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EXISTENCE AND STABILITY OF SOLUTIONS FOR A NONLINEAR BEAM EQUATION WITH INTERNAL DAMPING

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ABSTRACT

This manuscript deals with the global existence and asymptotic behavior of solutions for a Kirchhoff beam equation with internal damping. The existence of solutions is obtained by using the Faedo-Galerkin method. Exponential stability is proved by applying Nakao's theorem.

KEYWORDS

Nonlinear beam equation, internal damping, global solution, stability

MATHEMATICS SUBJECT CLASSIFICATION (2020)

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1. INTRODUCTION

In this paper, we deal with the existence and decay of solution for the nonlinear initial boundary value problem

$$u'' + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + |u|^\rho + u' = 0, \quad \text{in} \quad \Omega \times (0, T), \tag{1.1}$$

$$u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x, t) = \frac{\partial u}{\partial \eta}(x, t) = 0, \quad x \in \partial\Omega, \ t \geq 0, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $T > 0$ is a fixed but arbitrary real number, $\rho > 1$ is a real number, $M(s)$ is a continuous function on $[0, +\infty)$ and η is the unit outward normal on $\partial\Omega$. The physical meaning of (1.3) is that, with the natural boundary conditions, we imposed a priori conditions on the function space and it turns out that a weak solution automatically satisfies the boundary conditions.

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More physical justifications and the model background are discussed in [3, 6]. In addition, mathematical points of view are explained in the pioneer works [2, 7, 18]. In [9], an extensive list of references about the Kirchhoff equation can be found.

Now, we present a short overview with new contributions on the subject. In 1999, in [11], the initial-boundary value problem for the Kirchhoff equation was studied

$$\begin{aligned} u'' - M\Big(t, \int_\Omega |\nabla u|^2 dx\Big)\Delta u = f \quad &\text{in } \Omega \times (0,\infty),\\ u = 0 \quad &\text{on } \Gamma_0 \times (0,\infty),\\ \mu(t)\frac{\partial u}{\partial \nu} + \delta(x)u' = 0 \quad &\text{on } \Gamma_1 \times (0,\infty),\\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), \quad &x \in \Omega, \end{aligned} \tag{1.4}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with its boundary given by two disjoint parts Γ_0 and Γ_1, $\nu(x)$ is the unit exterior normal vector at $x \in \Gamma_1$. $\delta(x)$ is a real function defined in Γ_1 and $\mu(t)$ is a positive real function. By the construction of a special basis and the Galerkin method, the authors proved existence and uniqueness of solutions for (1.4).

Cavalcanti et al in 2004 [4], studied the equation

$$u'' + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + g(u') + f(u) = 0, \tag{1.5}$$

with $g(s) = |s|^{\rho-1}s$ and $f(s) = |s|^{\gamma-1}s$ where ρ and γ are positive constants such that $1 < \rho, \gamma \leq n/(n-2)$ if $n \geq 3$; $\rho, \gamma > 1$ if $n = 1, 2$. The global existence and asymptotic stability were proved by means of the fixed point theorem and continuity arguments.

In 2013, Zhijian [19], investigated the problem (1.5) more generally as follows

$$u'' + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + g(u') + f(u) = h(x), \tag{1.6}$$

where the source terms $f, g \in C^1(\mathbb{R})$, $|f'(s)| \leq C(1 + |s|^{p-1})$ and $K_0|s|^{q-1} < g'(s) \leq C(1 + |s|^{q-1})$, $K_0, C > 0$ with $1 \leq p < \infty, 1 \leq q < \infty$ if $n \leq 4$; $1 \leq p \leq p^* = (n+4)/(n-4)$ and $p \leq q$ if $n \geq 5$. By Galerkin approximation combined with the monotone arguments, the author proved the existence of global solution.

On the same conditions of (1.4), in 2017, Milla et al [12], investigated the existence and uniqueness of local solutions of the initial value problem for the nonlinear mixed problem of Kirchhoff type

$$\begin{aligned} u'' - M\Big(t, \int_\Omega |\nabla u|^2 dx\Big)\Delta u + |u|^\rho = f \quad &\text{in } \Omega \times (0,T_0),\\ u = 0 \quad &\text{on } \Gamma_0 \times (0,T_0),\\ \frac{\partial u}{\partial \nu} + \delta(x)h(u') = 0 \quad &\text{on } \Gamma_1 \times (0,T_0),\\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), \quad &x \in \Omega, \end{aligned} \tag{1.7}$$

where $\rho > 1$ is a real number, $h(s)$ is real function defined in $\mathbb{R}$ and $T_0 > 0$ represents a finite time. The authors used the Galerkin method with a special basis, a modification of the Tartar approach, compactness method and fixed-point theorem.

In 2019, Pereira et al [15], considered the following nonlinear beam equation with internal damping and source term

$$u'' + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + u' = |u|^{r-1}u,$$

where $r > 1$ is a constant, $M(s)$ is a continuous function on $[0, +\infty)$. The authors constructed the global solutions by using the Faedo-Galerkin approximations, taking into account that the initial data is in appropriate set of stability created from the Nehari manifold. The asymptotic behavior was obtained by the Nakao method.

The mathematical structure of the paper is organized as follows. In the section 2 we present some hypotheses needed in the proof of our results. In the section 3 the global solutions are constructed by means of the Galerkin approximations. Finally, in the section 4 we apply the results due to M. Nakao [13, 14] to prove the exponential stability.

2. PRELIMINARIES

In this section, we present some hypotheses needed in the proof of our results. We use the standard Lebesgue space $L^2(\Omega)$ and Sobolev space $H_0^2(\Omega)$ with their usual scalar products and norms. We consider the following hypothesis:

(H.1) $M \in C([0,\infty))$ with $M(\lambda) \geq -\beta$, $\forall\, \lambda \geq 0$, $0 < \beta < \lambda_1$, λ_1 the first eigenvalue of the problem

$$\Delta^2 u - \lambda(-\Delta u) = 0.$$

REMARK 2.1. Let λ_1 the first eigenvalue of $\Delta^2 u - \lambda(-\Delta u) = 0$ with the following boundary conditions

$$u\big|_{\partial\Omega} = \frac{\partial u}{\partial \eta}\bigg|_{\partial\Omega} = 0,$$

then (see Miklin [10]),

$$\lambda_1 = \inf_{u \in H_0^2(\Omega)} \frac{|\Delta u|^2}{|\nabla u|^2} > 0 \text{ and } |\nabla u|^2 \leq \frac{1}{\lambda_1}|\Delta u|^2. \tag{2.1}$$

(H.2) We suppose $1 < \rho < \dfrac{n}{n-2}$ if $n \geq 5$. Observe that

$$\frac{n}{n-2} \leq \frac{n+4}{n-4} \quad \text{if} \quad n \geq 5. \tag{2.2}$$

REMARK 2.2. By (H.2) and (2.2) we have that

$$H_0^2(\Omega) \hookrightarrow L^{q*}(\Omega) \hookrightarrow L^{\rho+1}(\Omega), \quad \text{where} \quad \frac{1}{q*} = \frac{1}{2} - \frac{1}{n}. \tag{2.3}$$

3. EXISTENCE OF GLOBAL SOLUTION

In this section, our goal is to prove existence of solutions to the system (1.1) - (1.3). This is the content of Theorem 3.1.

THEOREM 3.1. Let $u_0 \in H_0^2(\Omega)$, $u_1 \in L^2(\Omega)$. If (H.1), (H.2) hold and

$$|\Delta u_0| < \lambda^* \stackrel{\text{def}}{=} \left(\frac{(\lambda_1 - \beta)(\rho+1)}{2\lambda_1 C_0^{\rho+1}} \right)^{\frac{1}{\rho-1}}, \tag{3.1}$$

$$\frac{1}{2}|u_1|^2 + \frac{1}{2}\left(1 + \frac{m_0}{\lambda_1}\right)|\Delta u_0|^2 + \frac{1}{\rho+1}\int_\Omega |u_0|^\rho u_0 \, \mathrm{d}x \stackrel{\text{def}}{=} N < \frac{\lambda_1 - \beta}{2\lambda_1}(\lambda^*)^2, \tag{3.2}$$

then, there exists a function $u: [0,T] \longrightarrow L^2(\Omega)$ in the class

$$u \in L^\infty(0,T; H_0^2(\Omega)), \tag{3.3}$$

$$u' \in L^\infty(0,T; L^2(\Omega)), \tag{3.4}$$

such that, for all $w \in H_0^2(\Omega)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(u'(t), w) + (\Delta u(t), \Delta w) + M(|\nabla u(t)|^2)(-\Delta u(t), w) + (|u(t)|^\rho, w) + (u'(t), w) = 0 \text{ in } \mathcal{D}'(0,T) \tag{3.5}$$

and

$$u(0) = u_0, \qquad u'(0) = u_1, \tag{3.6}$$

where C_0 is the constant of the embedding from $H_0^2(\Omega)$ into $L^{\rho+1}(\Omega)$ and

$$m_0 = \max_{0 \leq s \leq |\nabla u_{0m}| \leq C} M(s),$$

and, C a positive constant independent of m and t.

Proof. We use the Faedo-Galerkin method and a modification of the Tartar [17] approach. The proof is divided into three steps: Approximated problem, a priori estimates and passage to the limit.

3.1. Approximated problem

Let $(w_j)_{j\in\mathbb{N}}$ be an orthogonal basis of space $H_0^2(\Omega)$, and $V_m = \text{span}\{w_1, w_2, \dots, w_m\}$. Let

$$u_m(t) = \sum_{j=1}^{m} g_{j_m}(t) w_j$$

be a solution of the approximated problem

$$\big(u_m''(t), w\big) + \big(\Delta u_m(t), \Delta w\big) + M\big(|\nabla u_m(t)|^2\big)\big(-\Delta u_m(t), w\big) + \big(|u_m(t)|^\rho, w\big) + \big(u_m'(t), w\big) = 0, \forall w \in V_m, \tag{3.7}$$

$$u_m(0) = u_{0m} \longrightarrow u_0 \quad \text{strongly in} \quad H_0^2(\Omega), \tag{3.8}$$

$$u_m'(0) = u_{1m} \longrightarrow u_1 \quad \text{strongly in} \quad L^2(\Omega). \tag{3.9}$$

The system (3.7)—(3.9) has a local solution in $[0, t_m)$, $0 < t_m \le T$, by virtue of Carathéodory's Theorem, see [5].

The extension of the solution to the whole interval $[0, T]$ is a consequence of the following estimates.

3.2. A priori estimates

Let $w = u_m'(t)$ in (3.7). We get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}|u_m'(t)|^2 + \frac{1}{2}|\Delta u_m(t)|^2 + \frac{1}{2}\hat{M}\left(|\nabla u_m(t)|^2\right)\right] + \int_\Omega |u_m(t)|^\rho u_m'(t)\,\mathrm{d}x + |u_m'(t)|^2 = 0,$$

where $\hat{M}(s) = \int_0^s M(\xi)\,\mathrm{d}\xi$.

Now, $\int_\Omega |u_m(t)|^\rho u_m'(t)\,\mathrm{d}t = \frac{1}{\rho+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_m(t)|^\rho u_m(t)\,\mathrm{d}x$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\left(|u_m'(t)|^2 + |\Delta u_m(t)|^2 + \hat{M}(|\nabla u_m(t)|^2)\right) + \frac{1}{\rho+1}\int_\Omega |u_m(t)|^\rho u_m(t)\,\mathrm{d}x\right] + |u_m'(t)|^2 = 0. \tag{3.10}$$

Integrating (3.10) on $[0, t]$, $0 \le t \le t_m$, we obtain

$$\begin{aligned}&\frac{1}{2}\left(|u_m'(t)|^2 + |\Delta u_m(t)|^2 + \hat{M}(|\nabla u_m(t)|^2)\right) + \frac{1}{\rho+1}\int_\Omega |u_m(t)|^\rho u_m(t)\,\mathrm{d}x \\ &+ \int_0^t |u_m'(t)|^2\,\mathrm{d}s = \frac{1}{2}\left(|u_{1m}|^2 + |\Delta u_{0m}|^2 + \hat{M}(|\nabla u_{0m}|^2)\right) + \frac{1}{\rho+1}\int_\Omega |u_{0m}|^\rho u_{0m}\,\mathrm{d}x.\end{aligned} \tag{3.11}$$

By (H.1) we have

$$\hat{M}\left(|\nabla u_m(t)|^2\right) \ge -\frac{\beta}{\lambda_1}|\Delta u_m(t)|^2 \tag{3.12}$$

and

$$\hat{M}\left(|\nabla u_{0m}(t)|^2\right) \le m_0|\nabla u_{0m}|^2 \le \frac{m_0}{\lambda_1}|\Delta u_{0m}|^2. \tag{3.13}$$

Substituting (3.12) and (3.13) in (3.11) we obtain

$$\begin{aligned}&\frac{1}{2}|u_m'(t)|^2 + \frac{1}{2}\left(1 - \frac{\beta}{\lambda_1}\right)|\Delta u_m(t)|^2 + \frac{1}{\rho+1}\int_\Omega |u_m(t)|^\rho u_m(t)\,\mathrm{d}x \\ &+ \int_\Omega |u_m'(s)|^2\,\mathrm{d}s \le \frac{1}{2}|u_{1m}|^2 + \frac{1}{2}\left(1 + \frac{m_0}{\lambda_1}\right)|\Delta u_{0m}|^2 + \frac{1}{\rho+1}\int_\Omega |u_{0m}|^\rho u_{0m}\,\mathrm{d}x.\end{aligned} \tag{3.14}$$

By convergences (3.8) and (3.9) the second member of (3.14) is finite and, $\forall\ m \ge m_1$, we obtain

$$\frac{1}{2}|u_m'(t)|^2 + J\left(u_m(t)\right) + \int_0^t |u_m'(s)|^2\,\mathrm{d}s \le \frac{1}{2}|u_1|^2 + \frac{1}{2}\left(1 + \frac{m_0}{\lambda_1}\right)|\Delta u_0|^2 + \frac{1}{\rho+1}\int_\Omega |u_0|^\rho u_0\,\mathrm{d}x = N, \tag{3.15}$$

where $J : H_0^2(\Omega) \longrightarrow \mathbb{R}$ is defined by

$$J(u) = \frac{1}{2}\left(1 - \frac{\beta}{\lambda_1}\right)|\Delta u|^2 + \frac{1}{\rho+1}\int_\Omega |u|^\rho u \, \mathrm{d}x. \tag{3.16}$$

Now we note that

$$\left|\int_\Omega |u|^\rho u \, \mathrm{d}x\right| \leq \int_\Omega |u|^{\rho+1} \, \mathrm{d}x = |u|_{\rho+1}^{\rho+1} \leq C_0^{\rho+1}|\Delta u|^{\rho+1}.$$

Thus, $\int_\Omega |u|^\rho u \, \mathrm{d}x \geq -C_0^{\rho+1}|\Delta u|^{\rho+1}$ and

$$J(u) \geq \frac{1}{2}\left(1 - \frac{\beta}{\lambda_1}\right)|\Delta u|^2 - \frac{C_0^{\rho+1}}{\rho+1}|\Delta u|^{\rho+1}. \tag{3.17}$$

We introduce the function

$$F(\lambda) = \frac{1}{2}\left(1 - \frac{\beta}{\lambda_1}\right)\lambda^2 - \frac{C_0^{\rho+1}}{\rho+1}\lambda^{\rho+1}, \quad \lambda \geq 0.$$

We are interested in $\lambda \geq 0$ such that $F(\lambda) \geq 0$, that is,

$$\frac{1}{2}\left(1 - \frac{\beta}{\lambda_1}\right)\lambda^2 - \frac{C_0^{\rho+1}}{\rho+1}\lambda^{\rho+1} \geq 0.$$

We have

$$\lambda^2\left[\frac{1}{2}\left(1 - \frac{\beta}{\lambda_1}\right) - \frac{C_0^{\rho+1}}{\rho+1}\lambda^{\rho-1}\right] \geq 0, \quad \text{therefore} \quad \frac{1}{2}\left(1 - \frac{\beta}{\lambda_1}\right) - \frac{C_0^{\rho+1}}{\rho+1}\lambda^{\rho-1} \geq 0.$$

Then

$$0 \leq \lambda \leq \left(\frac{(\lambda_1 - \beta)(\rho+1)}{2\lambda_1 C_0^{\rho+1}}\right)^{\frac{1}{\rho+1}} \overset{\text{def}}{=} \lambda^*.$$

Thus,

$$F(\lambda) \geq 0 \quad \text{for} \quad 0 \leq \lambda \leq \lambda^*. \tag{3.18}$$

As consequence of (3.18), we obtain

$$J(u_m(t)) \geq \frac{1}{2}\left(1 - \frac{\beta}{\lambda_1}\right)|\Delta u_m(t)|^2 - \frac{C_0^{\rho+1}}{\rho+1}|\Delta u_m(t)|^{\rho+1} \geq 0, \quad \text{for} \quad |\Delta u_m(t)| \leq \lambda^*, \; t \in [0, t_m). \tag{3.19}$$

From (3.15) an (3.19), we have

$$\frac{1}{2}|u_m'(t)|^2 + \frac{1}{2}\left(1 - \frac{\beta}{\lambda_1}\right)|\Delta u_m(t)|^2 - \frac{C_0^{\rho+1}}{\rho+1}|\Delta u_m(t)|^{\rho+1} + \int_0^t |u_m'(s)|^2 \, \mathrm{d}s \leq N, \quad t \in [0, t_m).$$

So,

$$|u_m'(t)| \leq \sqrt{2N} \quad \text{and} \quad |\Delta u_m(t)| \leq \sqrt{\frac{2\lambda_1 N}{\lambda_1 - \beta}}, \quad \text{for each} \quad t \in [0, t_m) \text{ and } |\Delta u_m(t)| \leq \lambda^*. \tag{3.20}$$

Now we will verify that the inequalityies (3.20) hold for all $t \in [0, T]$. We observe that from (3.1), we have $|\Delta u_{0m}| = |\Delta u_0| < \lambda^*$, $\forall \; m \geq m_1$. Reasoning by contradiction, we assume there exists $t_1 \in (0, t_m)$ such that $|\Delta u(t_1)| = \lambda^*$. Let $t^* = \inf\{t_1 \in (0, t_m);\; |\Delta u(t_1)| = \lambda^*\}$. By continuity of $|\Delta u_m(t)|$, we obtain $|\Delta u_m(t^*)| = \lambda^*$. Note that $0 < t^* < t_m$.

Consider $t \in [0, t^*)$. Then $|\Delta u_m(t)| < \lambda^*$. So the inequalities (3.20) provide $|\Delta u_m(t)| \leq \sqrt{\frac{2\lambda_1 N}{\lambda_1 - \beta}}$, $t \in [0, t^*)$. This implies that $\lambda^* = |\Delta u_m(t^*)| \leq \sqrt{\frac{2\lambda_1 N}{\lambda_1 - \beta}}$, but this is a contradiction because by hypothesis (3.2), $\sqrt{\frac{2\lambda_1 N}{\lambda_1 - \beta}} < \lambda^*$.

Then, the estimates

$$|u_m'(t)| \leq \sqrt{2N} \quad \text{and} \quad |\Delta u_m(t)| \leq \sqrt{\frac{2\lambda_1 N}{\lambda_1 - \beta}}, \tag{3.21}$$

hold for all $t \in [0, T]$, $\forall\ m \geq m_1$ and the approximated solution can be extended to al interval $[0, T]$.

3.3. Passage to the limit

The estimates (3.21) allow us to the find a function u and a subsequence denoted by (u_m) such that

$$u_m \overset{*}{\rightharpoonup} u \quad \text{weak star in} \quad L^\infty(0, T; H_0^2(\Omega)), \tag{3.22}$$

$$u_m' \overset{*}{\rightharpoonup} u' \quad \text{weak star in} \quad L^\infty(0, T; L^2(\Omega)). \tag{3.23}$$

Applying the Lions–Aubin–Simon compactness Lemma [1, 16], we get from (3.22) and (3.23)

$$u_m \longrightarrow u \quad \text{strongly} \quad L^\infty(0, T; H_0^1(\Omega)), \tag{3.24}$$

$$u_m \longrightarrow u \quad \text{a.e. in} \quad \Omega \times (0, T), \tag{3.25}$$

by continuity of M and (3.24), $M(|\nabla u_m|^2) \longrightarrow M(|\nabla u|^2)$ in $L^2(0, T)$.

Therefore

$$M(|\nabla u_m|^2)(-\Delta u_m) \rightharpoonup M(|\nabla u|^2)(-\Delta u) \text{ weakly in } L^2(0, T; L^2(\Omega)). \tag{3.26}$$

By (3.2) we have

$$|u_m|^\rho \longrightarrow |u|^\rho \text{ a.e. in } \Omega \times (0, T). \tag{3.27}$$

Now,

$$\int_\Omega |u_m|^\rho \, \mathrm{d}x = |u_m|_\rho^\rho \leq C|\Delta u_m|^\rho, \tag{3.28}$$

where $C > 0$ is a constant independent of m and t.

So, (3.27), (3.28) and Lemma 1.3 [8] implies

$$|u_m|^\rho \rightharpoonup |u|^\rho \quad \text{weakly in} \quad L^2(0, T; L^2(\Omega)). \tag{3.29}$$

Convergences (3.22), (3.23), (3.26) and (3.29) allow us to pass the limit in the approximate equation (3.7) and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(u'(t), w) + (\Delta u(t), \Delta w) + M(|\nabla u(t)|^2)(-\Delta u(t), w) + (|u(t)|^\rho, w) + (u'(t), w) = 0, \tag{3.30}$$

for all $w \in H_0^2(\Omega)$ in the sense of $\mathcal{D}'(0, T)$.

The proof of Theorem 3.1 of global existence is complete. □

4. EXPONENTIAL STABILITY

This section is dedicated to study the asymptotic behavior. The principal result in this section is Theorem 4.2. In order to prove Theorem 4.2 we apply the following result due to Nakao.

LEMMA 4.1. Suppose that $\phi(t)$ is a bounded nonnegative function on $\mathbb{R}^+$, satisfying

$$\sup_{t \leq s \leq t+1} \text{ess}\, \phi^{1+\beta}(s) \leq K[\phi(t) - \phi(t+1)], \quad \forall\ t \geq 0.$$

Then,

(a) If $\beta = 0$ then $\phi(t) \leq Ce^{-\gamma t}$, $\forall\ t \geq 0$,

(b) If $\beta > 0$ then $\phi(t) \leq C(1 + t)^{-\frac{1}{\beta}}$, $\forall\ t \geq 0$,

where C and γ are positive constants.

Proof. This lemma is a consequence of Theorem 1 [13] and Lemma 2 [14]. □

Now we show that the solution of problem (1.1)-(1.3) is exponentially stable. The main goal is to prove the following stability result.

THEOREM 4.2. Under the hypotheses of Theorem 3.1, the solution of problem (1.1)—(1.3) satisfies

$$\frac{1}{2}|u'(t)|^2+\frac{1}{2}\left(1-\frac{\beta}{\lambda_1}\right)|\Delta u(t)|^2+\frac{1}{\rho+1}\int_\Omega |u(t)|^\rho u(t)\,\mathrm{d}x+\int_t^{t+1}|u'(s)|^2\,\mathrm{d}s\le Ce^{-\alpha t},\quad \forall\ t\ge 0,$$

where C and α are positive constants.

Proof. Let $w=u'(t)$ in the equation (3.30). Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}|u'(t)|^2+\frac{1}{2}|\Delta u(t)|^2+\frac{1}{2}\hat{M}(|\nabla u(t)|^2)\right]+\int_\Omega |u(t)|^\rho u'(t)\,\mathrm{d}t+|u'(t)|^2=0.$$

Now,

$$\int_\Omega |u(t)|^\rho u'(t)\,\mathrm{d}t=\frac{1}{\rho+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u(t)|^\rho u(t)\,\mathrm{d}t.$$

Then $\frac{\mathrm{d}}{\mathrm{d}t}E(t)+|u'(t)|^2=0$, where

$$E(t)=\frac{1}{2}|u'(t)|^2+\frac{1}{2}|\Delta u(t)|^2+\frac{1}{2}\hat{M}(|\nabla u(t)|^2)+\frac{1}{\rho+1}\int_\Omega |u(t)|^\rho u(t)\,\mathrm{d}x. \tag{4.1}$$

Integrating from t to $t+1$, we obtain

$$\int_t^{t+1}|u'(s)|^2\,\mathrm{d}s\le E(t)-E(t+1)\overset{\text{def}}{=}F^2(t). \tag{4.2}$$

Then there exist $t_1\in\left[t,t+\frac{1}{4}\right]$ and $t_2\in\left[t+\frac{3}{4},t+1\right]$ such that

$$|u'(t_i)|\le 2F(t),\quad i=1,2. \tag{4.3}$$

Let $w=u(t)$ in the equation (3.30), then we obtain,

$$\frac{\mathrm{d}}{\mathrm{d}t}(u'(t),u(t))-|u'(t)|^2+|\Delta u(t)|^2+M(|\nabla u(t)|^2)|\nabla u(t)|^2+(u'(t),u(t))+\int_\Omega |u(t)|^\rho u(t)\,\mathrm{d}x=0.$$

Integrating from t_1 to t_2 and by hypotheses (H.1) and (4.2) we obtain,

$$\begin{aligned}
&\int_{t_1}^{t_2}\left[\left(1-\frac{\beta}{\lambda_1}\right)|\Delta u(s)|^2+\int_\Omega |u(s)|^\rho u(s)\,\mathrm{d}x\right]\mathrm{d}s\\
&\le |u'(t_1)||u(t_1)|+|u'(t_2)||u(t_2)|\int_{t_1}^{t_2}|u'(s)|^2\,\mathrm{d}s+\int_{t_1}^{t_2}|u'(s)||u(s)|\,\mathrm{d}s\\
&\le C\left(\operatorname*{sup\,ess}_{t\le s\le t+1}|\Delta u(s)|\right)\left[|u'(t_1)|+|u'(t_2)|\right]+F^2(t)+\int_{t_1}^{t_2}C|u'(s)||\Delta u(s)|\,\mathrm{d}s\\
&\le 4CF(t)\operatorname*{sup\,ess}_{t\le s\le t+1}|\Delta u(s)|+F^2(t)+\frac{C^2}{\delta}\int_{t_1}^{t_2}|u'(s)|^2\,\mathrm{d}s+\delta\int_{t_1}^{t_2}|\Delta u(s)|^2\,\mathrm{d}s,
\end{aligned}$$

where $0 < \delta < 1 - \frac{\beta}{\lambda_1}$ and C is a constant such that $|u(t)| \leq C|\Delta u(t)|$. Then

$$\int_{t_1}^{t_2} \left[\left(1 - \frac{\beta}{\lambda_1} - \delta\right) |\Delta u(s)|^2 + \int_\Omega |u(s)|^\rho u(s)\, dx \right] ds \leq 4CF(t) \sup_{t \leq s \leq t+1} \operatorname{ess} |\Delta u(s)| + \left(1 + \frac{C^2}{\delta}\right) F^2(t)$$
$$\leq C_1 \left[F(t) \sup_{t \leq s \leq t+1} \operatorname{ess} |\Delta u(s)| + F^2(t) \right] \stackrel{\text{def}}{=} G^2(t), \tag{4.4}$$

where $C_1 = \max\left\{4C, 1 + \frac{C^2}{\delta}\right\}$.

From (4.2) and (4.4) we have

$$\int_{t_1}^{t_2} \left[\left(1 - \frac{\beta}{\lambda_1} - \delta\right) |\Delta u(s)|^2 + \int_\Omega |u(s)|^\rho u(s)\, dx + |u'(s)|^2 \right] ds \leq F^2(t) + G^2(t).$$

Hence there exists $t^* \in [t_1, t_2]$ such that

$$|u'(t^*)|^2 + \left(1 - \frac{\beta}{\lambda_1} - \delta\right) |\Delta u(t^*)|^2 + \int_\Omega |u(t^*)|^\rho u(t^*)\, dx \leq 2[F^2(t) + G^2(t)],$$

or

$$|u'(t^*)|^2 + |\Delta u(t^*)|^2 + \int_\Omega |u(t^*)|^\rho u(t^*)\, dx \leq 2[F^2(t) + G^2(t)], \tag{4.5}$$

where $C_2 = \frac{2}{1 - \frac{\beta}{\lambda_1} - \delta}$.

Now,

$$\hat{M}(|\nabla u(t^*)|^2) \leq m_0 |\nabla u(t^*)|^2 \leq \frac{m_0}{\lambda_1} |\Delta u(t^*)|^2, \quad \text{where} \quad m_0 = \max_{0 \leq s \leq |\nabla u(t^*)|^2} M(s).$$

Therefore

$$\hat{M}(|\nabla u(t^*)|^2) \leq C_3[F^2(t) + G^2(t)], \tag{4.6}$$

where C_3 is a positive constant.

From (4.1), (4.5) and (4.6) we get

$$E(t^*) \leq C_4[F^2(t) + G^2(t)], \tag{4.7}$$

where C_4 is a positive constant.

Now, by applying (4.2) and (4.7), we have

$$\begin{aligned} \sup_{t \leq s \leq t+1} \operatorname{ess} E(s) &\leq E(t^*) + \int_{t_1}^{t_2} |u'(s)|\, ds \\ &\leq C_4[F^2(t) + G^2(t)] + F^2(t) \\ &\leq C_5 F^2(t) + C_4 F(t) \sup_{t \leq s \leq t+1} \operatorname{ess} |\Delta u(s)| \\ &\leq C_6 F^2(t) + \frac{1}{2} \sup_{t \leq s \leq t+1} \operatorname{ess} E(s). \end{aligned}$$

Therefore,

$$\sup_{t \leq s \leq t+1} \operatorname{ess} E(s) \leq K[E(t) - E(t+1)],$$

where K is a positive constant.

By Lemma 4.1 with $\phi(t) = E(t)$, we have

$$E(t) \leq Ce^{-\alpha t}, \quad t \geq 0, \tag{4.8}$$

where C and α are positive constants.

So, by (H.1), (4.1) and (4.8), it follows that

$$\frac{1}{2}|u'(t)|^2 + \frac{1}{2}\left(1 - \frac{\beta}{\lambda_1}\right)|\Delta u(t)|^2 + \frac{1}{\rho+1}\int_\Omega |u(t)|^\rho u(t)\,\mathrm{d}x + \int_t^{t+1} |u'(s)|^2\,\mathrm{d}s \le ce^{-\alpha t}, \quad \forall\ t \ge 0,$$

and the proof of Theorem 4.2 is complete. □

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