines. This determines the entire configuration, which is seen to depend on  $2 + 1 \times 3 + 2 \times 3 = 11$  parameters.

1 It should be mentioned that ‘Pascal’s hexagram’, which is sometimes called Pascal’s ‘*hexagrammum mysticum*’, is an imprecise term. In this paper, we have taken it to mean the whole collection of Pascal lines and Kirkman points arising out of six points on a conic. Sometimes it is assumed to include several more points and lines (e.g., Cayley lines and Steiner points) which will not appear here.

2 It is understood that 123 stands for the set  $\{1, 2, 3\}$ , and hence  $M[312]$  and  $M[123]$  refer to the same point. A similar convention will be true of lines.



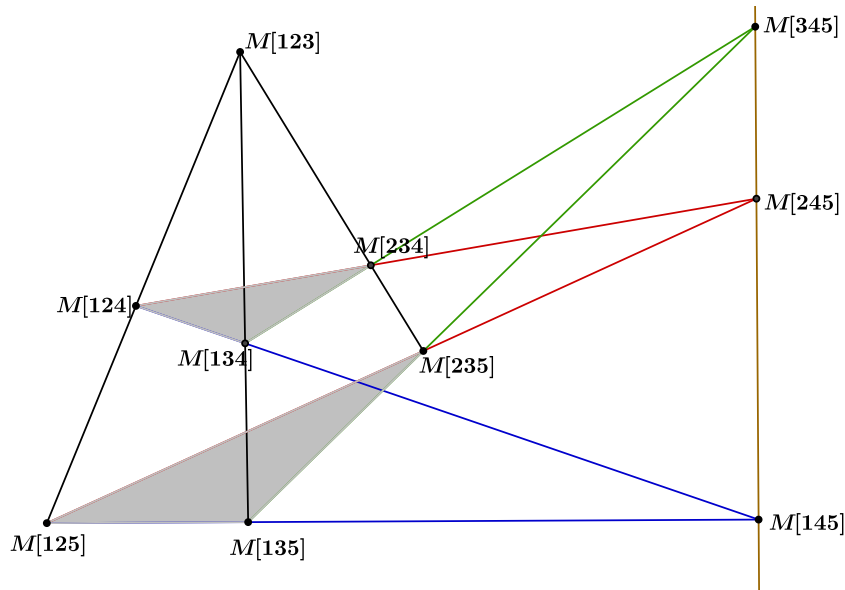


FIGURE 1. Desargues configuration. The line labels are omitted in order to avoid clutter.

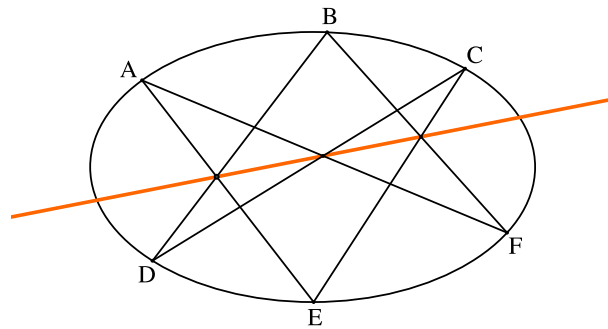


FIGURE 2. Pascal's theorem

## 2.2. Pascal's hexagram

Let  $\mathcal{K}$  be a nonsingular conic in  $\mathbb{P}^2$ , and choose six distinct points  $A, B, C, D, E, F$  on  $\mathcal{K}$ . If the points are arranged into an array, say  $\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix}$ , then Pascal's theorem says that the points

$$AE \cap BD, \quad AF \cap CD, \quad BF \cap CE$$

(corresponding to the three minors of the array) are collinear (see Diagram 2). The line containing them is usually called the Pascal line (or just the Pascal) of the array, which we will denote by

$\left\{ \begin{matrix} A & B & C \\ D & E & F \end{matrix} \right\}$ . The Pascal remains unchanged if we permute either the rows or the columns of the

array; for instance,  $\left\{ \begin{matrix} A & B & C \\ D & E & F \end{matrix} \right\} = \left\{ \begin{matrix} B & A & C \\ E & D & F \end{matrix} \right\} = \left\{ \begin{matrix} D & F & E \\ A & C & B \end{matrix} \right\}$  etc. Hence there are 12 different

ways of denoting the same Pascal. On the other hand, if we take any essentially different arrangement of the same points, say  $\left\{ \begin{matrix} A & C & F \\ D & B & E \end{matrix} \right\}$ , then *a priori* we get a different Pascal. Thus, the same sextuple

$A, \dots, F$  gives rise to  $6!/12 = 60$  notionally distinct Pascals. It is a theorem due to Pedoe [15], that these lines are pairwise distinct if the initial six points are chosen in general position.



Now it is a remarkable fact, perhaps first observed by Veronese [20, p. 660], that these sixty lines naturally separate themselves into six Desargues configurations. This is elegantly expressed by using the labelling schema for Pascals<sup>3</sup> introduced by Baker [3, Note II].

### 2.3. The exotic isomorphism

For any set  $X$ , let  $\mathfrak{S}(X)$  denote the group of bijections  $X \rightarrow X$ . Consider the set of numbers  $\text{SIX} = \{1, 2, 3, 4, 5, 6\}$  and the set of letters  $\text{LTR} = \{A, B, C, D, E, F\}$ . The following table, first discovered by Sylvester [18], defines an isomorphism

$$\zeta : \mathfrak{S}(\text{SIX}) \longrightarrow \mathfrak{S}(\text{LTR}).$$

It is a variant of the unique outer automorphism of the symmetric group  $\mathfrak{S}_6$  (see [12]).

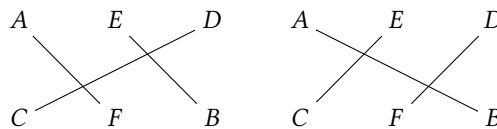
	2	3	4	5	6
1	$AE.BD.CF$	$AD.BF.CE$	$AB.CD.EF$	$AF.BC.DE$	$AC.BE.DF$
2		$AF.BE.CD$	$AC.BF.DE$	$AB.CE.DF$	$AD.BC.EF$
3			$AE.BC.DF$	$AC.BD.EF$	$AB.CF.DE$
4				$AD.BE.CF$	$AF.BD.CE$
5					$AE.BF.CD$

The table is to be interpreted as saying that  $\zeta$  takes the transposition (12) to the element  $(AE)(BD)(CF)$  of cycle type  $2 + 2 + 2$ .

### 2.4. The labelling schema

We can now construct a canonical label for each Pascal. For example, consider the array  $\begin{bmatrix} A & E & D \\ C & F & B \end{bmatrix}$ .

We can separate its six diagonal lines into two diagrams:



Read them respectively as group elements  $(AF)(BE)(CD)$  and  $(AB)(CE)(DF)$ . By the table above, they come from transpositions (23) and (25). Pick the common element 2, which alternately pairs with 3 and 5. Now we label the Pascal  $\begin{bmatrix} A & E & D \\ C & F & B \end{bmatrix}$  as<sup>4</sup>  $k(2, 35)$ . Thus, each of the sixty Pascals is labelled as  $k(z, uv) = k(z, vu)$  for some pairwise distinct elements  $u, v, z \in \text{SIX}$ . The reader may wish to check that this procedure can be reversed; for instance,  $k(1, 46)$  stands for  $\begin{bmatrix} A & D & E \\ F & B & C \end{bmatrix}$ . Now we have the following theorem due to Kirkman. (A proof may be found in any of the references given in Section 1.2.)

**THEOREM 2.1 (Kirkman).** For pairwise distinct elements  $z, u, v, w \in \text{SIX}$ , the three Pascals

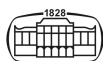
$$k(z, uv), \quad k(z, uw), \quad k(z, vw)$$

are concurrent.

Their common point, naturally called a Kirkman point, is labelled as  $K[z, uvw]$  (see Diagram 3). There are altogether 60 such points. If  $A, \dots, F$  are chosen generally, then there are no further incidences between the Pascal lines and Kirkman points except those captured by the theorem. For instance, in that case  $K[2, 136]$  will not lie on  $k(2, 14)$  or  $k(4, 13)$ .

<sup>3</sup> This schema is also explained in a somewhat different but equivalent form in [4].

<sup>4</sup> It is understood that 35 stands for the set  $\{3, 5\}$ , and hence  $k(2, 53)$  would mean the same thing. A similar convention prevails throughout; for instance,  $K[2, 146]$  is the same as  $K[2, 614]$ .



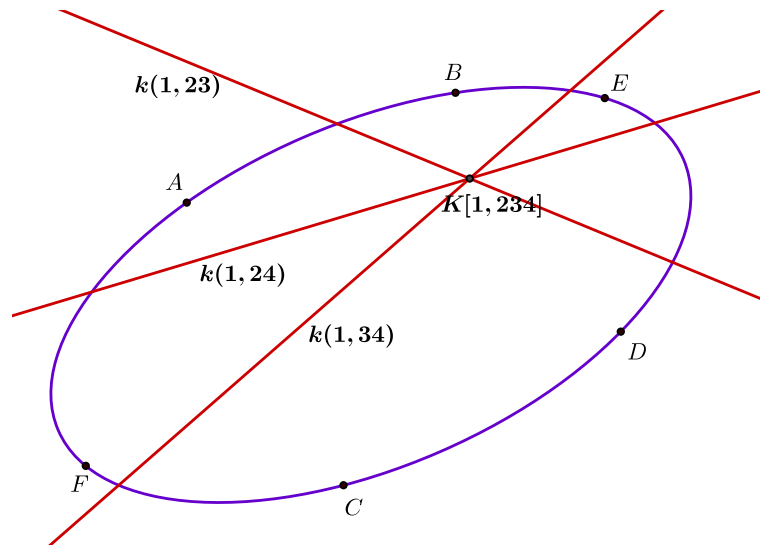


FIGURE 3. The Pascals  $k(1, 23)$ ,  $k(1, 24)$ ,  $k(1, 34)$  concur at the Kirkman point  $K[1, 234]$ .

### 2.5. The Veronese decomposition

Now fix an element  $z \in \text{SIX}$ , and consider the ten lines

$$\ell[uv] = k(z, uv), \quad u, v \in \text{SIX} \setminus \{z\},$$

together with the ten points

$$M[uvw] = K[z, uvw], \quad u, v, w \in \text{SIX} \setminus \{z\}.$$

The notation is parallel to the one in Section 2.1, except that the set  $\text{SIX} \setminus \{z\}$  plays the role of FIVE. Kirkman’s theorem immediately implies that these form a Desargues configuration, which we will denote by  $\Delta_z$ . In summary, a general set of six points  $A, \dots, F \in \mathcal{K}$  leads to six Desargues configurations  $\Delta_1, \dots, \Delta_6$ . This is sometimes called the Veronese decomposition of the totality of Pascal lines and Kirkman points. Diagram 4 shows the configuration  $\Delta_6$ . The two yellow triangles are in perspective from the point  $K[6, 245]$ , and the line of perspectivity (shown in green) is  $k(6, 13)$ .

### 3. MAIER TRIPLES AND THE VERONESE MAP

Our enumerative problem, as stated in Section 1.1, involves 11 parameters on each side. It is sensible to reduce this number by using the action of the group  $\text{Aut}(\mathbb{P}^2) = \text{PGL}(3, \mathbb{C})$ . Thus we will reformulate the problem by fixing the conic and considering Desargues configurations up to projectivity.

#### 3.1.

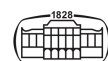
Fix coordinates  $[x, y, z]$  on  $\mathbb{P}^2$ , and let  $\mathcal{K}^\circ$  denote the ‘standard’ conic  $xz = y^2$ . Choose the following six points on  $\mathcal{K}^\circ$ :

$$\begin{aligned} A^\circ &= [1, 0, 0], & B^\circ &= [1, -1, 1], & C^\circ &= [0, 0, 1], \\ D^\circ &= [1, -p, p^2], & E^\circ &= [1, -q, q^2], & F^\circ &= [1, -r, r^2], \end{aligned}$$

where  $p, q, r$  are distinct from 0, 1. Given the isomorphism

$$\mathbb{P}^1 \xrightarrow{\sim} \mathcal{K}^\circ, \quad t \longrightarrow [1, -t, t^2],$$

these points respectively correspond to the sequence  $0, 1, \infty, p, q, r$ . By the fundamental theorem of projective geometry, any ordered set of six points on  $\mathcal{K}^\circ$  is projectively equivalent to  $A^\circ, \dots, F^\circ$  for unique values of  $p, q, r$ .



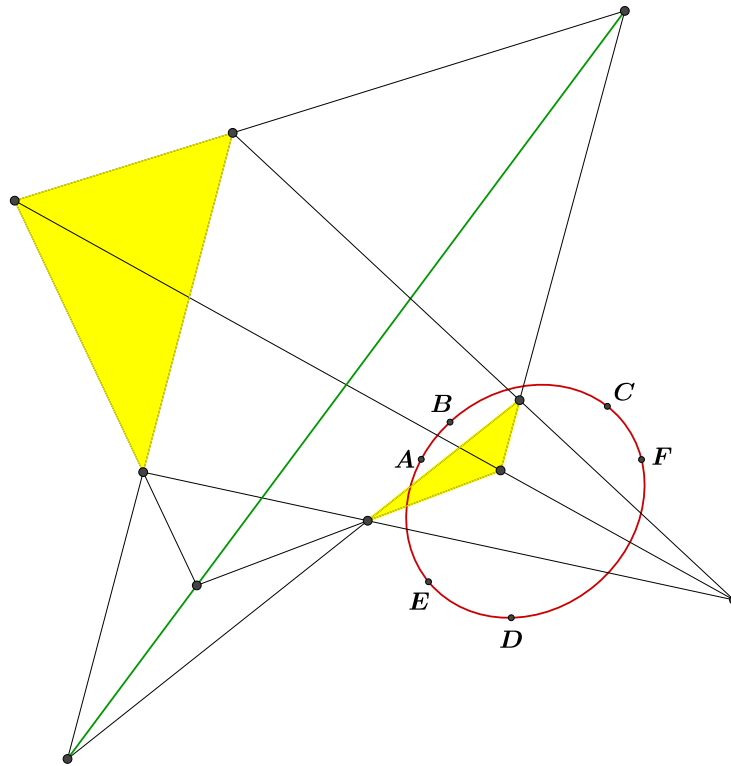


FIGURE 4. The configuration  $\Delta_6$

Now let  $\mathfrak{P}$  be the set of triples  $(p, q, r) \in \mathbb{C}^3$  such that  $\Delta_1, \dots, \Delta_6$  are Desargues configurations. The issue is that for special values of  $p, q, r$ , the Pascal lines and Kirkman points might have ‘unwanted’ incidences which would prevent any of the  $\Delta_i$  from being Desargues configurations. However, this will not happen for general  $p, q, r$ , and hence  $\mathfrak{P}$  is a dense open subset<sup>5</sup> of  $\mathbb{C}^3$ .

### 3.2. Ordered quadrangles and Maier triples

An ordered quadrangle is an ordered set of four points in  $\mathbb{P}^2$  such that no three of them are collinear. For instance,

$$[1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1] \tag{3.1}$$

is the ‘standard’ ordered quadrangle. By the fundamental theorem of projective geometry, there is a unique automorphism of  $\mathbb{P}^2$  taking any ordered quadrangle to the standard one.

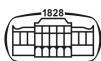
**Maier triples** were introduced in [13]; they are a method of identifying Desargues configurations up to projectivity. Let  $\Gamma$  be a Desargues configuration labelled as in Section 2.1, and consider the ordered quadrangle

$$Q : M[123], M[124], M[134], M[234], \tag{3.2}$$

with vertices in  $\Gamma$ . It determines the lines  $\ell[12], \ell[13], \ell[23], \ell[14], \ell[24], \ell[34]$ . Now observe that specifying the point  $M[125]$  is equivalent to specifying the cross-ratio

$$a = \langle M[123], M[124], \ell[12] \cap \ell[34], M[125] \rangle.$$

5 Since the number of such unwanted incidences is large, the complement of  $\mathfrak{P}$  is an algebraic subvariety of  $\mathbb{C}^3$  defined by several complicated equations. Fortunately, we won’t have to describe it explicitly.



Similarly let

$$b = \langle M[123], M[134], \ell[13] \cap \ell[24], M[135] \rangle, \quad c = \langle M[123], M[234], \ell[23] \cap \ell[14], M[235] \rangle.$$

Then  $\mu_Q(\Gamma) = (a, b, c)$  will be called the Maier triple of  $\Gamma$  with respect to the ordered quadrangle  $Q$ . It specifies the lower triangle  $M[125] - M[135] - M[235]$ , and hence all the remaining points and lines in the configuration.<sup>6</sup>

If we are only given the values of  $(a, b, c)$ , then we can take the points of  $Q$  to be those of the standard quadrangle and then complete the rest of the configuration. Hence, the triple determines  $\Gamma$  up to projectivity.

### 3.3.

There are altogether 120 ordered quadrangles in  $\Gamma$ , and a Maier triple associated to each of them. The former can be enumerated as follows: choose an element  $z \in \text{FIVE}$  and now list the 3-element subsets of  $\text{FIVE} \setminus \{z\}$ . For instance,  $Q$  was obtained by choosing 5, and then by successively omitting 4, 3, 2, 1 from the set  $\{1, 2, 3, 4\}$ . Thus  $Q$  can be written as  $Q[5, 4, 3, 2, 1]$ , and in general the ordered quadrangles in  $\Gamma$  are naturally labelled by permutations of FIVE. Hence the group  $\mathfrak{S}(\text{FIVE})$  will act on the set of ordered quadrangles. If we transfer this action to the corresponding Maier triples, then a straightforward computation shows that the elements  $(1\ 2)$  and  $(1\ 2\ 3\ 4\ 5)$  in  $\mathfrak{S}(\text{FIVE})$  will take  $(a, b, c)$  respectively to

$$\left( \frac{(1 - bc)(a - 1)}{bc - b - c + 1}, \frac{(1 - ac)(b - 1)}{ac - a - c + 1}, \frac{(1 - ab)(c - 1)}{ab - a - b + 1} \right), \quad \text{and}$$

$$\left( \frac{a + b - 2}{a - 1}, \frac{a + c - 2}{a - 1}, \frac{(bc - 1)(a - 1)}{abc - a - b - c + 2} \right).$$

This completely specifies the group action, and thus we can calculate all the 120 Maier triples of a Desargues configuration starting from any one of them. These triples are all distinct for general  $(a, b, c)$ . In other words, there are no non-identity automorphisms of  $\mathbb{P}^2$  mapping a general Desargues configuration  $\Gamma$  to itself.

The following lemma will be useful later.

**LEMMA 3.1.** For a general triple  $(p, q, r) \in \mathfrak{P}$ , the configurations  $\Delta_1, \dots, \Delta_6$  are pairwise projectively inequivalent. In other words, there is no automorphism of  $\mathbb{P}^2$  taking  $\Delta_i$  to  $\Delta_j$  for  $i \neq j$ .

**Proof.** Since this is an open condition on  $\mathfrak{P}$ , it suffices to make the calculation in one case. Choose  $(p, q, r) = (5, 11, 19)$  and find all the Maier triples for each of the  $\Delta_i$ . It turns out that for  $i \neq j$ , all the Maier triples of  $\Delta_i$  are distinct from those of  $\Delta_j$ . This completes the proof.  $\square$

### 3.4. The Veronese map

Since  $\text{SIX} \setminus \{6\} = \text{FIVE}$ , notice that  $\Delta_6$  is labelled exactly as in Section 2.1. For  $(p, q, r) \in \mathfrak{P}$ , let  $v(p, q, r) = \mu_Q(\Delta_6)$ , which is the Maier triple associated to  $\Delta_6$  with respect to the quadrangle  $Q$ . This defines the Veronese map

$$v : \mathfrak{P} \longrightarrow \mathbb{C}^3.$$

In effect, we have reduced the number of parameters in the original problem from 11 to 3 by using the action of the 8-dimensional group  $\text{PGL}(3, \mathbb{C})$ . The next result is at the heart of the main theorem.

**PROPOSITION 3.2.** For a general triple  $(a, b, c)$ , the set  $v^{-1}\{(a, b, c)\}$  consists of six elements.

This will follow from an elimination-theoretic computation. The necessary techniques are explained at length in [1] and [6].

<sup>6</sup> A somewhat different use of cross-ratios in this context may be found in [7].



**Proof.** Fix the polynomial ring  $R = \mathbb{Q}[p, q, r, a, b, c]$ , and write  $v(p, q, r) = (\varphi_1, \varphi_2, \varphi_3)$ , where  $\varphi_i$  are rational functions of  $p, q, r$ . Starting from  $A^\circ, \dots, F^\circ$ , we can calculate the coordinates of all Pascal lines and Kirkman points. This leads to explicit expressions for the  $\varphi_i$ . For instance, we have

$$\varphi_1 = \frac{(p-q)(p^2qr + pq^2r - 2pqr^2 - p^2q + pr^2 - q^2r + qr^2 + pq - pr)}{r(p-1)(p^2qr - pq^2r - p^2q + 2pq^2 - pr^2 - q^2r + qr^2 - pq + pr)},$$

with similar expressions for the other two  $\varphi_i$ . Let  $J$  be the ideal in  $R$  generated by the numerators of the expressions  $\varphi_1 - a, \varphi_2 - b, \varphi_3 - c$ . Then we have  $v(p, q, r) = (a, b, c)$ , exactly when all the elements in  $J$  vanish. Since the solutions  $p, q, r = 0, 1$  are disallowed, we replace  $J$  with the colon ideal

$$\tilde{J} = J : (p, q, r, p-1, q-1, r-1).$$

Now calculate the Gröbner basis of  $\tilde{J}$  using an elimination order with respect to the variables  $q, r$ . Although the basis has several elements, the most important one is of the form

$$f = u_0 p^6 + u_1 p^5 + \dots + u_5 p + u_6,$$

where  $u_i$  are polynomials in  $a, b, c$ . Unfortunately the  $u_i$  are too lengthy to write down in full; for instance,  $u_3$  is a sum of 65 monomials in  $a, b, c$ . However, the end terms are more compact. They are

$$u_0 = a^2(1-b)(b+c)(a+b+c-1)^2, \quad \text{and} \\ u_6 = b^4(c-1)(b+c-1).$$

Thus  $u_0, u_6 \neq 0$  for general values of  $a, b, c$ . Moreover, it is straightforward to check that the discriminant of  $f$  with respect to  $p$  is not identically zero. It follows that the equation  $f = 0$  has six distinct solutions in  $p$  for general values of  $a, b, c$ .

The Gröbner basis also contains two elements of the form

$$g = v_6 \square - \left( \sum_{i=0}^5 v_{5-i} p^i \right),$$

where  $v_i$  are polynomials in  $a, b, c$ , and  $\square$  stands for either  $q$  or  $r$ . Hence the values of  $q$  and  $r$  are completely determined by  $p$ . This completes the proof of the proposition.  $\square$

### 3.5.

It does not seem easy to find an explicit example over  $\mathbb{Q}$  of these six pre-images. However, a systematic search gives the following example over the finite field  $\mathbb{F}_{9547}$ . The following six triples

$$(p, q, r) = (1621, 3235, 7813), \quad (3378, 1408, 3921), \quad (3514, 3675, 5136) \\ (4525, 8263, 5987), \quad (8040, 5802, 2266), \quad (9281, 5342, 1486)$$

are sent to  $(5, 11, 13)$  by the Veronese map  $v$ .

### 3.6.

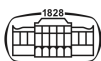
We now regard  $a, b, c$  as indeterminates, and consider  $f$  as a univariate polynomial over the field  $\mathbb{Q}(a, b, c)$ .

**PROPOSITION 3.3.** The Galois group of  $f$  is  $\mathfrak{S}_6$ .

**Proof.** Since the Galois group is *a priori* a subgroup of  $\mathfrak{S}_6$ , it is enough to do the computation for a polynomial obtained by specialising  $a, b, c$  (see [19, §61]). If we let  $a = 2, b = 3, c = 5$ , then  $f$  specialises to

$$\tilde{f} = 5184 p^6 - 3816 p^5 - 13988 p^4 + 34088 p^3 - 35736 p^2 + 15786 p - 2268.$$

Now a MAPLE computation shows that  $\text{Gal}(\tilde{f}) \simeq \mathfrak{S}_6$ , which proves the proposition.  $\square$





3.7.

Let  $v^{-1}\{(a, b, c)\} = \{(p_1, q_1, r_1), \dots, (p_6, q_6, r_6)\}$ , and let  $S_i = \{A^\circ, B^\circ, C^\circ, D_i^\circ, E_i^\circ, F_i^\circ\}$  be the point sets in  $\mathcal{K}^\circ$  corresponding to these values for  $1 \leq i \leq 6$ .

**PROPOSITION 3.4.** Assume that  $(a, b, c)$  is a general point in  $\mathbb{C}^3$ . Then these point sets are projectively inequivalent, i.e., there exists no automorphism of  $\mathcal{K}^\circ$  which takes  $S_i$  to  $S_j$  if  $i \neq j$ .

**Proof.** This is most naturally proved using the invariant theory of binary sextics, as explained in the treatises by Olver [14] or Grace and Young [9]. Recall that the generic binary sextic  $\mathcal{F}$  has invariants<sup>7</sup>

$$I_2 = (\mathcal{F}, \mathcal{F})_6, \quad I_4 = ((\mathcal{F}, \mathcal{F})_4, (\mathcal{F}, \mathcal{F})_4)_4,$$

in degrees 2, 4 respectively (see [9, p. 156]). If two sextics are projectively equivalent, then the ratio  $\theta_{\mathcal{F}} = I_4/I_2^2$  is the same for both of them. Now let

$$H_i = x_1(x_1 - x_2)x_2(x_1 - p_i x_2)(x_1 - q_i x_2)(x_1 - r_i x_2), \quad 1 \leq i \leq 6$$

be the homogeneous sextics in variables  $\{x_1, x_2\}$  corresponding to the  $S_i$ . If two of the  $\theta_{H_i}$  were to be equal, then they would remain equal modulo any prime. However, if we evaluate them modulo 9547 on the solutions in Section 3.5, then they respectively come out to be

$$8718, \quad 3419, \quad 7235, \quad 6900, \quad 6613, \quad 5320.$$

This shows that the  $S_i$  are pairwise inequivalent for (5, 11, 13). Since this is an open condition on triples, the same is true of a general  $(a, b, c)$  in  $\mathbb{C}^3$ . □

3.8.

Now we are ready to state the final version of the main theorem. Define a *conical sextuple* to be a set of six distinct points in  $\mathbb{P}^2$  which lie on a nonsingular conic. Given a general conical sextuple  $X$ , let  $\delta(X)$  denote the set of six Desargues configurations determined by  $X$ .

**THE MAIN THEOREM.** Let  $\Gamma$  be a general Desargues configuration in  $\mathbb{P}^2$ . Then there are six conical sextuples  $X$  such that  $\Gamma \in \delta(X)$ .

This is shown in Diagram 5, but the depiction is merely suggestive and not meant to be geometrically accurate. By contrast, all the other diagrams are indeed honest; for example, the configuration shown in Diagram 4 is the true  $\Delta_6$  for those points  $A, \dots, F$ .

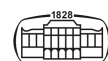
**Proof.** Label  $\Gamma$  as in Diagram 1, and let  $(a, b, c) = \mu_Q(\Gamma)$ . For  $1 \leq i \leq 6$ , let  $S_i \subset \mathcal{K}^\circ$  be the sextuples obtained as above, and  $\Delta_6^{(i)}$  the corresponding Desargues configurations. Let  $\tau^{(i)}$  denote the unique automorphism of  $\mathbb{P}^2$  taking  $\Delta_6^{(i)}$  to  $\Gamma$ . Then  $X^{(i)} = \tau^{(i)}(S_i)$  are conical sextuples such that  $\Gamma \in \delta(X^{(i)})$ .

It remains to show that these are all. Let  $Y$  be a conical sextuple on a conic  $\mathcal{K}$  such that  $\Gamma \subseteq \delta(Y)$ . At the outset, label the points of  $Y$  arbitrarily as  $A, \dots, F$ , so that  $\Gamma = \Delta_j$  for some  $1 \leq j \leq 6$ . If  $j \neq 6$ , then we relabel the points as dictated by the table in Section 2.3. For example, assume  $j = 3$ . Since the transposition (3 6) goes to  $AB.CF.DE$ , we will end up with  $\Gamma = \Delta_6$  by interchanging  $A$  with  $B$ ,  $C$  with  $F$  and  $D$  with  $E$ . Now there is a unique automorphism  $\sigma$  of  $\mathbb{P}^2$  which takes  $\mathcal{K}$  to  $\mathcal{K}^\circ$  and  $A, B, C$  respectively to  $A^\circ, B^\circ, C^\circ$ . Let  $p_*, q_*, r_*$  be the values corresponding to  $\sigma(D), \sigma(E), \sigma(F)$ . Since  $v(p_*, q_*, r_*) = (a, b, c)$ , we must have  $(p_*, q_*, r_*) = (p_i, q_i, r_i)$  and thus  $\sigma(Y) = S_i$  for some  $i$ . But then  $\sigma^{-1} = \tau^{(i)}$  and  $Y = X^{(i)}$ . This completes the proof. □

3.9.

We end with an algebro-geometric interpretation of the main theorem. This needs some moderately advanced concepts in algebraic geometry, all of which are covered in [10] and [11]. Let  $\mathcal{M}_2$  denote the moduli space of isomorphism classes of curves of genus 2. In our set-up, this space appears in the following two ways:

7 Here  $(G_1, G_2)_r$  denotes the  $r$ -th transvectant of forms  $G_1$  and  $G_2$ .



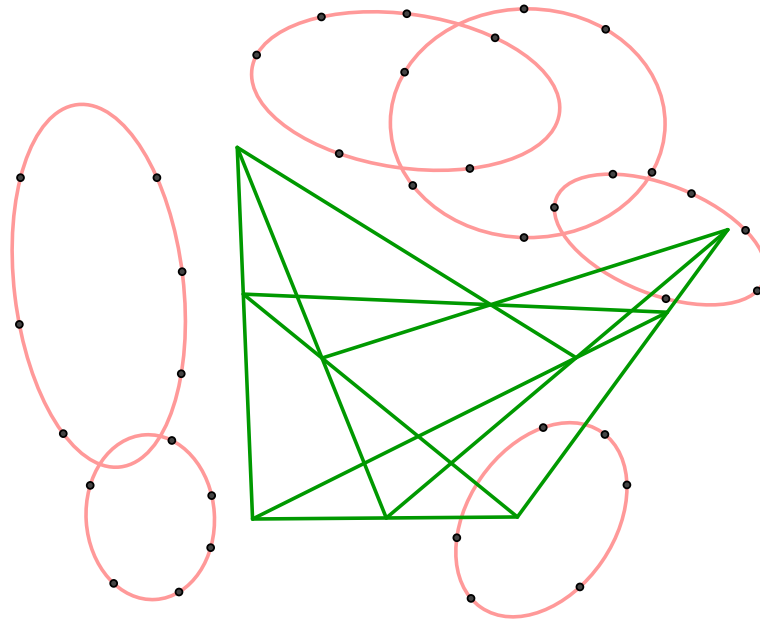


FIGURE 5. Conical sextuples associated to a Desargues configuration

- Given six points on a conic  $\mathcal{K} \simeq \mathbb{P}^1$ , there is a canonically determined double cover  $C \rightarrow \mathbb{P}^1$  ramified over these points such that  $C$  has genus 2. In short, a conical sextuple determines a point in  $\mathcal{M}_2$ .
- A Desargues configuration in the plane also determines a point in  $\mathcal{M}_2$  by the so-called Stephanos map (see [2, §3]).

Now, via the construction

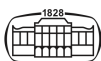
$$A, \dots, F \in \mathcal{K} \rightsquigarrow \Delta_1, \dots, \Delta_6,$$

a general point on  $\mathcal{M}_2$  gives rise to 6 points on  $\mathcal{M}_2$ . On the other hand, the main theorem gives a passage from a general point on  $\mathcal{M}_2$  to 6 points on  $\mathcal{M}_2$  in the *reverse* direction. In summary, we have a (6, 6)-correspondence  $\mathcal{Z} \subseteq \mathcal{M}_2 \times \mathcal{M}_2$ . We may think of  $\mathcal{Z}$  as a binary relation  $z \sim z'$  on the set  $\mathcal{M}_2$ , where a general  $z$  is related to six values of  $z'$  and conversely.

It would be interesting to investigate the intersection-theoretic properties as well as the intrinsic geometry of  $\mathcal{Z}$ . Since  $\mathcal{M}_2$  is known to be rational, it is natural to ask whether the same is true of  $\mathcal{Z}$ .

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