

ON A SPECIAL GINI MEAN

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ABSTRACT

We offer new properties of the special Gini mean $S(a, b) = a^{a/(a+b)} \cdot b^{b/(a+b)}$, in connections with other special means of two arguments.

KEYWORDS

Gini means, special means of two arguments, inequalities

MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 26D15; Secondary 26D99

1. INTRODUCTION

Let $a, b > 0$ be two positive real numbers. The classical logarithmic and identric means of a and b are given by

$$L(a, b) = \begin{cases} \frac{a-b}{\ln a - \ln b}, & a \neq b \\ a, & a = b \end{cases} \quad \text{and} \quad I(a, b) = \begin{cases} \frac{1}{e} (b^b/a^a)^{1/(b-a)}, & a \neq b \\ a, & a = b \end{cases}. \quad (1.1)$$

By studying certain properties of the means L and I , in 1990 [5] the author has introduced the following mean:

$$S(a, b) = (a^a \cdot b^b)^{1/(b+a)}. \quad (1.2)$$

Later [6], by studying identities and inequalities for various means, he discovered the identity

$$S(a, b) = \frac{I(a^2, b^2)}{I(a, b)}. \quad (1.3)$$

Let us introduce also the classical arithmetic and geometric means

$$A(a, b) = \frac{a+b}{2}, \quad G(a, b) = \sqrt{ab}.$$

Many new properties, involving also identities and inequalities have been proved by I. Raşa and the author [7]. We quote only the following results:

$$\ln \frac{S}{I} = 1 - \frac{G^2}{AL}; \quad (1.4)$$

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$$S(a, b) < C(a, b), \quad \text{for } a \neq b \tag{1.5}$$

where

$$C(a, b) = \frac{a^2 + b^2}{a + b}$$

$$S(a, b) > Q(a, b), \quad \text{for } a \neq b \tag{1.6}$$

where

$$Q(a, b) = \sqrt{\frac{a^2 + b^2}{2}}.$$

In [7] the authors have proved also that

$$\frac{S - G}{S - A} > \sqrt{2}. \tag{1.7}$$

The mean S is in fact a special case of the Gini mean introduced in 1938 [1] as follows:

$$G_{x,y}(a, b) = \begin{cases} \left(\frac{a^x + b^x}{a^y + b^y} \right)^{\frac{1}{x-y}}, & x \neq y \\ \exp \left(\frac{a^x \ln a + b^x \ln b}{a^x + b^x} \right), & x = y \neq 0 \\ \sqrt{ab}, & x = y = 0. \end{cases} \tag{1.8}$$

In fact, one has $G_{1,1}(a, b) = S(a, b)$.

But the mean S has a central role, since by denoting $S_t(a, b) = (S(a^t, b^t))^{1/t}$, one has the following integral representation (see [2])

$$\ln G_{x,y} = \frac{1}{y-x} \cdot \int_x^y \ln S_t dt. \tag{1.9}$$

In paper [2], E. Neuman and the author have used identity (1.9), as well as inequality (1.6) in the study of the general Gini mean $G_{x,y}$. Particularly, convexity properties, comparison with the general Stolarsky means, and Ky Fan type inequalities have been established.

An interesting application of inequality (1.6) has been provided in another paper by E. Neuman and the author [3].

One has the inequality

$$L < L(A, G) \quad (a \neq b). \tag{1.10}$$

Indeed, by applying (1.6) for \sqrt{a} in place of a and \sqrt{b} in place of b , and taking logarithms, we get

$$\ln A < \frac{\sqrt{a} \ln a + \sqrt{b} \ln b}{\sqrt{a} + \sqrt{b}},$$

thus

$$\ln A - \ln G < \frac{\sqrt{a} \ln a + \sqrt{b} \ln b}{\sqrt{a} + \sqrt{b}} - \frac{1}{2}(\ln a + \ln b) = \frac{(\sqrt{b} - \sqrt{a})(\ln b - \ln a)}{2(\sqrt{a} + \sqrt{b})} = \frac{A - G}{L},$$

so (1.10) follows.

The aim of this paper is to prove refinement of inequalities (1.6) and (1.7) by applying new methods.

Connections with certain exponential means will be pointed out, too.

2. MAIN RESULTS

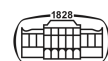
The mean S is a special Gini mean, but it is also a special weighted geometric mean of a and b , the weights being $\frac{a}{a+b}$ and $\frac{b}{a+b}$.

Thus, using the classical weighted arithmetic mean – geometric mean inequality, we get relation (1.5). By using the weighted geometric mean – harmonic mean inequality, we get $S > A$, but (1.6) is a refinement of this relation, as clearly $Q > A$.

The following result shows connections between the means G, A and C :

THEOREM 1. One has the double inequality:

$$\frac{C + 2G}{3} < A < Q < \frac{C + G}{2} \quad (a \neq b). \tag{2.1}$$



Proof. By dividing both sides of (2.1) with b , and putting $\frac{a}{b} = t$, (2.1) may be rewritten as

$$\frac{1}{3} \left(\frac{t^2 + 1}{t + 1} + 2\sqrt{t} \right) < \frac{t + 1}{2} < \sqrt{\frac{t^2 + 1}{2}} < \frac{\frac{t^2 + 1}{t + 1} + \sqrt{t}}{2}.$$

Clearly, one must prove the first and the last inequality. The left side can be rewritten equivalently as

$$t^2 + 6t + 1 > 4(t + 1) \cdot \sqrt{t}$$

and this follows by the classical relation $u + v > 2\sqrt{uv}$ ($u, v > 0$, $u \neq v$) for $u = t^2 + 2t + 1$, $v = 4t$.

For the right side inequality, put $t = p^2$. The inequality becomes

$$(p^2 + 1)\sqrt{2(p^4 + 1)} > (p^2 + 1)p + p^4 + 1 = p^4 + p^3 + p + 1,$$

and by taking squares, it becomes

$$f(p) = p^8 - 2p^7 + 3p^6 - 2p^5 - 2p^3 + 3p^2 - 2p + 1 > 0.$$

Now, it is immediate that $f(p)$ can be written as

$$f(p) = (p - 1)^2 \cdot (p^6 + 2p^4 + 2p^3 + 2p^2 + 1) > 0,$$

so the result follows. \square

THEOREM 2. One has

$$S < C < \frac{A\sqrt{2} - G}{\sqrt{2} - 1} \quad (a \neq b). \quad (2.2)$$

Proof. The first inequality of (2.2) is exactly relation (1.5). Now, the second inequality can be written as

$$C(\sqrt{2} - 1) + G < A \cdot \sqrt{2}. \quad (2.3)$$

Now, by the left side of (2.1) one has $G < \frac{3A - C}{2}$. It is sufficient to prove that

$$\frac{3A - C}{2} + C(\sqrt{2} - 1) < A\sqrt{2},$$

or equivalently $A \cdot (3 - 2\sqrt{2}) < C(3 - 2\sqrt{2})$. Since $3 - 2\sqrt{2} > 0$, this becomes $A < C$, which is trivially true. Thus (2.3) follows. \square

REMARK 1. Remark that inequality (1.7) may be rewritten as

$$S < \frac{A\sqrt{2} - G}{\sqrt{2} - 1}.$$

Therefore, (2.2) is refinement of (1.7). It is also of interest to be noted that the proof of (1.7) from [7] is quite complicated, involving more auxiliary functions.

THEOREM 3. One has

$$S > \frac{A + C}{2} > Q \quad (a \neq b). \quad (2.4)$$

Proof. First remark that, the second inequality of (2.4) follows by

$$\frac{A + C}{2} > \sqrt{A \cdot C} = Q.$$

For the proof of the first inequality of (2.4), divide both sides with b and put $a/b = t > 1$. Then using definition (1.2), the inequality becomes

$$t^{t/(t+1)} > \frac{1}{2} \cdot \left(\frac{t + 1}{2} + \frac{t^2 + 1}{t + 1} \right). \quad (2.5)$$

This can be written equivalently as

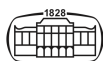
$$4 \cdot (t + 1) \cdot t^{t/(t+1)} > 3t^2 + 2t + 3,$$

and by taking logarithms as

$$F(t) = 2 \ln 2 + \ln(t + 1) + \frac{t}{t + 1} \ln t - \ln(3t^2 + 2t + 3) > 0.$$

This function is defined also for $t = 1$ and $F(1) = 0$. By computing the first derivative of this function, one gets

$$F'(t) = \frac{\ln t}{(t + 1)^2} - \frac{4(t - 1)}{(t + 1)(3t^2 + 2t + 3)},$$



after elementary computations, which we omit here.

Now, by the basic inequality between the logarithmic and arithmetic inequality, namely $L < A$ (see e.g. [5] for references) one has $L(t, 1) < A(t, 1)$, so

$$\frac{t-1}{\ln t} < \frac{t+1}{2}. \tag{2.6}$$

Now, by (2.6), for $t > 1$ we have

$$(3t^2 + 2t + 3) \ln t > 2(t-1) \cdot \frac{3t^2 + 2t + 3}{t+1}.$$

We will prove that

$$2(t-1) \cdot \frac{3t^2 + 2t + 3}{t+1} > 4(t-1)(t+1), \tag{2.7}$$

and this clearly will imply that $F'(t) > 0$ for $t > 1$. Now, for $t > 1$, clearly (2.7) becomes $3t^2 + 2t + 3 > 2t^2 + 4t + 2$, i.e. $t^2 - 2t + 1 > 0$, or $(t-1)^2 > 0$. As $F'(t) > 0$ for $t > 1$, the function $F(t)$ will be strictly increasing, implying that $F(t) > F(1) = 0$, so (2.5) follows. This finishes the proof of Theorem 3. \square

REMARK 2. By (2.1), (2.4) we get the following chain of inequalities

$$S > \frac{A+C}{2} > Q > \frac{G+C}{2} > A > \frac{C+2G}{3} > G. \tag{2.8}$$

3. ON CERTAIN EXPONENTIAL MEANS

The exponential mean

$$E(a, b) = \frac{be^b - ae^a}{e^b - e^a} - 1 \quad (a \neq b); \quad E(a, a) = 0 \tag{3.1}$$

has been introduced by G. Toader [10], and studied also in [4], [9].

Another exponential mean is

$$F(a, b) = \frac{ae^a + be^b}{e^a + e^b}. \tag{3.2}$$

It is immediate by definitions (1.1) and (1.2) that one has

$$E(a, b) = \ln I(e^a, e^b) \tag{3.3}$$

$$F(a, b) = \ln S(e^a, e^b).$$

It is also immediate that from identity (1.3) and (3.3) we get

$$F(a, b) = E(2a, 2b) - E(a, b). \tag{3.4}$$

Let us define, as in [9] the means

$$\mathcal{L}(a, b) = \frac{e^b - e^a}{b - a} = \ln(L(e^a, e^b)) \tag{3.5}$$

$$\mathcal{A}(a, b) = \ln \left(\frac{e^b + e^a}{2} \right) = \ln(A(e^a, e^b)). \tag{3.6}$$

By using identity (1.4), we get also

$$F - E = 1 - e^{2A - (A + \mathcal{L})}. \tag{3.7}$$

If one defines

$$\beta = \ln Q(e^a, e^b), \quad C = \ln C(e^a, e^b), \tag{3.8}$$

then we get by $C = Q^2/A$ that

$$C = 2\beta - A. \tag{3.9}$$

Now, by using the theorems of the first two paragraphs, we can state:

THEOREM 4.

$$2\beta - A > F > \ln \left(\frac{e^{2\beta - A} + e^{\ln A}}{2} \right) > \beta > A > E > A \quad (a \neq b). \tag{3.10}$$



Proof. The first inequality follows by (3.9) and (1.5), while the second and third ones by (2.4). The inequality $\mathcal{A} > E$ follows by the classical inequality $A > I$, combined with (3.3), while the last inequality of (3.10), i.e. $E > A$ is a classical inequality, due to Toader. We give here a new proof of this result, based on the Cebisev integral inequality:

$$(b-a) \int_a^b f(x)g(x)dx > \int_a^b f(x)dx \cdot \int_a^b g(x)dx, \quad (3.11)$$

where $f(x)$ and $g(x)$ are of the same monotonicity functions.

Let $f(x) = x$, $g(x) = e^x$. As

$$E(a, b) = \frac{\int_a^b x e^x dx}{\int_a^b e^x dx},$$

a simple computation gives $E > A$. Improvements of this inequality may be found e.g. in [6], [8], [9]. □

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