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AKADÉMIAI KIADÓ

HYPERSPHERE HAVING $\Delta^{II}\mathbf{x} = \mathcal{A}\mathbf{x}$ IN $\mathbb{E}^4$

Erhan Güler[1,*] and Kübra Yılmaz[2]

[1] Department of Mathematics, Faculty of Sciences, Bartın University, 74100 Bartın, Turkey
[2] Graduate School of Natural and Applied Sciences, Bartın University, 74100 Bartın, Turkey

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ABSTRACT

We consider hypersphere $\mathbf{x} = \mathbf{x}(u, v, w)$ in the four dimensional Euclidean space. We calculate the Gauss map, and the curvatures of it. Moreover, we compute the second Laplace–Beltrami operator the hypersphere satisfying $\Delta^{II}\mathbf{x} = \mathcal{A}\mathbf{x}$, where $\mathcal{A} \in Mat(4, 4)$.

KEYWORDS

Four space, Euclidean space, the second Laplace–Beltrami operator, hypersphere, Gauss map, curvatures

MATHEMATICS SUBJECT CLASSIFICATION (2020)

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1. INTRODUCTION

With Chen's works [10, 11, 12, 13], the studies of submanifolds of finite type whose immersion into $\mathbb{E}^m$ (or $\mathbb{E}^m_\nu$) by using a finite number of eigenfunctions of their Laplacian have been studied for almost a half century.

Takahashi [43] gave a connected Euclidean submanifold is of 1-type, iff it is either minimal in $\mathbb{E}^m$ or minimal in some hypersphere of $\mathbb{E}^m$. 2-type spherical closed submanifolds were given by [7, 8, 11]. Garay studied [25] the extension of Takahashi's theorem in $\mathbb{E}^m$. Cheng and Yau [16] introduced the hypersurfaces with constant scalar curvature; Chen and Piccinni [14] focused the submanifolds with finite type Gauss map in $\mathbb{E}^m$. Dursun [20] considered the hypersurfaces with pointwise 1-type Gauss map in $\mathbb{E}^{n+1}$.

In $\mathbb{E}^3$; Takahashi [43] proved the minimal surfaces and spheres are the only surfaces satisfying the condition $\Delta r = \lambda r$, $\lambda \in \mathbb{R}$; Ferrandez, Garay, and Lucas [22] found the surfaces satisfying $\Delta H = AH$, $A \in Mat(3, 3)$ are either minimal, or an open piece of sphere or of a right circular cylinder; Choi and Kim [17] classified the minimal helicoid in terms of pointwise 1-type Gauss map of the first kind; Garay [24] studied the class of finite type surfaces of revolution; Dillen, Pas, and Verstraelen [18] obtained the only surfaces satisfying $\Delta r = Ar + B$, $A \in Mat(3, 3)$, $B \in Mat(3, 1)$ are the minimal surfaces, the spheres and the circular cylinders; Stamatakis and Zoubi [42] focused the surfaces of

* Corresponding author. E-mail: eguler@bartin.edu.tr

revolution satisfying $\Delta^{III}x = Ax$; Senoussi and Bekkar [41] gave the helicoidal surfaces M^2 which are of finite type with respect to the fundamental forms I, II and III, i.e., their position vector field $r(u, v)$ satisfies the condition $\Delta^J r = Ar$, $J = I, II, III$, where $A \in Mat(3, 3)$; Kim, Kim, and Kim [36] introduced the Cheng-Yau operator and Gauss map of surfaces of revolution.

In $\mathbb{E}^4$; Moore [39, 40] considered the general rotational surfaces; Hasanis and Vlachos [33] studied the hypersurfaces with harmonic mean curvature vector field; Cheng and Wan [15] gave the complete hypersurfaces with CMC; Kim and Turgay [37] worked the surfaces with L_1-pointwise 1-type Gauss map; Arslan et al. [2] introduced the Vranceanu surface with pointwise 1-type Gauss map; Arslan et al. [3] worked the generalized rotational surfaces; Aksoyak and Yaylı [34] studied the flat rotational surfaces with pointwise 1-type Gauss map; Güler, Magid, and Yaylı [29] introduced the helicoidal hypersurfaces; Güler, Hacısalihoğlu, and Kim [28] worked the Gauss map and the third Laplace–Beltrami operator of the rotational hypersurface; Güler and Turgay [30] worked the Cheng-Yau operator and Gauss map of rotational hypersurfaces; Güler [27] obtained the rotational hypersurfaces satisfying $\Delta^I R = AR$, where $A \in Mat(4, 4)$. He [26] also worked the fundamental form IV and curvature formulas of the hypersphere.

In Minkowski 4-space $\mathbb{E}^4_1$; Ganchev and Milousheva [23] studied the similar surfaces of [39, 40]; Arvanitoyeorgos, Kaimakamais, and Magid [6] indicated that if the mean curvature vector field of M^3_1 satisfies the equation $\Delta H = \alpha H$ (α a constant), then M^3_1 has CMC; Arslan and Milousheva [4] introduced the meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map; Turgay [44] considered the classifications of Lorentzian surfaces with finite type Gauss map; Dursun and Turgay [21] worked the space-like surfaces in with pointwise 1-type Gauss map. Aksoyak and Yaylı [35] gave general rotational surfaces with pointwise 1-type Gauss map in $\mathbb{E}^4_2$. Bektaş, Canfes, and Dursun [9] obtained the surfaces in a pseudo-sphere with 2-type pseudo-spherical Gauss map in $\mathbb{E}^5_2$.

Arslan, Sütveren, and Bulca [5] worked the rotational λ-hypersurfaces in Euclidean spaces. Güler, Yaylı, and Hacısalihoğlu [31, 32] introduced the bi-rotational hypersurfaces in $\mathbb{E}^4$ and $\mathbb{E}^4_2$, respectively.

In this work, we consider the hypersphere in four dimensional Euclidean space $\mathbb{E}^4$. In Section 2, we give the notions of the four dimensional Euclidean geometry. We consider the curvature formulas of the hypersurface in $\mathbb{E}^4$ in Section 3. In Section 4, we define the hypersphere. Finally, we study the hypersphere having $\Delta^{II}\mathbf{x} = \mathcal{A}\mathbf{x}$ for some 4×4 matrix $\mathcal{A}$ in $\mathbb{E}^4$ in the last section.

2. PRELIMINARIES

In this section, giving some of basic facts and definitions, we describe notations used whole paper. Let $\mathbb{E}^m$ denote the Euclidean m-space with the canonical Euclidean metric tensor given by $\tilde{g} = \langle\ ,\ \rangle = \sum_{i=1}^{m} dx_i^2$, where $(x_1, x_2, \dots, x_m)$ is a rectangular coordinate system in $\mathbb{E}^m$. Consider an m-dimensional Riemannian submanifold of the space $\mathbb{E}^m$. We denote the Levi–Civita connections of $\mathbb{E}^m$ and M by $\tilde{\nabla}$ and ∇, respectively. We shall use letters X, Y, Z, W (resp., ξ, η) to denote vectors fields tangent (resp., normal) to M. The Gauss and Weingarten formulas are given, respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi, \tag{2.2}$$

where h, D and A are the second fundamental form, the normal connection and the shape operator of M, respectively.

For each $\xi \in T_p^\perp M$, the shape operator A_ξ is a symmetric endomorphism of the tangent space $T_p M$ at $p \in M$. The shape operator and the second fundamental form are related by

$$\langle h(X, Y), \xi \rangle = \left\langle A_\xi X, Y \right\rangle .$$

The Gauss and Codazzi equations are given, respectively, by

$$\langle R(X, Y,)Z, W \rangle = \langle h(Y, Z), h(X, W) \rangle - \langle h(X, Z), h(Y, W) \rangle, \tag{2.3}$$

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z), \tag{2.4}$$

where R, R^D are the curvature tensors associated with connections ∇ and D, respectively, and $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

2.1. Hypersurfaces of Euclidean space

Now, let M be an oriented hypersurface in the Euclidean space $\mathbb{E}^{n+1}$, S its shape operator (i.e., Weingarten map) and x its position vector. We consider a local orthonormal frame field $\{e_1, e_2, \dots, e_n\}$ of consisting of principal directions of M corresponding from the principal curvature k_i for $i = 1, 2, \dots n$. Let the dual basis of this frame field be $\{\theta_1, \theta_2, \dots, \theta_n\}$. Then the first structural equation of Cartan is

$$d\theta_i = \sum_{i=1}^{n} \theta_j \wedge \omega_{ij}, \quad i, j = 1, 2, \dots, n, \tag{2.5}$$

where ω_{ij} denotes the connection forms corresponding to the chosen frame field. We denote the Levi–Civita connection of M and $\mathbb{E}^{n+1}$ by ∇ and $\tilde{\nabla}$, respectively. Then, from the Codazzi equation (2.3), we have

$$e_i(k_j) = \omega_{ij}(e_j)(k_i - k_j), \tag{2.6}$$

$$\omega_{ij}(e_l)(k_i - k_j) = \omega_{il}(e_j)(k_i - k_l) \tag{2.7}$$

for distinct $i, j, l = 1, 2, \dots, n$.

We put $s_j = \sigma_j(k_1, k_2, \dots, k_n)$, where σ_j is the j-th elementary symmetric function given by

$$\sigma_j(a_1, a_2, \dots, a_n) = \sum_{1 \le i_1 < i_2 < \dots < i_j \le n} a_{i_1} a_{i_2} \dots a_{i_j}.$$

We use following notation

$$r_i^j = \sigma_j(k_1, k_2, \dots, k_{i-1}, k_{i+1}, k_{i+2}, \dots, k_n).$$

By the definition, we have $r_i^0 = 1$ and $s_{n+1} = s_{n+2} = \dots = 0$. We call the function s_k as the k-th mean curvature of M. We would like to note that functions $H = \frac{1}{n}s_1$ and $K = s_n$ are called the mean curvature and Gauss-Kronecker curvature of M, respectively. In particular, M is said to be j-minimal if $s_j \equiv 0$ on M.

In $\mathbb{E}^{n+1}$, to find the i-th curvature formulas $\mathfrak{C}_i$ (Curvature formulas sometimes are represented as mean curvature H_i, and sometimes as Gaussian curvature K_i by different writers, such as [1] and [38]. We will call it just i-th curvature $\mathfrak{C}_i$ in this paper.), where $i = 0, \dots, n$, firstly, we use the characteristic polynomial of S:

$$P_{\mathrm{S}}(\lambda) = 0 = \det(\mathrm{S} - \lambda \mathcal{I}_n) = \sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k}, \tag{2.8}$$

where $i = 0, \dots, n$, $\mathcal{I}_n$ denotes the identity matrix of order n. Then, we get curvature formulas $\binom{n}{i}\mathfrak{C}_i = s_i$. That is, $\binom{n}{0}\mathfrak{C}_0 = s_0 = 1$ (by definition), $\binom{n}{1}\mathfrak{C}_1 = s_1, \dots, \binom{n}{n}\mathfrak{C}_n = s_n = K$.

k-th fundamental form of M is defined by $I\left(\mathrm{S}^{k-1}(X), Y\right) = \left\langle \mathrm{S}^{k-1}(X), Y \right\rangle$. So, we have

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathfrak{C}_i I\left(\mathrm{S}^{n-i}(X), Y\right) = 0. \tag{2.9}$$

In particular, one can get classical result $\mathfrak{C}_0 III - 2\mathfrak{C}_1 II + \mathfrak{C}_2 I = 0$ of surface theory for $n = 2$. See [38] for details.

For a Euclidean submanifold $x : M \longrightarrow \mathbb{E}^m$, the immersion (M, x) is called *finite type*, if x can be expressed as a finite sum of eigenfunctions of the Laplacian Δ of (M, x), i.e. $x = x_0 + \sum_{i=1}^{k} x_i$, where x_0 is a constant map, $x_1, \dots, x_k$ non-constant maps, and $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, \dots, k$. If λ_i are different, M is called *k-type*. See [11] for details.

2.2. Rotational hypersurfaces

We will obtain a rotational hypersurface (rot-hypface for short) in Euclidean 4-space. Note that the definition of rot-hypfaces in Riemannian space forms were defined in [19]. A rot-hypface $M \subset \mathbb{E}^{n+1}$ generated by a curve C around an axis $\mathfrak{r}$ that does not meet C is obtained by taking the orbit of C under those orthogonal transformations of $\mathbb{E}^{n+1}$ that leaves $\mathfrak{r}$ pointwise fixed (See [19, Remark 2.3]).

Throughout the paper, we shall identify a vector (a, b, c, d) with its transpose. Consider the case $n = 3$, and let C be the curve parametrized by

$$\gamma(u) = (f(u), 0, 0, \varphi(u)) . \tag{2.10}$$

If $\mathfrak{r}$ is the x_4-axis, then an orthogonal transformations of $\mathbb{E}^{n+1}$ that leaves $\mathfrak{r}$ pointwise fixed has the form

$$\mathbf{Z}(v, w) = \begin{pmatrix} \cos v \cos w & -\sin v & -\cos v \sin w & 0 \\ \sin v \cos w & \cos v & -\sin v \sin w & 0 \\ \sin w & 0 & \cos w & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad v, w \in \mathbb{R}.$$

Therefore, the parametrization of the rot-hypface generated by a curve C around an axis $\mathfrak{r}$ is given by

$$\mathbf{x}(u, v, w) = \mathbf{Z}(v, w)\gamma(u). \tag{2.11}$$

DEFINITION 2.1. Let $\mathbf{x} = \mathbf{x}(u, v, w)$ be an isometric immersion from $M^3 \subset \mathbb{E}^3$ to $\mathbb{E}^4$. Triple vector product of $\overrightarrow{x} = (x_1, x_2, x_3, x_4), \overrightarrow{y} = (y_1, y_2, y_3, y_4), \overrightarrow{z} = (z_1, z_2, z_3, z_4)$ of $\mathbb{E}^4$ is defined by as follows:

$$\begin{aligned} \overrightarrow{x} \times \overrightarrow{y} \times \overrightarrow{z} = (&x_2 y_3 z_4 - x_2 y_4 z_3 - x_3 y_2 z_4 + x_3 y_4 z_2 + x_4 y_2 z_3 - x_4 y_3 z_2, \\ &- x_1 y_3 z_4 + x_1 y_4 z_3 + x_3 y_1 z_4 - x_3 y_4 z_1 - x_4 y_1 z_3 + x_4 y_3 z_1, \\ &+ x_1 y_2 z_4 - x_1 y_4 z_2 - x_2 y_1 z_4 + x_2 y_4 z_1 + x_4 y_1 z_2 - x_4 y_2 z_1, \\ &- x_1 y_2 z_3 + x_1 y_3 z_2 + x_2 y_1 z_3 - x_2 y_3 z_1 - x_3 y_1 z_2 + x_3 y_2 z_1). \end{aligned}$$

DEFINITION 2.2. For a hypface $\mathbf{x}$ in 4-space, we have

$$I = \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix}, \qquad II = \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix}, \tag{2.12}$$

and

$$\det I = (EG - F^2)C - EB^2 + 2FAB - GA^2,$$
$$\det II = \left(LN - M^2\right) V - LT^2 + 2MPT - NP^2,$$

where I and II are the first and the second fundamental form matrices, respectively, where $E = \mathbf{M}_u \cdot \mathbf{M}_u$, $F = \mathbf{M}_u \cdot \mathbf{M}_v$, $G = \mathbf{M}_v \cdot \mathbf{M}_v$, $A = \mathbf{M}_u \cdot \mathbf{M}_w$, $B = \mathbf{M}_v \cdot \mathbf{M}_w$, $C = \mathbf{M}_w \cdot \mathbf{M}_w$, $L = \mathbf{M}_{uu} \cdot e$, $M = \mathbf{M}_{uv} \cdot e$, $N = \mathbf{M}_{vv} \cdot e$, $P = \mathbf{M}_{uw} \cdot e$, $T = \mathbf{M}_{vw} \cdot e$, $V = \mathbf{M}_{ww} \cdot e$. Here,

$$e = \frac{\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w}{\|\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w\|} \tag{2.13}$$

is the unit normal (i.e., the Gauss map) of hypface $\mathbf{x}$.

DEFINITION 2.3. The product matrices $I^{-1} \cdot II$ gives the matrix of the shape operator S of hypface $\mathbf{x}$ in 4-space as follows

$$\mathrm{S} = \frac{1}{\det I} \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}, \tag{2.14}$$

where

$$\begin{aligned} \det I &= (EG - F^2)C - A^2 G + 2ABF - B^2 E, \\ s_{11} &= ABM - CFM - AGP + BFP + CGL - B^2 L, \\ s_{12} &= ABN - CFN - AGT + BFT + CGM - B^2 M, \\ s_{13} &= ABT - CFT - AGV + BFV + CGP - B^2 P, \end{aligned}$$

$$s_{21} = ABL - CFL + AFP - BPE + CME - A^2M,$$
$$s_{22} = ABM - CFM + AFT - BTE + CNE - A^2N,$$
$$s_{23} = ABP - CFP + AFV - BVE + CTE - A^2T,$$
$$s_{31} = -AGL + BFL + AFM - BME + GPE - F^2P,$$
$$s_{32} = -AGM + BFM + AFN - BNE + GTE - F^2T,$$
$$s_{33} = -AGP + BFP + AFT - BTE + GVE - F^2V.$$

See [28, 29, 30] for details.

3. *i*-TH CURVATURES

To compute the i-th mean curvature formula $\mathfrak{C}_i$, where $i = 0, \ldots, 3$, we use characteristic polynomial $P_{\mathrm{S}}(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d = 0$:

$$P_{\mathrm{S}}(\lambda) = \det(\mathrm{S} - \lambda \mathcal{I}_3) = 0.$$

Then, obtain $\mathfrak{C}_0 = 1$ (by definition), $\binom{3}{1}\mathfrak{C}_1 = \binom{3}{1}H = -\frac{b}{a}$, $\binom{3}{2}\mathfrak{C}_2 = \frac{c}{a}$, $\binom{3}{3}\mathfrak{C}_3 = K = -\frac{d}{a}$, and $\mathcal{I}_3$ is the identity matrix of order 3.

Therefore, we find i-th curvature formulas depend on the coefficients of the fundamental forms I and II in 4-space. See [26] for details.

THEOREM 3.1. A hypface $\mathbf{x}$ in $\mathbb{E}^4$ has the following curvature formulas, $\mathfrak{C}_0 = 1$ (by definition),

$$\mathfrak{C}_1 = \frac{\left\{\begin{array}{c}(EN + GL - 2FM)C + (EG - F^2)V - LB^2 - NA^2 \\ -2(APG - BPF - ATF + BTE - ABM)\end{array}\right\}}{3\left[(EG - F^2)C - EB^2 + 2FAB - GA^2\right]}, \tag{3.1}$$

$$\mathfrak{C}_2 = \frac{\left\{\begin{array}{c}(EN + GL - 2FM)\,V + \left(LN - M^2\right)C - ET^2 - GP^2 \\ -2\,(APN - BPM - ATM + BTL - PTF)\end{array}\right\}}{3\left[(EG - F^2)C - EB^2 + 2FAB - GA^2\right]}, \tag{3.2}$$

$$\mathfrak{C}_3 = \frac{\left(LN - M^2\right)V - LT^2 + 2MPT - NP^2}{(EG - F^2)C - EB^2 + 2FAB - GA^2}. \tag{3.3}$$

Proof. Solving $\det(\mathrm{S} - \lambda I_3) = 0$ with some algebraic computations, we obtain coefficients a, b, c, d of polynomial $P_{\mathrm{S}}(\lambda)$. □

A hypersurface $\mathbf{x}$ in $\mathbb{E}^4$ is $\mathfrak{C}_i$-minimal, when $\mathfrak{C}_i = 0$ identically on $\mathbf{x}$.

4. HYPERSPHERE

In this section, we define the hypersphere, then find its differential geometric properties in $\mathbb{E}^4$.

For an open interval $I \subset \mathbb{R}$, let $\gamma : I \longrightarrow \Pi$ be a curve in a plane Π in $\mathbb{E}^4$, and let ℓ be a straight line in Π.

DEFINITION 4.1. A *rotational* hypface in $\mathbb{E}^4$ is called hypersphere, when a curve

$$\gamma(u) = (r\cos u, 0, 0, r\sin u)$$

rotates around a line $\ell = (0, 0, 0, 1)$ (these are called the *profile curve* and the *axis*, respectively).

So, the hypersphere spanned by the vector ℓ, is given by as follows

$$\mathbf{x}(u, v, w) = Z(v, w)\gamma(u)^t \tag{4.1}$$

in $\mathbb{E}^4$, where $r \in \mathbb{R} \setminus \{0\}$, $0 \leq u, v, w < 2\pi$. Therefore, the more clear form of (4.1) is given by as follows

$$\mathbf{x}(u, v, w) = \begin{pmatrix} r\cos u\cos v\cos w \\ r\sin u\cos v\cos w \\ r\sin v\cos w \\ r\sin w \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \end{pmatrix}. \tag{4.2}$$

When $v = 0$, we have the sphere in $\mathbb{E}^4$.

Next, we obtain the curvatures and the Gaussian curvature of the hypersphere (4.2).

We get the following first derivatives of (4.2) with respect to u, v, w, respectively,

$$\mathbf{x}_u = \begin{pmatrix} -r\sin u\cos v\cos w \\ r\cos u\cos v\cos w \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{x}_v = \begin{pmatrix} -r\cos u\sin v\cos w \\ -r\sin u\sin v\cos w \\ r\cos v\cos w \\ 0 \end{pmatrix},$$

and

$$\mathbf{x}_w = \begin{pmatrix} -r\cos u\sin v\sin w \\ -r\sin u\sin v\sin w \\ -r\cos v\sin w \\ r\cos w \end{pmatrix}.$$

The first quantities of (4.2) are given by as follows

$$I = \begin{pmatrix} r^2\cos^2 v\cos^2 w & 0 & 0 \\ 0 & r^2\cos^2 w & 0 \\ 0 & 0 & r^2 \end{pmatrix}. \tag{4.3}$$

We have $\det I = r^6\cos^2 v\cos^4 w$. By using (2.13), we get the Gauss map of the hypersphere (4.2) as follows

$$e = \begin{pmatrix} \cos u\cos v\cos w \\ \sin u\cos v\cos w \\ \sin v\cos w \\ \sin w \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}. \tag{4.4}$$

By using the second derivatives of (4.2) with respect to u, v, w, and the Gauss map (4.4) of the hypersphere (4.2), we have the second quantities as follows

$$II = \begin{pmatrix} -r\cos^2 v\cos^2 w & 0 & 0 \\ 0 & -r\cos^2 w & 0 \\ 0 & 0 & -r \end{pmatrix}. \tag{4.5}$$

So, we get $\det II = -r^3\cos^2 v\cos^4 w$. We calculate the shape operator matrix of the hypersphere (4.2), using (2.14), as follows

$$\mathbf{S} = \begin{pmatrix} -\frac{1}{r} & 0 & 0 \\ 0 & -\frac{1}{r} & 0 \\ 0 & 0 & -\frac{1}{r} \end{pmatrix}.$$

Finally, using (3.1), (3.2) and (3.3), with (4.3), (4.5), respectively, we find the curvatures of the hypersphere (4.2) as follows.

COROLLARY 4.2. Let $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ be an immersion given by (4.2). Then, M^3 has the following constant ith-curvatures

$$\mathfrak{C}_1 = H = -\frac{1}{r},$$

$$\mathfrak{C}_2 = \frac{1}{r^2},$$

$$\mathfrak{C}_3 = K = -\frac{1}{r^3}.$$

5. HYPERSPHERE SATISFYING $\Delta^{II}\mathrm{X} = \mathcal{A}\mathrm{X}$

In this section, we give the second Laplace–Beltrami operator of a smooth function, then we calculate it taking the hypersphere (4.2).

The inverse of the matrix

$$\left(h_{ij}\right) = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}$$

is as follows

$$\left(h^{ij}\right) = \frac{1}{h}\begin{pmatrix} h_{22}h_{33} - h_{23}h_{32} & -(h_{12}h_{33} - h_{13}h_{32}) & h_{12}h_{23} - h_{13}h_{22} \\ -(h_{21}h_{33} - h_{31}h_{23}) & h_{11}h_{33} - h_{13}h_{31} & -(h_{11}h_{23} - h_{21}h_{13}) \\ h_{21}h_{32} - h_{22}h_{31} & -(h_{11}h_{32} - h_{12}h_{31}) & h_{11}h_{22} - h_{12}h_{21} \end{pmatrix},$$

where

$$\begin{aligned} h &= \det\left(h_{ij}\right) \\ &= h_{11}h_{22}h_{33} - h_{11}h_{23}h_{32} + h_{12}h_{31}h_{23} - h_{12}h_{21}h_{33} + h_{21}h_{13}h_{32} - h_{13}h_{22}h_{31}. \end{aligned}$$

By considering above matrices, and determinant, we give the following.

DEFINITION 5.1. The second Laplace–Beltrami operator of a smooth function $\phi = \phi(x^1, x^2, x^3)|_{\mathbf{D}}$ ($\mathbf{D} \subset \mathbb{R}^3$) of class C^3 with respect to the second fundamental form of a hypersurface $\mathbf{M}$ is the operator Δ^{II} which is defined by as follows

$$\Delta^{II}\phi = \frac{1}{\sqrt{h}}\sum_{i,j=1}^{3}\frac{\partial}{\partial x^i}\left(\sqrt{h}h^{ij}\frac{\partial\phi}{\partial x^j}\right). \tag{5.1}$$

where $\left(h^{ij}\right) = (h_{kl})^{-1}$ and $h = \det\left(h_{ij}\right)$.

Clearly, we can write (5.1) as follows

$$\Delta^{II}\phi = \frac{1}{\sqrt{h}}\begin{pmatrix} \frac{\partial}{\partial x^1}\left(\sqrt{h}h^{11}\frac{\partial\phi}{\partial x^1}\right) + \frac{\partial}{\partial x^1}\left(\sqrt{h}h^{12}\frac{\partial\phi}{\partial x^2}\right) + \frac{\partial}{\partial x^1}\left(\sqrt{h}h^{13}\frac{\partial\phi}{\partial x^3}\right) \\ +\frac{\partial}{\partial x^2}\left(\sqrt{h}h^{21}\frac{\partial\phi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\sqrt{h}h^{22}\frac{\partial\phi}{\partial x^2}\right) + \frac{\partial}{\partial x^2}\left(\sqrt{h}h^{23}\frac{\partial\phi}{\partial x^3}\right) \\ +\frac{\partial}{\partial x^3}\left(\sqrt{h}h^{31}\frac{\partial\phi}{\partial x^1}\right) + \frac{\partial}{\partial x^3}\left(\sqrt{h}h^{32}\frac{\partial\phi}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(\sqrt{h}h^{33}\frac{\partial\phi}{\partial x^3}\right) \end{pmatrix}. \tag{5.2}$$

So, we get the inverse of (4.3)

$$II^{-1} = \frac{1}{\det II}\begin{pmatrix} NV - T^2 & PT - MV & MT - NP \\ PT - MV & LV - P^2 & MP - LT \\ MT - NP & MP - LT & LN - M^2 \end{pmatrix},$$

where $\det II = \left(LN - M^2\right)V - P^2N + 2PTM - T^2L$. Hence, more clear notation of (5.2) for a smooth function $\phi = \phi(u, v, w)$ is as follows

$$\Delta^{II}\phi = \frac{1}{\sqrt{|\det II|}}\begin{pmatrix} \frac{\partial}{\partial u}\left(\frac{(NV-T^2)\phi_u + (PT-MV)\phi_v + (MT-NP)\phi_w}{\sqrt{|\det II|}}\right) \\ +\frac{\partial}{\partial v}\left(\frac{(PT-MV)\phi_u + (LV-P^2)\phi_v + (MP-LT)\phi_w}{\sqrt{|\det II|}}\right) \\ +\frac{\partial}{\partial w}\left(\frac{(MT-NP)\phi_u + (MP-LT)\phi_v + (LN-M^2)\phi_w}{\sqrt{|\det II|}}\right) \end{pmatrix}. \tag{5.3}$$

We continue our calculations to find the second Laplace–Beltrami operator $\Delta^{\mathbf{II}}\mathbf{x}$ of the hypersphere $\mathbf{x}$ using (4.2) and (5.3).

The second Laplace–Beltrami operator of the hypersphere (4.2) is given by

$$\Delta^{II}\mathbf{x} = \frac{1}{\sqrt{|\det II|}}\left(\frac{\partial}{\partial u}\mathcal{U} + \frac{\partial}{\partial v}\mathcal{V} + \frac{\partial}{\partial w}\mathcal{W}\right), \tag{5.4}$$

where

$$\mathcal{U} = \frac{\left(NV - T^2\right)\mathbf{x}_u + (PT - MV)\mathbf{x}_v + (MT - NP)\mathbf{x}_w}{\sqrt{|\det II|}},$$
$$\mathcal{V} = \frac{(PT - MV)\mathbf{x}_u + \left(LV - P^2\right)\mathbf{x}_v + (MP - LT)\mathbf{x}_w}{\sqrt{|\det II|}},$$
$$\mathcal{W} = \frac{(MT - NP)\mathbf{x}_u + (MP - LT)\mathbf{x}_v + \left(LN - M^2\right)\mathbf{x}_w}{\sqrt{|\det II|}}.$$

Here $M = P = T = 0$. Hence, we briefly can write $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$, as follows

$$\mathcal{U} = \frac{NV}{\sqrt{|\det II|}}\mathbf{x}_u, \qquad \mathcal{V} = \frac{LV}{\sqrt{|\det II|}}\mathbf{x}_v, \qquad \mathcal{W} = \frac{LN}{\sqrt{|\det II|}}\mathbf{x}_w.$$

Differentiating $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$, with respect to u, v, w, respectively, we obtain

$$\frac{\partial}{\partial u}\mathcal{U} = \begin{pmatrix} -r^{3/2}\cos u\cos w \\ -r^{3/2}\sin u\cos w \\ 0 \\ 0 \end{pmatrix},$$

$$\frac{\partial}{\partial v}\mathcal{V} = \begin{pmatrix} r^{3/2}\left(\cos^2 v - \sin^2 v\right)\cos u\cos w \\ r^{3/2}\sin u\left(\cos^2 v - \sin^2 v\right)\cos w \\ 2r^{3/2}\sin v\cos v\cos w \\ 0 \end{pmatrix},$$

$$\frac{\partial}{\partial w}\mathcal{W} = r^{3/2}\left(-2\sin w\cos w\right)\begin{pmatrix} -\cos u\cos^2 v\sin w \\ -\sin u\cos^2 v\sin w \\ -\sin v\cos v\sin w \\ \cos v\cos w \end{pmatrix} + r^{3/2}\cos^2 w\begin{pmatrix} -\cos u\cos^2 v\cos w \\ -\sin u\cos^2 v\cos w \\ -\sin v\cos v\cos w \\ -\cos v\sin w \end{pmatrix}.$$

Finally, substituting $\frac{\partial}{\partial u}\mathcal{U}$, $\frac{\partial}{\partial v}\mathcal{V}$, $\frac{\partial}{\partial w}\mathcal{W}$ into (5.4), we get

$$\Delta^{II}\mathbf{x} = \begin{pmatrix} \Delta^{II}\mathbf{x}_1 \\ \Delta^{II}\mathbf{x}_2 \\ \Delta^{II}\mathbf{x}_3 \\ \Delta^{II}\mathbf{x}_4 \end{pmatrix} = \begin{pmatrix} -\frac{3}{r}\cos u\cos v\cos w \\ -\frac{3}{r}\sin u\cos v\cos w \\ -\frac{3}{r}\sin v\cos w \\ -\frac{3}{r}\sin w \end{pmatrix} = -\frac{3}{r}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix},$$

where e_i are the elements of the Gauss map (4.4).

That is,

$$\Delta^{II}\mathbf{x}_i = -\frac{3}{r}e_i.$$

Therefore, we have some results between the second Laplace–Beltrami operator and the curvatures of the hypersphere:

COROLLARY 5.2. Let $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ be an immersion given by (4.2). Then $\mathbf{x}$ has

$$\Delta^{II}\mathbf{x} = 3r\mathfrak{C}_1 e = -3r^2\mathfrak{C}_2 e = 3r^3\mathfrak{C}_3 e,$$

where $\mathfrak{C}_{i=1,2,3}$, and e are the ith-curvature and the Gauss map, respectively, of the hypersphere $\mathbf{x}$.

COROLLARY 5.3. Let $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ be an immersion given by (4.2). Then, the hypersphere $\mathbf{x}$ has

$$\Delta^{II}\mathbf{x} = \mathcal{A}\mathbf{x},$$

where

$$\mathcal{A} = 3\mathfrak{C}_1\mathcal{I}_4 = -3\left(\mathfrak{C}_2\right)^{1/2}\mathcal{I}_4 = 3\left(\mathfrak{C}_3\right)^{1/3}\mathcal{I}_4,$$

and $\mathcal{A} \in Mat\,(4,4)$, $\mathcal{I}_4$ is the identity matrix of order 4.

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