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HAMILTON TRIANGLE OF A TRIANGLE IN THE ISOTROPIC PLANE

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ABSTRACT

In this paper we introduce the concept of the Hamilton triangle of a given triangle in an isotropic plane and investigate a number of important properties of this concept. We prove that the Hamilton triangle is homological with the observed triangle and with its contact and complementary triangles. We also consider some interesting statements about the relationships between the Hamilton triangle and some other significant elements of the triangle, like e.g. the Euler and the Feuerbach line, the Steiner ellipse and the tangential triangle.

KEYWORDS

standard triangle, Hamilton triangle, isotropic plane

MATHEMATICS SUBJECT CLASSIFICATION (2020)

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1. INTRODUCTION

The isotropic (or Galilean) plane is a projective–metric plane, where the absolute consists of one line, absolute line ω, and one point on that line, absolute point Ω. The lines through the point Ω are isotropic lines, and the points on the line ω are isotropic points. Points with the same abscissa, i.e., which lie on the same isotropic line, are called *parallel*.

For two non-parallel points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ the *isotropic distance* is defined by $d(P_1, P_2) := x_2 - x_1$. The isotropic distance is directed. For two parallel points $P_1(x_1, y_1)$ and $P_2(x_1, y_2)$, the isotropic span is defined by $s(P_1, P_2) := y_2 - y_1$.

A triangle is called allowable if none of its sides is isotropic ([9]). If we choose the coordinate system in such a that the circumscribed circle of an allowable triangle ABC has the equation $y = x^2$, and therefore its vertices are the points $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$, where

$$a + b + c = 0, \tag{1.1}$$

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then we say that the triangle ABC is in the standard position, or that ABC is a standard triangle, for short. Its sides BC, CA, and AB have equations $y = -ax - bc$, $y = -bx - ca$, and $y = -cx - ab$. Denoting

$$p := abc, \qquad q := bc + ca + ab, \tag{1.2}$$

we get e.g.

$$q = bc - a^2. \tag{1.3}$$

To prove geometric facts for all allowable triangles it is sufficient to provide a proof for a standard triangle [6].

The midpoints A_m, B_m, and C_m of the sides BC, CA, and AB determine the so-called *complementary triangle* $A_mB_mC_m$ of the triangle ABC (Figure 1). In [6], it is shown that for the standard triangle ABC we get

$$A_m = \left(-\frac{a}{2}, -\frac{1}{2}(q + bc)\right), \ B_m = \left(-\frac{b}{2}, -\frac{1}{2}(q + ca)\right), \ C_m = \left(-\frac{c}{2}, -\frac{1}{2}(q + ab)\right) \tag{1.4}$$

and e.g. the line B_mC_m has the equation

$$y = -ax - q + \frac{bc}{2}. \tag{1.5}$$

The standard triangle ABC has the *centroid* $G = \left(0, -\frac{2}{3}q\right)$ and its *Euler line* $\mathcal{E}$ is given by the equation $x = 0$.

Isotropic altitudes h_a, h_b, and h_c associated with sides BC, CA, and AB are isotropic lines passing through vertices A, B, and C. The points $A_h = BC \cap h_a$, $B_h = CA \cap h_b$, and $C_h = AB \cap h_c$ are vertices of the triangle called the *orthic triangle* of the triangle ABC. In case of the standard triangle ABC, the line B_hC_h is given by the equation

$$y = 2ax - q + 2bc. \tag{1.6}$$

The lines AA_h, BB_h, CC_h pass through the absolute point Ω, the points $BC \cap B_hC_h$, $CA \cap C_hA_h$, $AB \cap A_hB_h$ lie on a line, call it $\mathcal{H}$, and we say that the triangles ABC and $A_hB_hC_h$ are *homological*. The *center* of this homology is the absolute point, and the line $\mathcal{H}$ is the *axis* of homology (Figure 1). The line $\mathcal{H}$ is the *orthic line* of the triangle ABC and in case of the standard triangle ABC its equation is $y = -\frac{q}{3}$.

The *Euler circle* $\mathcal{K}_e$ of the triangle ABC touches its inscribed circle $\mathcal{K}_i$ at the so-called *Feuerbach point* of this triangle ([1]). Their common tangent at this point is the so-called *Feuerbach line* $\mathcal{F}$ of the triangle ABC (Figure 1). In case of the standard triangle ABC, the Feuerbach point is $\Phi = (0, -q)$, the Feuerbach line $\mathcal{F}$ is given by $y = -q$, and its inscribed circle $\mathcal{K}_i$ is given by the equation

$$y = \frac{1}{4}x^2 - q. \tag{1.7}$$

Tangent lines of the circle circumscribed to the triangle ABC at vertices A, B, C form the so-called *tangential triangle* $A_tB_tC_t$ of the triangle ABC. These two triangles are homological. The center of this homology is the *symmedian center* K and its axis of homology is the *Lemoine line* $\mathcal{L}$ of the triangle ABC (Figure 1). The isotropic line through the symmedian center is the *Brocard diameter* $\mathcal{B}$ of the triangle ABC. These facts are considered in [5], where it is shown that for the standard triangle ABC the Brocard diameter and the Lemoine line are given by $x = \frac{3p}{2q}$ and

$$y = \frac{3p}{q}x + \frac{q}{3}, \tag{1.8}$$

respectively.

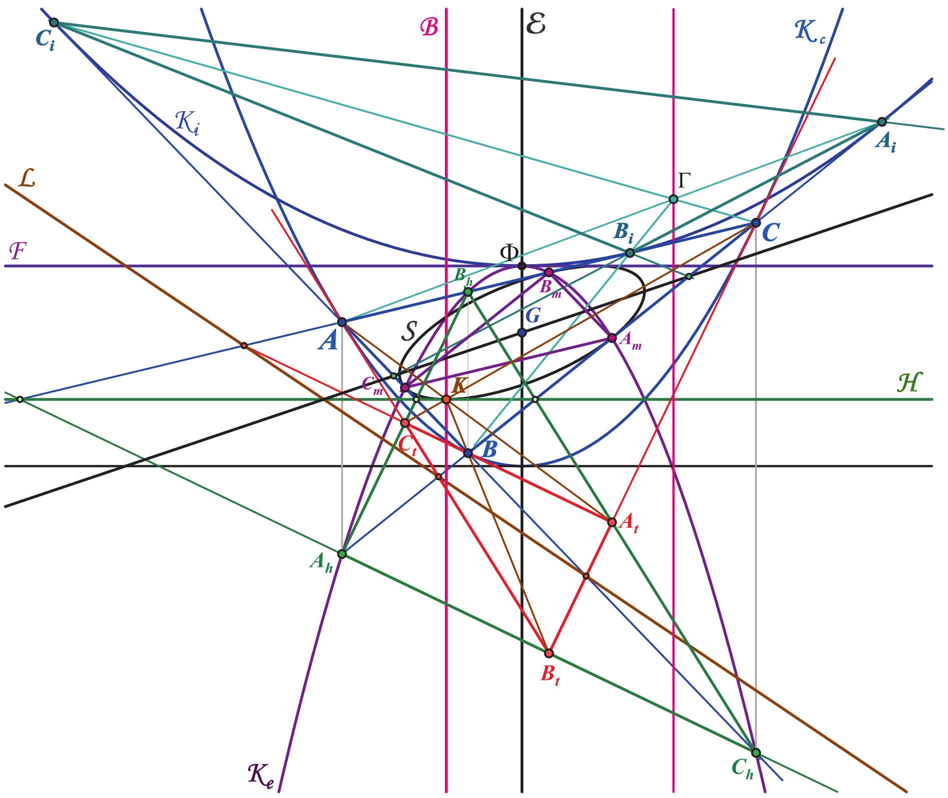

FIGURE 1.

If G is the centroid of the triangle ABC, then the homothecy $(G, -2)$ maps each line to its anticomplementary line. As $G = (0, -\frac{2}{3}q)$, the line $x = -\frac{3p}{q}$ is the anticomplementary line of the Brocard diameter.

Points A_i, B_i, C_i, where the incircle touches BC, CA and AB respectively, determine the *contact triangle* $A_iB_iC_i$ of the triangle ABC. According to [2], for the standard triangle ABC, we have e.g.

$$A_i = (-2a, bc - 2q) \tag{1.9}$$

and the line B_iC_i has the equation

$$y = \frac{a}{2}x - q - bc. \tag{1.10}$$

The triangles ABC and $A_iB_iC_i$ are homological. The center of this homology is the *Gergonne point* Γ of the triangle ABC, while the axis of homology is the harmonic line of the point Γ. The Gergonne point Γ is

$$\Gamma = \left(-\frac{3p}{q}, -\frac{4}{3}q\right), \tag{1.11}$$

its harmonic line is given by the equation

$$y = -\frac{3p}{2q}x - \frac{2}{3}q, \tag{1.12}$$

and it is also the *Steiner axis* of the triangle ABC ([12]).

For each triangle ABC there is an ellipse which touches the lines BC, CA, AB at points A_m, B_m, C_m respectively. This is the *inscribed Steiner ellipse* $\mathcal{S}$ of the triangle ABC (Figure 1). According to [12], for the standard triangle ABC this ellipse has the equation

$$4q^2x^2 - 36pxy - 12qy^2 - 24pqx - 16q^2y + 9p^2 - 4q^3 = 0. \tag{1.13}$$

Owing to [7], two points are inverse to each other with respect to a circle if and only if they parallel and the circle contains their midpoint.

2. HAMILTON TRIANGLE AND SOME OTHER SIGNIFICANT ELEMENTS OF A TRIANGLE

By analogy to the Euclidean case, the triangle UVW, whose vertices are the intersections of the corresponding sides of the complementary and the contact triangle of the triangle ABC, $U = B_mC_m \cap B_iC_i$, $V = C_mA_m \cap C_iA_i$, $W = A_mB_m \cap A_iB_i$, will be called the *Hamilton triangle* of the triangle ABC, and the lines A_iU, B_iV, C_iW which join the corresponding vertices of the contact and the Hamilton triangle will be called the *Hamilton lines* of the triangle ABC (Figure 2).

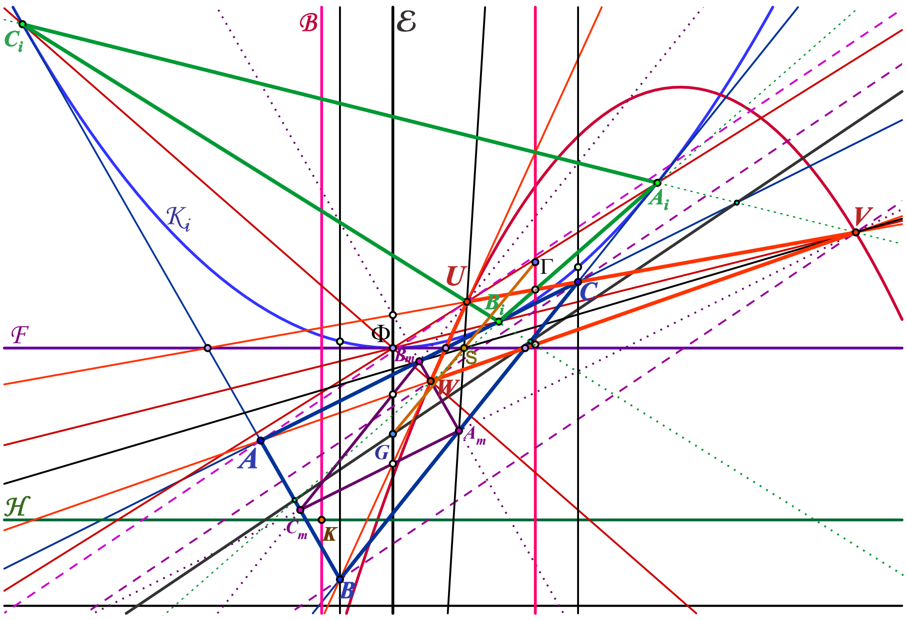

FIGURE 2.

THEOREM 2.1. Vertices of the Hamilton triangle UVW of the standard triangle ABC are

$$U = \left(\frac{bc}{a}, -q - \frac{bc}{2}\right), V = \left(\frac{ca}{b}, -q - \frac{ca}{2}\right), W = \left(\frac{ab}{c}, -q - \frac{ab}{2}\right), \tag{2.1}$$

the sides VW, WU, UV are given by

$$y = \frac{bc}{2a}x + \frac{bc}{2} - q, \quad y = \frac{ca}{2b}x + \frac{ca}{2} - q, \quad y = \frac{ab}{2c}x + \frac{ab}{2} - q, \tag{2.2}$$

and the Hamilton lines of the triangle ABC have the following equations

$$y = -\frac{a}{2}x - q, \quad y = -\frac{b}{2}x - q, \quad y = -\frac{c}{2}x - q. \tag{2.3}$$

Proof. From equations (1.5) of B_mC_m and (1.10) of B_iC_i we obtain the coordinates of the point U

$$x = \frac{bc}{a}, \quad y = -q - \frac{bc}{2}.$$

The obtained point U lies e.g. on the second line (2.2). The first line in (2.3) passes through points A_i from (1.9), and U from (2.1). Analogous facts hold for the remaining equalities (2.1) – (2.3). □

THEOREM 2.2. Hamilton lines of a triangle pass through its Feuerbach point (Figure 2).

Proof. The lines in (2.3) obviously pass through the point $\Phi = (0, -q)$. □

Theorem 2.2 states that the contact and the Hamilton triangle of the triangle ABC are homological with the center of homology Φ. We would like to answer the following question: What is the axis of this homology?

THEOREM 2.3. The axis of homology of the contact and the Hamilton triangle of the allowable triangle ABC is the line anticomplementar to the Brocard diameter of the triangle ABC (Figure 2).

Proof. From the first equalities (1.10) and (2.2) we obtain the equation for the abscissa of the point $B_iC_i \cap VW$

$$\left(\frac{a}{2} - \frac{bc}{2a}\right) x = \frac{3}{2}bc, \quad \text{i.e,} \quad (a^2 - bc)x = 3p,$$

which, because of (1.3), gives $x = -\frac{3p}{q}$. Therefore, the axis of homology is the line with equation $x = -\frac{3p}{q}$, anticomplementary to the Brocard diameter of the triangle ABC. □

THEOREM 2.4. The standard triangle ABC and its Hamilton triangle are homological. The center of homology is the point at infinity of the Steiner axis of the triangle ABC, and the axis of homology is the Feuerbach line of the considered triangle. The triangle ABC is inscribed in its Hamilton triangle UVW, i.e., the points A, B, and C lie on the lines VW, WU, and UV respectively (Figure 2).

Proof. Due to (1.3) and (1.2), the line AU through $A = (a, a^2)$ and U from (2.1) has the slope $-\frac{3p}{2q}$ as does the line (1.12). By (1.1), $B = (b, b^2)$ and $C = (c, c^2)$ obviously lie on the line $y = -ax - bc$. According to (1.3), the point $(-a, -q)$ also lies on this line as well as on the line VW with the first equation in (2.2), and on the Feuerbach line of the triangle ABC. The point $A = (a, a^2)$ also satisfies the first equation in (2.2) because of (1.3). □

For the Euclidean case see [3] and [11].

The following theorem is a generalization of the Euclidean case, see [11].

THEOREM 2.5. The complementary and the Hamilton triangle of the standard triangle ABC are homological. The midpoint of the centroid and the Gergonne point of the triangle ABC is the center of homology and its Euler line is the axis of homology (Figure 2).

Proof. Let us consider the line

$$2(q - 3bc)y = 2aqx - 2q^2 + 3bcq + 3b^2c^2.$$

The points A_m and U from (1.4) and (2.1) lie on this line since, due to (1.3),

$$\begin{aligned} 2aq\left(-\frac{a}{2}\right) - 2q^2 + 3bcq + 3b^2c^2 &= q(q - bc) - 2q^2 + 3bcq + 3b^2c^2 \\ &= -(q - 3bc)(q + bc), \\ 2aq \cdot \frac{bc}{a} - 2q^2 + 3bcq + 3b^2c^2 &= -2q^2 + 5bcq + 3b^2c^2 \\ &= -(q - 3bc)(2q + bc). \end{aligned}$$

However, by (1.3), we get

$$\begin{aligned} 2aq\left(-\frac{3p}{2q}\right) - 2q^2 + 3bcq + 3b^2c^2 &= -3bc(bc - q) - 2q^2 + 3bcq + 3b^2c^2 \\ &= -2q(q - 3bc) \end{aligned}$$

and the point

$$S = \left(-\frac{3p}{2q}, -q\right) \tag{2.4}$$

lies on this line. The point S is the midpoint of points $G = (0, -\frac{2}{3}q)$ and Γ from (1.11). With $x = 0$ from equation (1.5) and the first equation in (2.2) of lines B_mC_m and VW we get $y = \frac{bc}{2} - q$, and therefore

$$B_mC_m \cap VW = \left(0, \frac{bc}{2} - q\right).$$

This point obviously lies on the Euler line of the triangle ABC. □

Similarly, as in the Euclidean case ([10]), we have:

THEOREM 2.6. Let $A_mB_mC_m$ be the complementary triangle and UVW the Hamilton triangle of the standard triangle ABC. If B'_m and C'_m are points symmetrical to B_m and C_m with respect to points W and V, then the centroid G_a of points A_m, B'_m, C'_m lies on the line VW.

Proof. Since $B'_m = 2W - B_m$ from (1.4) and (2.1) we get the coordinates of the point B'_m

$$\frac{2ab}{c} + \frac{b}{2} = \frac{bc + 4ab}{2c}, \; -2q - ab + \frac{1}{2}q + \frac{1}{2}ca = \frac{1}{2}(ca - 2ab - 3q),$$

so we have

$$B'_m = \left(\frac{bc + 4ab}{2c}, \frac{1}{2}(ca - 2ab - 3q)\right), \; C'_m = \left(\frac{bc + 4ca}{2b}, \frac{1}{2}(ab - 2ca - 3q)\right). \quad (2.5)$$

Coordinates of the centroid G_a of points A_m, B'_m, and C'_m from (1.4) and (2.5), are given by (1.1) and (1.3)

$$\begin{aligned}
3x &= -\frac{a}{2} + \frac{1}{2bc}(b^2c + 4ab^2 + bc^2 + 4ac^2)\\
&= -\frac{a}{2} + \frac{1}{2bc}\left(bc(b + c) + 4a(b + c)^2 - 8abc\right)\\
&= -\frac{a}{2} + \frac{1}{2bc}(4a^3 - 9abc) = \frac{2a}{bc}(bc - q) - 5a = -3a - \frac{2aq}{bc},\\
3y &= \frac{1}{2}(-q - bc + ca - 2ab - 3q + ab - 2ca - 3q)\\
&= \frac{1}{2}(-bc - ca - ab - 7q) = -4q,
\end{aligned}$$

and therefore

$$G_a = \left(-a - \frac{2aq}{3bc}, -\frac{4}{3}q\right).$$

The point G_a lies on the first line (2.2) because

$$\frac{bc}{2a}\left(-a - \frac{2aq}{3bc}\right) + \frac{bc}{2} - q = -\frac{1}{3}q - q = -\frac{4}{3}q.$$

□

Similarly, as in the Euclidean case ([3]), we have:

THEOREM 2.7. The Hamilton triangle of the standard triangle is an autopolar triangle with respect to its inscribed circle and its inscribed Steiner ellipse (Figure 3).

Proof. The polar line of the point (x_o, y_o) with respect to the circle (1.7) is the line $y + y_o = \frac{1}{2}x_ox - 2q$, which in the case of the point U from (2.1) becomes

$$y - q - \frac{bc}{2} = \frac{bc}{2a}x - 2q,$$

and it is the first equation in (2.2), i.e., the polar of the point U is the line VW. The polar of the point (x_o, y_o) with respect to the ellipse (1.13) is the line

$$4q^2x_ox - 18p(y_ox + x_oy) - 12qy_oy - 12pq(x + x_o) - 8q^2(y + y_o) + 9p^2 - 4q^3 = 0.$$

In particular, for the point U from (2.1), because of (1.2) and (1.3), the coefficients of x, y and 1 in this equation are

$$\begin{aligned}
4q^2x_o - 18py_o - 12pq &= 4\frac{bc}{a}q^2 + 18p\left(q + \frac{bc}{2}\right) - 12pq \\
&= \frac{bc}{a}(4q^2 + 6a^2q + 9ap) \\
&= \frac{bc}{a}[4q^2 + 6q(bc - q) + 9bc(bc - q)] \\
&= -\frac{bc}{a}(2q^2 + 3bcq - 9b^2c^2), \\
-18px_o - 12qy_o - 8q^2 &= -18b^2c^2 + 12q\left(q + \frac{bc}{2}\right) - 8q^2 \\
&= 2(2q^2 + 3bcq - 9b^2c^2), \\
-12pqx_o - 8q^2y_o + 9p^2 - 4q^3 &= -12b^2c^2q + 8q^2\left(q + \frac{bc}{2}\right) + 9p^2 - 4q^3 \\
&= 4q^3 + 4bcq^2 - 12b^2c^2q + 9b^2c^2(bc - q) \\
&= 4q^3 + 4bcq^2 - 21b^2c^2q + 9b^3c^3 \\
&= (2q - bc)(2q^2 + 3bcq - 9b^2c^2),
\end{aligned}$$

respectively, so this equation becomes

$$-\frac{bc}{a}x + 2y + 2q - bc = 0,$$

and this is the first equation in (2.2). □

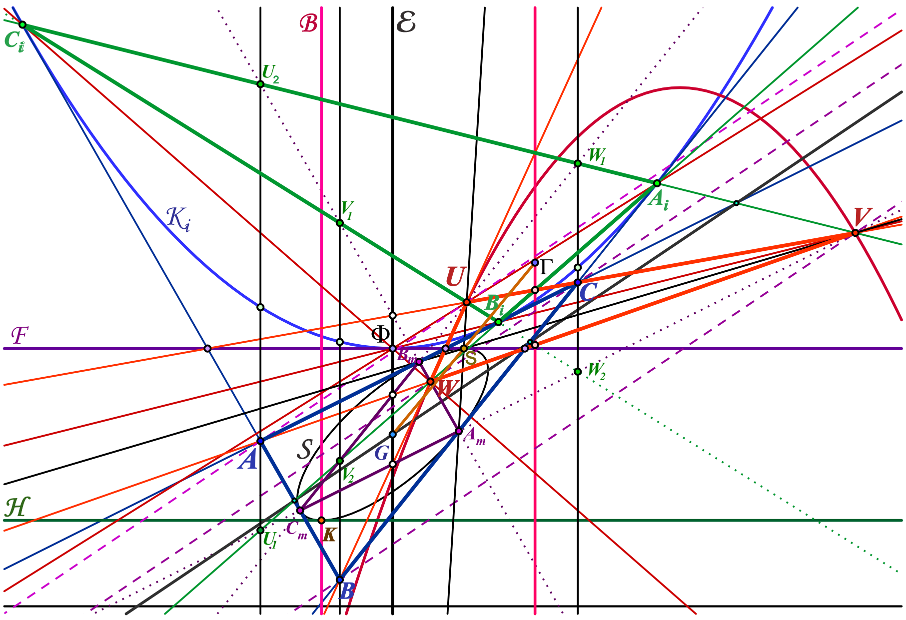

FIGURE 3.

THEOREM 2.8. Lengths of the sides of the allowable triangle ABC and its Hamilton triangle UVW satisfy the equality $d(V, W) \cdot d(W, U) \cdot d(U, V) = d(B, C) \cdot d(C, A) \cdot d(A, B)$, and the ratio between their areas is $-2 : 1$.

Proof. From (2.1) we obtain for example

$$d(V,W) = \frac{ab}{c} - \frac{ca}{b} = \frac{a}{bc}(b^2 - c^2) = -\frac{a^2}{bc}(b-c) = \frac{a^2}{bc} \cdot d(B,C),$$

and analogously

$$d(W,U) = \frac{b^2}{ca} \cdot d(C,A), \quad d(U,V) = \frac{c^2}{ab} \cdot d(A,B),$$

which implies $d(V,W) \cdot d(W,U) \cdot d(U,V) = d(B,C) \cdot d(C,A) \cdot d(A,B)$. For the areas $\triangle$ and $\triangle'$ of triangles ABC and UVW we obtain

$$2\triangle = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = (b-c)(c-a)(a-b),$$

$$\begin{aligned} 2\triangle' &= \begin{vmatrix} \frac{bc}{a} & -q-\frac{bc}{2} & 1 \\ \frac{ca}{b} & -q-\frac{ca}{2} & 1 \\ \frac{ab}{c} & -q-\frac{ab}{2} & 1 \end{vmatrix} = -\begin{vmatrix} \frac{bc}{a} & -\frac{bc}{2} & 1 \\ \frac{ca}{b} & -\frac{ca}{2} & 1 \\ \frac{ab}{c} & -\frac{ab}{2} & 1 \end{vmatrix} = -\frac{1}{8abc}\begin{vmatrix} 2bc & p & 2a \\ 2ca & p & 2b \\ 2ab & p & 2c \end{vmatrix} \\ &= -\frac{1}{2}\begin{vmatrix} bc & 1 & a \\ ca & 1 & b \\ ab & 1 & c \end{vmatrix} = \frac{1}{2}\left(bc(b-c) + ca(c-a) + ab(a-b)\right) \\ &= -\frac{1}{2}(b-c)(c-a)(a-b) = -\frac{1}{2} \cdot 2\triangle = -\triangle\,. \end{aligned}$$

□

THEOREM 2.9. The circumscribed circle of the Hamilton triangle of the standard triangle ABC (Figure 3) is given by the equation

$$y = -\frac{1}{2}x^2 + \frac{q^2}{2p}x - \frac{q}{2}.$$

Proof. If we consider for example the point U from (2.1), then because of (1.3) we get

$$\begin{aligned} -\frac{1}{2} \cdot \frac{b^2c^2}{a^2} + \frac{q^2}{2p} \cdot \frac{bc}{a} - \frac{q}{2} &= -\frac{1}{2a^2}(b^2c^2 - q^2 + a^2q) \\ &= -\frac{1}{2a^2}\left((bc+q)a^2 + a^2q\right) \\ &= -\frac{1}{2}(bc+2q) = -q - \frac{bc}{2}. \end{aligned}$$

□

We will mention two more results concerning the Hamilton triangle of the triangle ABC.

THEOREM 2.10. Let $U_1 = C_mA_m \cap A_iB_i$, $U_2 = A_mB_m \cap C_iA_i$, $V_1 = A_mB_m \cap B_iC_i$, $V_2 = B_mC_m \cap A_iB_i$, $W_1 = B_mC_m \cap C_iA_i$, and $W_2 = C_mA_m \cap B_iC_i$ be the intersections of the non-corresponding sides of the complementary triangle $A_mB_mC_m$ and the contact triangle $A_iB_iC_i$ of the allowable triangle ABC. Points U_1 and U_2, V_1 and V_2, W_1 and W_2, are inverse to each other, with respect to the inscribed circle of the triangle, and are parallel to points A, B, and C respectively (Figure 3).

For the Euclidean case, see [4].

Proof. The lines C_mA_m and A_iB_i, analogous to lines B_mC_m and B_iC_i with equations (1.5) and (1.10), are given by

$$y = -bx - q + \frac{ca}{2}, \quad y = \frac{c}{2}x - q - ab,$$

from where $x = a$, $y = \frac{ca}{2} - ab - q$, so we get

$$U_1 = \left(a, \frac{ca}{2} - ab - q\right), \quad U_2 = \left(a, \frac{ab}{2} - ca - q\right).$$

The points U_1 and U_2 are parallel to the point A, and the ordinate of their midpoint is

$$\frac{1}{2}\left(-\frac{ca}{2}-\frac{ab}{2}-2q\right)=\frac{1}{2}\left(-\frac{q}{2}+\frac{bc}{2}-2q\right)=\frac{1}{4}(bc-5q)=\frac{1}{4}(a^2-4q),$$

and this midpoint $\left(a,\frac{1}{4}a^2-q\right)$ lies on the circle (1.7), i.e., the points U_1 and U_2 are inverse to each other. □

THEOREM 2.11. If L, M, N are the intersections of the corresponding sides of the orthic triangle $A_hB_hC_h$ and the contact triangle $A_iB_iC_i$ of the allowable triangle ABC, then the triangles ABC and LMN are homological. The center of this homology lies on the orthic line of the triangle ABC and its axis is parallel to the Lemoine line of this triangle.

In the Euclidean case, the center of homology is the Feuerbach point of the triangle ABC (see [8]).

Proof. First of the three analogous points

$$\begin{aligned} L&=\left(-\frac{2bc}{a},-q-2bc\right),\\ M&=\left(-\frac{2ca}{b},-q-2ca\right),\\ N&=\left(-\frac{2ab}{c},-q-2ab\right) \end{aligned} \tag{2.6}$$

lies on lines (1.6) and (1.10) and therefore $L=B_hC_h\cap B_iC_i$. The line

$$(3bc-q)y=3px-a^2q$$

passes through points $A=(a,a^2)$ and L, since, from (1.2) and (1.3) we obtain

$$\begin{aligned} &(3bc-q)a^2-3pa+a^2q=0,\\ &(3bc-q)(-q-2bc)-3p\left(-\frac{2bc}{a}\right)+a^2q=q^2-bcq-6b^2c^2+6\frac{p}{a}bc+a^2q\\ &\qquad=q(q-bc+a^2)=0. \end{aligned}$$

Also, this line obviously passes through the point

$$T=\left(-\frac{2q^2}{9p},-\frac{q}{3}\right) \tag{2.7}$$

because

$$(3bc-q)\left(-\frac{q}{3}\right)-3p\left(-\frac{2q^2}{9p}\right)+a^2q=q(q-bc+a^2)=0.$$

Likewise, the lines BM and CN pass through the point T. This point lies on the orthic line of the triangle ABC given by $y=-\frac{q}{3}$. The line

$$y=-\frac{bc}{a}x-q+2bc$$

passes through points M and N from (2.6) because e.g. by (1.1), for the point M we get

$$-\frac{bc}{a}\left(-\frac{2ca}{b}\right)-q+2bc=-q+2bc+2c^2=-q-2ca.$$

The point

$$\left(\frac{3p}{q}-a,-q-\frac{3ap}{q}\right) \tag{2.8}$$

lies on this line as well as on the line BC because, according to (1.2) and (1.3), we get

$$\frac{bc}{a}\left(\frac{3p}{q}-a\right)-q+2bc=-\frac{3b^2c^2}{q}-q+3bc$$
$$=-\frac{3bc}{q}(q+a^2)-q+3bc=-q-\frac{3ap}{q},$$

and therefore

$$-a\left(\frac{3p}{q}-a\right)-bc=-q-\frac{3ap}{q}.$$

Hence, the point (2.8) is the intersection $BC \cap MN$. It lies on the line

$$y=\frac{3p}{q}x-q-\frac{9p^2}{q^2} \tag{2.9}$$

because

$$\frac{3p}{q}\left(\frac{3p}{q}-a\right)-q-\frac{9p^2}{q^2}=-q-\frac{3ap}{q},$$

and analogous points $CA \cap NL$ and $AB \cap LM$ lie on this line as well. Line (2.9) is parallel to the line (1.8), i.e., to the Lemoine line of the triangle ABC. □

COROLLARY 2.12. In case of the standard triangle ABC, the center and the axis of homology of the triangles ABC and LMN from Theorem 2.11 are given by (2.7) and (2.9), and the vertices of the triangle LMN are given by (2.6).

REFERENCES

[1] Beban-Brkić, J., Kolar-Šuper, R., Kolar-Begović, Z., and Volenec, V. On Feuerbach's theorem and a pencil of circles in the isotropic plane. *Journal for Geometry and Graphics 10,* 2 (2006), 125–132.

[2] Beban-Brkić, J., Kolar-Šuper, R., Kolar-Begović, Z., and Volenec, V. Gergonne's point of a triangle in I_2. *Rad Hrvat. Akad. Znan. Umjet. 515,* 2 (2013), 95–106.

[3] Fontené, G. Correspondance. *Nouv. Ann. Math. 3,* 4 (1903), 183–184.

[4] Guba, V. S. Zadača no. 2049. *Matematika v škole 4,* (1979), 69.

[5] Kolar-Begović, Z., Kolar-Šuper, R., Beban-Brkić, J., and Volenec, V. Symmedians and the symmedian center of the triangle in an isotropic plane. *Mathematica Pannonica 17,* 2 (2006), 287–301.

[6] Kolar-Šuper, R., Kolar-Begović, Z., Volenec, V., and Beban-Brkić, J. Metrical relationships in a standard triangle in an isotropic plane. *Mathematical Communications 10,* 2 (2005), 149–157.

[7] Kolar-Šuper, R., Kolar-Begović, Z., Volenec, V., and Beban-Brkić, J. Isogonality and inversion in an isotropic plane. *International Journal of Pure and Applied Mathematics 44,* 2 (2008), 339–346.

[8] Pelletier, A. Problem 3440. *Amer. Math. Monthly 38,* (1932).

[9] Sachs, H. *Ebene isotrope Geometrie.* Vieweg–Verlag, Braunschweig/Wiesbaden, 1987.

[10] Thebault, V. Question 1963. *Mathesis 14,* 4 (1914), 88.

[11] Thebault, V. Contributions a la géométrie du triangle. *Mathesis 46,* (1932), suppl. 15.

[12] Volenec, V., Kolar-Begović, Z., and Kolar-Šuper, R. Steiner's ellipses of the triangle in an isotropic plane. *Mathematica Pannonica 21,* 2 (2010), 229–238.