Mathematica Pannonica ***New Series*** **29 /NS 3/ (2023) 1, 106-110**
DOI: 10.1556/314.2023.00009

AKADÉMIAI KIADÓ

A CHARACTERIZATION OF T_1 SPACES VIA LIMIT SETS OF NETS

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Communicated by Jörg Thuswaldner

Original Research Paper
Received: Dec 21, 2022 • Accepted: Feb 25, 2023
First published online: Mar 16, 2023

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ABSTRACT

This article indicates another set-theoretic formula, solely in terms of union and intersection, for the set of the limits of any given sequence (net, in general) in an arbitrary T_1 space; this representation in particular gives a new characterization of a T_1 space.

KEYWORDS

Fréchet–Urysohn space, limit, net, sequence, T_1 space, T_0 space

MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 54A20; Secondary 54G10

Throughout, a topological space is assumed to possess a property if and only if a corresponding declaration is made.

We are concerned with that notion of a *limit* x *of a net* $(x_\theta)_{\theta\in\Theta}$ in a topological space X, where x is an element of X with the property that for every neighborhood G of x in X there exists some $\gamma \in \Theta$ such that if θ (nonstrictly) succeeds γ then $x_\theta \in G$; a *net* in a topological space means by definition precisely a map from a *directed set* (i.e. a set equipped with a reflexive transitive relation [i.e. a *preorder*] such that any two elements of the set admit a common [nonstrict] successor) to the space. Evidently, the set of positive integers can trivially be made into a directed set, and so every sequence in any given topological space is a net in the space. And the notion of a net is in particular convenient in many settings; as well-known, the definition of Riemann integrability may be restated in terms of the convergence of some suitable net in the real field $\mathbb{R}$.

We add that there is the elementary fact that the set of the limits of a sequence in a non-Hausdorff space may very well have cardinal greater than one; the real field $\mathbb{R}$ receiving instead the cofinite topology, which is at least always T_1, is an example.

For reference, a *limit point of a net* (x_θ) in a topological space is defined, in the same fashion as in the case of sequences, precisely as an element x of the space such that for every neighborhood G of x and for every γ there exists some (nonstrict) successor θ of γ such that $x_\theta \in G$.

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It is well-known that, given any net $(x_\theta)_{\theta\in\Theta}$, with $\widetilde{<}$ denoting the given preorder over Θ, in any given topological space, the set of the limit points of $(x_\theta)_{\theta\in\Theta}$ may be represented as the closed set $\bigcap_{\gamma\in\Theta} \mathrm{cl}(\{x_\theta \mid \theta \geq \gamma\})$; on the other hand, by Theorem 4.7 in [1] we can express the set of the limits of $(x_\theta)_{\theta\in\Theta}$ as the closed set $\bigcap_{\varphi\,:\,\Theta\to\Theta \text{ cofinal}} \mathrm{cl}(\{x_{\varphi(\theta)} \mid \theta \in \Theta\})$, where the intersection ranges over all $\varphi : \Theta \to \Theta$ such that for every $\gamma \in \Theta$ there exists some $\theta \in \Theta$ for which $\varphi(\theta) \widetilde{>} \gamma$.

For our purposes, we make the following

DEFINITION 1. Let $(x_\theta)_{\theta\in\Theta}$ be a net in an arbitrary topological space X. We denote by $\lim_X((x_\theta)_{\theta\in\Theta})$, or simply by $\lim((x_\theta))$ when the other things under consideration are clear, the set of the limits of $(x_\theta)_{\theta\in\Theta}$. For brevity, we refer to $\lim_X((x_\theta)_{\theta\in\Theta})$ as the *limit set* of the net $(x_\theta)_{\theta\in\Theta}$ in X. □

Thus if (x_n) is a convergent sequence in a Hausdorff space X, then

$$\left\{\lim_{n\to\infty} x_n\right\} = \lim((x_n)).$$

In a metric space X, the limit set of a net $(x_\theta)_{\theta\in\Theta}$ in X, necessarily having at most one element, admits another representation. As immediately seen, for all sequences (x_n) in a metric space (X, d) we certainly have

$$\lim((x_n)) = \bigcap_{\varepsilon>0} \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} B_d(x_n, \varepsilon).$$

Here $B_d(x_n, \varepsilon)$ denotes the open d-ball in X of radius ε and of center x_n for all ε and all n. Manifestly, since the open balls in a metric space form by definition a basis, the equality holds for nets as well.

It is natural to try and do the same for the limit set of a net in a topological space as general as possible. Given any T_1 space, we will give a set-theoretic representation for the limit set of any given net in the space such that this representation induces a characterization of T_1 spaces. We add that those T_1 spaces that are not necessarily Hausdorff can certainly still be "geometrically meaningful"; geometric complexes (topologized coherently with respect to the geometric simplexes therein), even if not point-finitely triangulated, serve as well-known (e.g. [4]) examples.

If $A \subset X \times Y$ and if $(x, y) \in X \times Y$, denote by $(A)^x$ [resp. $(A)_y$] the cross section $\{z \in Y \mid (x, z) \in A\}$ [resp. $\{z \in X \mid (z, y) \in A\}$]. Throughout, the boundary □ simply serves as a symbol signifying for readability the end of a block but not always implies Q.E.D.; this will not cause any confusion. And $\mathbb{N} := \omega \setminus \{\emptyset\}$ for concreteness and "commonality".

One reason that the limit set of a net in a metric space is easily expressed via union and intersection over open balls is owing to the fact that formally we have $x \in B(y, \varepsilon)$ if and only if $y \in B(x, \varepsilon)$, which allows one to be concerned only with the radius ε. This convenience is not shared by just any collection of open sets in just any space; but this does not prevent one too much from proceeding:

THEOREM 2. If X is a T_1 space, with $\mathscr{T}_X(x)$ denoting the collection of all neighborhoods of x in X for all $x \in X$, and if $(x_\theta)_{\theta\in\Theta}$ is a net in X, with $\widetilde{<}$ denoting the given preorder over Θ, then

$$\lim((x_\theta)_{\theta\in\Theta}) = \bigcup_{x\in X} \bigcap_{G\in\mathscr{T}_X(x)} \bigcup_{\gamma\in\Theta} \bigcap_{\theta\widetilde{>}\gamma} (G \times G)^{x_\theta}.$$

Proof. Evidently, if y is a limit of (x_θ), then y is an element of X and has the property that for all $G \in \mathscr{T}_X(y)$ there is some $\gamma \in \Theta$ such that for all $\theta \in \Theta$ with $\theta \widetilde{>} \gamma$ we have $x_\theta \in G$, and hence $(x_\theta, y) \in G^2$; so $y \in (G^2)^{x_\theta}$. This proves the inclusion relation $\subset$.

To prove the converse, let y lie in the right-hand-side set. Then we can choose some $x \in X$ such that for every $G \in \mathscr{T}_X(x)$ there is some $\gamma \in \Theta$ such that for all $\theta \widetilde{>} \gamma$ we have $y \in (G^2)^{x_\theta}$, which in particular implies that y lies in every neighborhood of x, implying that $x \in \mathrm{cl}(\{y\})$. Since X is by assumption T_1, we have $\mathrm{cl}(\{y\}) = \{y\}$ and so $y = x$. But then, since x is a limit of (x_θ), so is y; this completes the proof. □

REMARK 3. We keep the notation of Theorem 2; fix G and x_θ. In the equality, the set $(G^2)^{x_\theta}$ is not the only choice. Manifestly, the set $(G^2)_{x_\theta}$ is by symmetry equally effective.

Moreover, since we formally have $(G^2)^{x_\theta} = \pi_2^{\rightarrow}\left((\{x_\theta\} \times G) \cap G^2\right)$, the latter [which is an image under the natural projection $(x, y) \mapsto y$] being an open set in X and $\{x_\theta\}$ being a singleton having x_θ as its unique element, the set $(G^2)^{x_\theta}$ is open in X; the same holds for $(G^2)_{x_\theta}$. □

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NOTATION 4. In what follows, the symbols $\mathscr{T}_X(x)$ and $\widetilde{\lesssim}$ are used as in Theorem 2. □

With T_0-ness the property that every sequence satisfies the equality in Theorem 2 suffices to characterize T_1-ness; even more:

THEOREM 5. Let X be a topological space. Then the following statements are equivalent:

i) X is T_1;

ii) X is T_0, and for every net $(x_\theta)_{\theta\in\Theta}$ in X we have

$$\lim((x_\theta)_{\theta\in\Theta}) = \bigcup_{x\in X}\bigcap_{G\in\mathscr{T}_X(x)}\bigcup_{\gamma\in\Theta}\bigcap_{\theta\widetilde{\gtrsim}\gamma}\left(G^2\right)^{x_\theta};$$

iii) X is T_0, and for every sequence $(x_n)_{n\in\mathbb{N}}$ in X we have

$$\lim((x_n)_{n\in\mathbb{N}}) = \bigcup_{x\in X}\bigcap_{G\in\mathscr{T}_X(x)}\bigcup_{N\in\mathbb{N}}\bigcap_{n\geq N}(G^2)^{x_n}.$$

Proof. That i) implies ii) follows directly from Theorem 2; that ii) implies iii) is trivial.

To prove that iii) implies i), suppose X is T_0 but not T_1. Since X is not T_1, we can choose from definition some distinct $x, y \in X$ such that, without loss of generality, every neighborhood of x contains y. Then y lies in the right-hand-side set associated with the constant sequence $(x)_{n\in\mathbb{N}}$ that has x as one of its limits. But as X is T_0, at least one of x and y has some neighborhood not containing the other, which in this case is necessarily y; it follows that $y \notin \lim((x)_{n\in\mathbb{N}})$, and so the equality fails for the sequence $(x)_{n\in\mathbb{N}}$. The proof is complete. □

The following is a simple example of a T_0 non-T_1 space that, as ensured by Theorem 5, does not satisfy the statement iii) in Theorem 5:

EXAMPLE 6. Give the set $X := \{0, 1, 2\}$ the topology $\{\emptyset, \{0\}, \{0, 1\}, X\}$. Then X is T_0; but since $\{0\}$ is not closed, the space X is not T_1.

The constant sequence (x_n) with $x_n := 1$ for all n has 1 and 2 as its only limits. But X is the only closed set containing 0, and so

$$0 \in \bigcup_{x\in X}\bigcap_{G\in\mathscr{T}_X(x)}\bigcup_{N}\bigcap_{n\geq N}(G^2)^{x_n}.$$

Therefore, we have

$$\{1, 2\} = \lim((x_n)) \subsetneq \bigcup_{x\in X}\bigcap_{G\in\mathscr{T}_X(x)}\bigcup_{N}\bigcap_{n\geq N}(G^2)^{x_n} = X.$$

Moreover, it would be conceptually worthwhile to remark that, in a T_0 space, a set of the form of the right-hand-side set occasionally equals the whole space.

In general, as immediately verifiable, every partially ordered set $(X, <')$, with a least element x_0 and of cardinal at least two, receiving the topology generated by the collection $\{\{y \in X \mid x_0 <' y <' x\} \mid x \in X\}$, in particular the Sierpiński space, is such an example; every topological space of cardinal at least two and with a dense open singleton is such an example. □

The T_0-ness assumption in Theorem 5 is essential; the other property in the statement ii) or iii) in Theorem 5 alone does not imply T_1-ness:

EXAMPLE 7. Let $X := \{0, 1, 2, 3\}$ be topologized with $\{\emptyset, \{0, 1\}, \{2, 3\}, X\}$. By considering the distinct elements $0, 1$ of X, the space X is not T_0.

Let (x_θ) be a net in X. Suppose first that $\lim((x_\theta)) = \emptyset$. If y belongs to the right-hand-side set associated with (x_θ) of the equality in the statement iii) in Theorem 5, then $\mathrm{cl}(\{y\})$ contains some limit of (x_θ), a contradiction; so the equality holds for all nonconvergent nets in X.

If (x_θ) is convergent, then $\lim((x_\theta)) = \{0, 1\}$ or $\lim((x_\theta)) = \{2, 3\}$ by construction. We claim that in either case the equality in the statement iii) in Theorem 5 always holds. Without loss of generality, suppose $\lim((x_\theta)) = \{0, 1\}$. If y belongs to the right-hand-side set associated with (x_θ) in the statement iii), then $0 \in \mathrm{cl}(\{y\})$ [without loss of generality]. Since $\mathrm{cl}(\{2\})$ and $\mathrm{cl}(\{3\})$ are both equal to $\{2, 3\}$ by construction, it follows that $y \in \{0, 1\}$. This proves the claim.

Thus the non-T_0, and hence non-T_1, space X satisfies the statement iii) in Theorem 5. □

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In view of Theorem 2 and Theorem 4.7 in [1], we record

PROPOSITION 8. If X is a T_1 space, and if $(x_\theta)_{\theta\in\Theta}$ is a net in X, then

$$\bigcap_{\varphi\,:\,\Theta\to\Theta\text{ cofinal}} \mathrm{cl}(\{x_{\varphi(\theta)}\mid\theta\in\Theta\}) = \bigcup_{x\in X}\bigcap_{G\in\mathscr{T}_X(x)}\bigcup_{\gamma\in\Theta}\bigcap_{\theta\tilde{\geq}\gamma}\left(G^2\right)^{x_\theta}$$

and, in particular, the right-hand-side set is closed in X. □

For reference, by a G_δ *set* in a topological space is meant precisely a countable intersection of open sets in the space. A countable union of G_δ sets is called a $G_{\delta\sigma}$ *set*, and so forth. A net is called *countable* if and only if its index set is countable; thus every sequence is a countable net.

THEOREM 9.

i) If X is a first countable T_1 space of countable cardinal, then the limit set of any given countable net in X is a $G_{\delta\sigma\delta\sigma}$ set.

ii) If X is a first countable Hausdorff space, then every singleton in X is a union of $G_{\delta\sigma\delta}$ sets.

Proof. i) Let $(x_\theta)_{\theta\in\Theta}$ be a countable net in X; define

$$A_G((x_\theta)) := \bigcup_{\gamma\in\Theta}\bigcap_{\theta\tilde{\geq}\gamma}\left(G^2\right)^{x_\theta}$$

for all open G in X. Since X is first countable by assumption, for every $x\in X$ we can, by acknowledging the Axiom of Choice, choose some countable neighborhood basis $\widehat{\mathscr{T}_X}(x)$ at x; then

$$\bigcap_{G\in\mathscr{T}_X(x)} A_G((x_\theta)) = \bigcap_{G\in\widehat{\mathscr{T}_X}(x)} A_G((x_\theta)).$$

The space X is assumed to be of countable cardinal, and so the openness of every $(G^2)^{x_\theta}$ and Theorem 2 now jointly imply that $\lim((x_\theta))$ is a $G_{\delta\sigma\delta\sigma}$ set.

ii) By the proof of the statement i), the desired result follows from considering the constant sequences in X. This completes the proof. □

REMARK 10.

i) A T_1 (even normal Hausdorff) space of countable cardinal need not be first countable; one may recall the Arens-Fort topologization (see, e.g. [5]): The set $\mathbb{N}^2$ is topologized such that every point other than $(1,1)$ is isolated and $(1,1)$ receives as a basic neighborhood every subset G of $\mathbb{N}^2$ for which $G\ni(1,1)$ and $G\ni(n,m)$ for some N, all $n\geq N$, some $\widehat{N}$, and all $m\geq\widehat{N}$.

ii) A non-Hausdorff first countable T_1 space of countable cardinal is readily available; any countably infinite set receiving the cofinite topology is a handy example. □

For reference, a *Fréchet–Urysohn space* is by definition precisely a topological space where a point lies in the closure of any given set A in the space if and only if there is some sequence in A converging to the point. Thus every first countable space is Fréchet–Urysohn.

PROPOSITION 11. If X is a T_1 Fréchet–Urysohn space, then

$$\mathrm{cl}(A) = \bigcup_{(x_n)\in A^{\mathbb{N}}}\bigcup_{x\in X}\bigcap_{G\in\mathscr{T}_X(x)}\bigcup_{N\in\mathbb{N}}\bigcap_{n\geq N}\left(G^2\right)^{x_n}$$

for all $A\subset X$.

Proof. For all $A\subset X$, the right-hand-side set is by Theorem 2 the union of every limit set of a sequence in A; the result then follows immediately from the definition of a Fréchet–Urysohn space. The proof is complete. □

REMARK 12. We draw a generic remark. One way to go about the intersections present in the formula of limit sets is to consider topological spaces such as P-spaces (i.e. spaces where every countable intersection of open sets is still open) or Alexandroff spaces (i.e. spaces where every intersection of open sets is still open).

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Although the former class of spaces is well-studied (in particular those Tychonoff ones) in general topology and is even natural ([3]) in some areas of analysis, and although the latter class plays a role in category theory and admits applications in digital topology ([2]), these spaces would not sit well simultaneously with T_1-ness and first countableness in terms of curiousness. Every T_1 Alexandroff space is simply a discrete space; and, since the intersection of all neighborhoods of a point equals the intersection of any given countable neighborhood basis at the point, so is every T_1 first countable P-space.

Therefore, while there is the fact that the limit set of a net in either a T_1 Alexandroff space or a T_1 first countable P-space is always clopen, it is merely due to the fact that these spaces are discrete spaces indeed. □

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