

## ON THE ALMOST SURE CENTRAL LIMIT THEOREM ALONG SUBSEQUENCES

István Berkes<sup>1\*</sup> and Endre Csáki<sup>1</sup>

<sup>1</sup> Alfréd Rényi Institute of Mathematics, Budapest, Hungary

Communicated by László Tóth

Original Research Paper

Received: Oct 17, 2021 · Accepted: Oct 27, 2021

First published online: Feb 18, 2022

© 2021 The Author(s)



### ABSTRACT

Generalizing results of Schatte [11] and Atlagh and Weber [2], in this paper we give conditions for a sequence of random variables to satisfy the almost sure central limit theorem along a given sequence of integers.

### KEYWORDS

almost sure central limit theorem, summation methods, subsequences

### MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 60F05, 60F15

## 1. INTRODUCTION

Schatte [11] proved that if  $(X_n)$  is a sequence of i.i.d. random variables with  $EX_1 = 0$ ,  $EX_1^2 = 1$ ,  $E|X_1|^3 < +\infty$ , then setting  $S_k = \sum_{i=1}^k X_i$  we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N I \left\{ \frac{S_{2^k}}{\sqrt{2^k}} < x \right\} = \Phi(x) \quad \text{a.s. for any } x,$$

where  $I$  denotes indicator function. Atlagh and Weber [2] proved that this result remains valid without  $E|X_1|^3 < +\infty$ . In other words, if we consider  $S_n/\sqrt{n}$  only along the subsequence  $2^k$ , then the logarithmic averages in the pointwise central limit theorem can be replaced by ordinary (Cesàro) averages. The following theorem shows how to choose the weights in the a.s. central limit theorem when we consider the partial sums only along a given subsequence  $(n_k)$  of integers. In what follows, we set  $\log_+ x = \log(x \vee e)$  for  $x > 0$ .

**THEOREM.** Let  $X_1, X_2, \dots$  be independent, real valued random variables and put  $S_n = \sum_{k=1}^n X_k$ . Assume that

$$(S_n - a_n)/b_n \xrightarrow{d} G \quad \text{as } n \rightarrow \infty \quad (1.1)$$

\* Corresponding author. E-mail: berkes.istvan@renyi.hu  
Research supported by NKFIH grant K 125569.

and

$$E \left( \log_+ \log_+ \left| \frac{S_n - a_n}{b_n} \right| \right)^{1+\delta} \leq M \quad (n = 1, 2, \dots) \quad (1.2)$$

hold for some sequences  $(a_n)$ ,  $(b_n)$ , constants  $M > 0$ ,  $\delta > 0$  and a distribution function  $G$  where  $(b_n)$  is a positive nondecreasing sequence satisfying  $b_n \rightarrow \infty$ . Let  $(n_k)$  be an increasing sequence of positive integers such that

$$b_{n_{k+1}}/b_{n_k} = O(1) \quad (1.3)$$

and set

$$d_k = \log(b_{n_{k+1}}/b_{n_k}), \quad D_n = \sum_{k \leq n} d_k. \quad (1.4)$$

Then we have

$$\lim_{N \rightarrow \infty} \frac{1}{D_N} \sum_{k=1}^N d_k I \left\{ \frac{S_{n_k} - a_{n_k}}{b_{n_k}} \leq x \right\} = G(x) \quad \text{a.s. for any } x \in C_G \quad (1.5)$$

where  $C_G$  denotes the set of continuity points of  $G$ .

In the special case  $n_k = k$ , this result was proved by Lifshits [8], who also proved that condition (1.2) is optimal. If in addition to  $n_k = k$ ,  $(b_n)$  grows at least with polynomial speed, relation (1.5) holds with weights  $d_k = 1/k$ , see Berkes and Dehling [3]. See also Atlagh [1], Major [9], [10], Ibragimov and Lifshits [7] for a.s. central limit theorems with the weights  $d_k = \log(b_{k+1}/b_k)$ .

To understand the significance of weight sequences in almost sure central limit theory, we mention a few facts from summation theory. Any sequence  $\mathbf{D} = (d_1, d_2, \dots)$  of positive numbers with  $\sum d_n = \infty$  defines a linear summation method (Riesz summation of order 1) as follows. Given a real sequence  $(x_n)$ , put

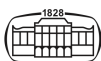
$$\sigma_n^{(\mathbf{D})} = D_n^{-1} \sum_{k \leq n} d_k x_k \quad \text{where} \quad D_n = \sum_{k \leq n} d_k.$$

We say that  $(x_n)$  is  $\mathbf{D}$ -summable if  $\sigma_n^{(\mathbf{D})}$  has a finite limit. By a classical theorem of Hardy (see e.g. [5], p. 35; see also [5], pp. 37-38 for a more general version due to Hirst), if two sequences  $\mathbf{D} = (d_n)$  and  $\mathbf{D}^* = (d_n^*)$  with partial sums  $D_n$  and  $D_n^*$  satisfy  $D_n^* = O(D_n)$  then, under mild regularity conditions, the summation procedure defined by  $\mathbf{D}^*$  is stronger (i.e. more effective) than the procedure defined by  $\mathbf{D}$  in the sense that if a sequence  $(x_n)$  is  $\mathbf{D}$ -summable then it is also  $\mathbf{D}^*$ -summable and to the same limit. Moreover, if  $D_n^\alpha \leq D_n^* \leq D_n^\beta$  for some  $0 < \alpha < \beta$  and sufficiently large  $n$ , then by a theorem of Zygmund (see also [5], p. 35) the summation procedures defined by  $\mathbf{D}$  and  $\mathbf{D}^*$  are equivalent, i.e.,  $\sigma_n^{(\mathbf{D})}$  converges for some  $(x_n)$  iff  $\sigma_n^{(\mathbf{D}^*)}$  does. Finally, if  $D_n^* = O(D_n^\gamma)$  for all  $\gamma > 0$  then the summation method defined by  $\mathbf{D}^*$  is strictly stronger than the method defined by  $\mathbf{D}$ . (For further results see [5], Chapter 2; we refer also to [4] for various connections between summation methods and probability theory.) For example, logarithmic summation defined by  $d_k = 1/k$  is stronger than Cesàro (or  $(C, 1)$ ) summation defined by  $d_k = 1$ ; on the other hand, all summation procedures defined by  $d_k = (\log k)^\alpha/k$ ,  $\alpha > -1$  are equivalent to logarithmic summation.

Returning to the Theorem, if  $n_k = [k^\alpha]$  ( $\alpha \geq 1$ ) then  $d_k \sim \alpha/k$ , i.e. the weights are still logarithmic. If  $n_k = [e^{k^\alpha}]$  ( $0 < \alpha \leq 1$ ), then  $d_k \sim \alpha/k^{1-\alpha}$ , which, as the previous discussion shows, corresponds to a weaker summation procedure. The case  $n_k = 2^k$  covers the results in [2], [11] mentioned above. As the discussion above also shows, the faster  $(n_k)$  grows, the weaker the summation procedure in the Theorem (and thus the stronger the result itself) becomes. For example, in the case  $n_k = 2^k$  one can use a weaker summation procedure (namely Cesàro summation) to get the a.s. convergence of indicators than in the case  $n_k = k$ , when logarithmic summation is needed.

**Proof of the theorem.** We prove that under the assumptions of the Theorem, relation (1.5) is equivalent to

$$\lim_{N \rightarrow \infty} \frac{1}{D_N} \sum_{k=1}^N d_k P \left\{ \frac{S_{n_k} - a_{n_k}}{b_{n_k}} \leq x \right\} = G(x) \quad \text{for any } x \in C_G. \quad (1.6)$$



Since (1.1) obviously implies (1.6), this will prove the theorem. We will also see that instead of the weights  $d_k$  in (1.4), the theorem remains valid with

$$0 \leq d_k \leq \log(b_{n_{k+1}}/b_{n_k}), \quad D_n = \sum_{k=1}^n d_k \longrightarrow \infty. \tag{1.7}$$

Replacing  $X_n$  by  $X_n - c_n$  where  $c_n = a_n - a_{n-1}$ , we can assume  $a_n = 0$ . Let  $BL = BL(0, 1)$  denote the set of functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that for some  $K > 0$

$$|f(x) - f(y)| \leq K|x - y|, \quad |f(x)| \leq K \quad \text{for all } x, y \in \mathbb{R}. \tag{1.8}$$

Applying Theorem 8.3 of Dudley [6] it follows that relations (1.5) and (1.6) in the Theorem are equivalent, respectively, to

$$\lim_{N \rightarrow \infty} \frac{1}{D_N} \sum_{k=1}^N d_k f\left(\frac{S_{n_k}}{b_{n_k}}\right) = \int f dG \quad \text{a.s. for all } f \in BL$$

and

$$\lim_{N \rightarrow \infty} \frac{1}{D_N} \sum_{k=1}^N d_k E f\left(\frac{S_{n_k}}{b_{n_k}}\right) = \int f dG \quad \text{for all } f \in BL.$$

Hence to prove the Theorem, it suffices to show that for any  $f \in BL$  we have

$$\lim_{N \rightarrow \infty} \frac{1}{D_N} \sum_{k=1}^N d_k \xi_k = 0 \quad \text{a.s.} \tag{1.9}$$

where

$$\xi_k = f\left(\frac{S_{n_k}}{b_{n_k}}\right) - E f\left(\frac{S_{n_k}}{b_{n_k}}\right).$$

Put  $g(x) = (\log_+ \log_+ x)^{1+\delta}$ . To prove (1.9) we first note that

$$|E(\xi_k \xi_\ell)| \leq 4K^2 \frac{Eg(|S_{n_k}|/b_{n_k})}{g(b_{n_\ell}/b_{n_k})} \quad (1 \leq k < \ell) \tag{1.10}$$

where  $K$  denotes a constant such that (1.8) holds. Indeed, letting  $\lambda = b_{n_\ell}/b_{n_k}$  and using the independence of  $S_{n_k}$  and  $S_{n_\ell} - S_{n_k}$  and the fact that  $x/y \leq g(x)/g(y)$  for  $x \leq y$ , we get

$$\begin{aligned} |E(\xi_k \xi_\ell)| &= \left| \text{Cov} \left( f\left(\frac{S_{n_k}}{b_{n_k}}\right), f\left(\frac{S_{n_\ell}}{b_{n_\ell}}\right) \right) \right| = \left| \text{Cov} \left( f\left(\frac{S_{n_k}}{b_{n_k}}\right), f\left(\frac{S_{n_\ell}}{b_{n_\ell}}\right) - f\left(\frac{S_{n_\ell} - S_{n_k}}{b_{n_\ell}}\right) \right) \right| \\ &\leq 2K^2 E \left( \frac{|S_{n_k}|}{b_{n_\ell}} \wedge 2 \right) = 4K^2 E \left( \frac{1}{\lambda} \left( \frac{|S_{n_k}|}{2b_{n_k}} \wedge \lambda \right) \right) \leq \\ &\leq \frac{4K^2}{g(\lambda)} E g \left( \frac{|S_{n_k}|}{2b_{n_k}} \wedge \lambda \right) \leq \frac{4K^2}{g(\lambda)} E g \left( \frac{|S_{n_k}|}{b_{n_k}} \right), \end{aligned}$$

proving (1.10). Now

$$E \left( \sum_{k \leq N} d_k \xi_k \right)^2 \leq 2 \sum_{1 \leq k < \ell \leq N} d_k d_\ell |E(\xi_k \xi_\ell)|. \tag{1.11}$$

Let  $N$  be so large that  $D_N \geq 4$ . By (1.2) and (1.10), the contribution of those terms in the sum on the right side of (1.11) where

$$b_{n_\ell}/b_{n_k} \geq \exp(D_N^{1/2})$$

is at most

$$\text{const} \cdot (\log D_N)^{-1-\delta} \sum_{1 \leq k < \ell \leq N} d_k d_\ell \leq \text{const} \cdot (\log D_N)^{-1-\delta} D_N^2.$$

On the other hand, letting  $M = \sup_{k \geq 1} (b_{n_{k+1}}/b_{n_k})$ , the relation  $b_{n_\ell}/b_{n_k} \leq \exp(D_N^{1/2})$  implies

$$\log b_{n_{\ell+1}} - \log b_{n_k} \leq \log b_{n_\ell} - \log b_{n_k} + \log M$$



and thus (1.7) and the trivial estimate  $|E(\xi_k \xi_\ell)| \leq 4K^2$  show that the contribution of those terms on the right-hand side of (1.11) where  $b_{n_\ell}/b_{n_k} < \exp(D_N^{1/2})$  is

$$\begin{aligned} &\leq 8K^2 \sum_{k=1}^N d_k \sum_{\{k \leq \ell \leq N : b_{n_\ell} \leq b_{n_k} \exp(D_N^{1/2})\}} d_\ell \\ &\leq \text{const} \cdot \sum_{k=1}^N d_k \sum_{\{k \leq \ell \leq N : b_{n_\ell} \leq b_{n_k} \exp(D_N^{1/2})\}} (\log b_{n_{\ell^*}} - \log b_{n_k}) \\ &\leq \text{const} \cdot \sum_{k=1}^N d_k (\log b_{n_{\ell^*}} - \log b_{n_k}) \leq \text{const} \cdot \sum_{k=1}^N d_k (\log M + \log b_{n_{\ell^*}} - \log b_{n_k}) \\ &\leq \text{const} \cdot \sum_{k=1}^N d_k (\log M + D_N^{1/2}) \leq \text{const} \cdot D_N^{3/2}, \end{aligned}$$

where  $\ell^*$  denotes the largest index  $k \leq \ell \leq N$  such that  $b_{n_\ell}/b_{n_k} < \exp(D_N^{1/2})$ . (If there is no such  $\ell$ , the sums above are empty.) Hence setting

$$T_N = \frac{1}{D_N} \sum_{k \leq N} d_k \xi_k$$

we get

$$ET_N^2 \leq \text{const} \cdot (\log D_N)^{-1-\delta}.$$

Let  $\eta > 0$  be so small that  $(1 + \delta)(1 - \eta) > 1$ . Since  $D_{n+1}/D_n \rightarrow 1$  by the assumptions of the Theorem, we can choose a nondecreasing sequence  $(N_k)$  of positive integers such that

$$D_{N_k} \sim e^{k^{1-\eta}} \quad (1.12)$$

and consequently

$$ET_{N_k}^2 \leq \text{const} \cdot k^{-1-\varrho}$$

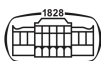
for some  $\varrho > 0$ . Hence we have  $\sum_{k=1}^{\infty} ET_{N_k}^2 < +\infty$  and consequently  $\sum_{k=1}^{\infty} T_{N_k}^2 < +\infty$  a.s., implying  $T_{N_k} \rightarrow 0$  a.s. Now for  $N_k < N \leq N_{k+1}$  we have

$$|T_N| \leq |T_{N_k}| + \frac{\text{const}}{D_N} \sum_{i=N_k+1}^N d_i = |T_{N_k}| + \text{const} \left(1 - \frac{D_{N_k}}{D_N}\right).$$

Since  $D_{N_{k+1}}/D_{N_k} \rightarrow 1$  by (1.12), it follows that  $T_N \rightarrow 0$  a.s., completing the proof of the theorem.  $\square$

## REFERENCES

- [1] ATLAGH, M. Théorème central limite presque sûr et loi du logarithme itéré pour des sommes de variables aléatoires indépendantes. C. R. Acad. Sci. Paris Sér. I Math. 316 (1993), 929–933.
- [2] ATLAGH, M., and WEBER, M. Un théorème central limite presque sûr relatif à des sous-suites. C. R. Acad. Sci. Paris Sér. I 315 (1992), 203–206.
- [3] BERKES, I., and DEHLING, H. Some limit theorems in log density, Ann. Probab. 21 (1993), 1640–1670.
- [4] BINGHAM, N. H., and ROGERS, L. C. G. Summability methods and almost sure convergence. In: Bellow, A., Jones, R., editors, Almost Everywhere Convergence II, Academic Press, New York, 1991, pages 69–83.
- [5] CHANDRASEKHARAN, K., and MINAKSHISUNDARAM, S. Typical Means. Oxford University Press, Oxford, 1952.
- [6] DUDLEY, R. M. Probabilities and metrics. Lecture Notes Series 45, Aarhus Universitet, 1976.
- [7] IBRAGIMOV, I. A., and LIFSHITS, M. A. On limit theorems of “almost sure” type. (Russian) Teor. Veroyatnost. i Primenen. 44 (1999), 328–350; translation in Theory Probab. Appl. 44 (2000), 254–272.



- [8] LIFSHITS, M. The almost sure limit theorem for sums of random vectors, *J. Math. Sciences* 109 (2002), 2166–2178.
- [9] MAJOR, P. Almost sure functional limit theorems. I. The general case. *Studia Sci. Math. Hungar.* 34 (1998), 273–304.
- [10] MAJOR, P. Almost sure functional limit theorems. II. The case of independent random variables. *Studia Sci. Math. Hungar.* 36 (2000), 231–273.
- [11] SCHATTE, P. On strong versions of the central limit theorem. *Math. Nachr.* 137 (1988), 249–256.

---

**Open Access statement.** This is an open-access article distributed under the terms of the Creative Commons Attribution-NonCommercial 4.0 International License (<https://creativecommons.org/licenses/by-nc/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium for non-commercial purposes, provided the original author and source are credited, a link to the CC License is provided, and changes – if any – are indicated.

