

A CHARACTERIZATION OF T_1 SPACES VIA LIMIT SETS OF NETS

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ABSTRACT

This article indicates another set-theoretic formula, solely in terms of union and intersection, for the set of the limits of any given sequence (net, in general) in an arbitrary T_1 space; this representation in particular gives a new characterization of a T_1 space.

KEYWORDS

Fréchet–Urysohn space, limit, net, sequence, T_1 space, T_0 space

MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 54A20; Secondary 54G10

Throughout, a topological space is assumed to possess a property if and only if a corresponding declaration is made.

We are concerned with that notion of a *limit* x of a *net* $(x_\theta)_{\theta \in \Theta}$ in a topological space X , where x is an element of X with the property that for every neighborhood G of x in X there exists some $\gamma \in \Theta$ such that if θ (nonstrictly) succeeds γ then $x_\theta \in G$; a *net* in a topological space means by definition precisely a map from a *directed set* (i.e. a set equipped with a reflexive transitive relation [i.e. a *preorder*] such that any two elements of the set admit a common [nonstrict] successor) to the space. Evidently, the set of positive integers can trivially be made into a directed set, and so every sequence in any given topological space is a net in the space. And the notion of a net is in particular convenient in many settings; as well-known, the definition of Riemann integrability may be restated in terms of the convergence of some suitable net in the real field \mathbb{R} .

We add that there is the elementary fact that the set of the limits of a sequence in a non-Hausdorff space may very well have cardinal greater than one; the real field \mathbb{R} receiving instead the cofinite topology, which is at least always T_1 , is an example.

For reference, a *limit point* of a *net* (x_θ) in a topological space is defined, in the same fashion as in the case of sequences, precisely as an element x of the space such that for every neighborhood G of x and for every γ there exists some (nonstrict) successor θ of γ such that $x_\theta \in G$.

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It is well-known that, given any net $(x_\theta)_{\theta \in \Theta}$, with \lesssim denoting the given preorder over Θ , in any given topological space, the set of the limit points of $(x_\theta)_{\theta \in \Theta}$ may be represented as the closed set $\bigcap_{\gamma \in \Theta} \text{cl}(\{x_\theta \mid \theta \geq \gamma\})$; on the other hand, by Theorem 4.7 in [1] we can express the set of the limits of $(x_\theta)_{\theta \in \Theta}$ as the closed set $\bigcap_{\varphi : \Theta \rightarrow \Theta \text{ cofinal}} \text{cl}(\{x_{\varphi(\theta)} \mid \theta \in \Theta\})$, where the intersection ranges over all $\varphi : \Theta \rightarrow \Theta$ such that for every $\gamma \in \Theta$ there exists some $\theta \in \Theta$ for which $\varphi(\theta) \gtrsim \gamma$.

For our purposes, we make the following

DEFINITION 1. Let $(x_\theta)_{\theta \in \Theta}$ be a net in an arbitrary topological space X . We denote by $\lim_X((x_\theta)_{\theta \in \Theta})$, or simply by $\lim((x_\theta))$ when the other things under consideration are clear, the set of the limits of $(x_\theta)_{\theta \in \Theta}$. For brevity, we refer to $\lim_X((x_\theta)_{\theta \in \Theta})$ as the *limit set* of the net $(x_\theta)_{\theta \in \Theta}$ in X . \square

Thus if (x_n) is a convergent sequence in a Hausdorff space X , then

$$\left\{ \lim_{n \rightarrow \infty} x_n \right\} = \lim((x_n)).$$

In a metric space X , the limit set of a net $(x_\theta)_{\theta \in \Theta}$ in X , necessarily having at most one element, admits another representation. As immediately seen, for all sequences (x_n) in a metric space (X, d) we certainly have

$$\lim((x_n)) = \bigcap_{\varepsilon > 0} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} B_d(x_n, \varepsilon).$$

Here $B_d(x_n, \varepsilon)$ denotes the open d -ball in X of radius ε and of center x_n for all ε and all n . Manifestly, since the open balls in a metric space form by definition a basis, the equality holds for nets as well.

It is natural to try and do the same for the limit set of a net in a topological space as general as possible. Given any T_1 space, we will give a set-theoretic representation for the limit set of any given net in the space such that this representation induces a characterization of T_1 spaces. We add that those T_1 spaces that are not necessarily Hausdorff can certainly still be “geometrically meaningful”; geometric complexes (topologized coherently with respect to the geometric simplexes therein), even if not point-finitely triangulated, serve as well-known (e.g. [4]) examples.

If $A \subset X \times Y$ and if $(x, y) \in X \times Y$, denote by $(A)^x$ [resp. $(A)_y$] the cross section $\{z \in Y \mid (x, z) \in A\}$ [resp. $\{z \in X \mid (z, y) \in A\}$]. Throughout, the boundary \square simply serves as a symbol signifying for readability the end of a block but not always implies Q.E.D.; this will not cause any confusion. And $\mathbb{N} := \omega \setminus \{\emptyset\}$ for concreteness and “commonality”.

One reason that the limit set of a net in a metric space is easily expressed via union and intersection over open balls is owing to the fact that formally we have $x \in B(y, \varepsilon)$ if and only if $y \in B(x, \varepsilon)$, which allows one to be concerned only with the radius ε . This convenience is not shared by just any collection of open sets in just any space; but this does not prevent one too much from proceeding:

THEOREM 2. If X is a T_1 space, with $\mathcal{T}_X(x)$ denoting the collection of all neighborhoods of x in X for all $x \in X$, and if $(x_\theta)_{\theta \in \Theta}$ is a net in X , with \lesssim denoting the given preorder over Θ , then

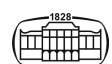
$$\lim((x_\theta)_{\theta \in \Theta}) = \bigcup_{x \in X} \bigcap_{G \in \mathcal{T}_X(x)} \bigcup_{\gamma \in \Theta} \bigcap_{\theta \gtrsim \gamma} (G \times G)^{x_\theta}.$$

Proof. Evidently, if y is a limit of (x_θ) , then y is an element of X and has the property that for all $G \in \mathcal{T}_X(y)$ there is some $\gamma \in \Theta$ such that for all $\theta \in \Theta$ with $\theta \gtrsim \gamma$ we have $x_\theta \in G$, and hence $(x_\theta, y) \in G^2$; so $y \in (G^2)^{x_\theta}$. This proves the inclusion relation \subset .

To prove the converse, let y lie in the right-hand-side set. Then we can choose some $x \in X$ such that for every $G \in \mathcal{T}_X(x)$ there is some $\gamma \in \Theta$ such that for all $\theta \gtrsim \gamma$ we have $y \in (G^2)^{x_\theta}$, which in particular implies that y lies in every neighborhood of x , implying that $x \in \text{cl}(\{y\})$. Since X is by assumption T_1 , we have $\text{cl}(\{y\}) = \{y\}$ and so $y = x$. But then, since x is a limit of (x_θ) , so is y ; this completes the proof. \square

REMARK 3. We keep the notation of Theorem 2; fix G and x_θ . In the equality, the set $(G^2)^{x_\theta}$ is not the only choice. Manifestly, the set $(G^2)_{x_\theta}$ is by symmetry equally effective.

Moreover, since we formally have $(G^2)^{x_\theta} = \pi_2^{\rightarrow}((\{x_\theta\} \times G) \cap G^2)$, the latter [which is an image under the natural projection $(x, y) \mapsto y$] being an open set in X and $\{x_\theta\}$ being a singleton having x_θ as its unique element, the set $(G^2)^{x_\theta}$ is open in X ; the same holds for $(G^2)_{x_\theta}$. \square



NOTATION 4. In what follows, the symbols $\mathcal{T}_X(x)$ and $\widetilde{<}$ are used as in Theorem 2. \square

With T_0 -ness the property that every sequence satisfies the equality in Theorem 2 suffices to characterize T_1 -ness; even more:

THEOREM 5. Let X be a topological space. Then the following statements are equivalent:

- i) X is T_1 ;
- ii) X is T_0 , and for every net $(x_\theta)_{\theta \in \Theta}$ in X we have

$$\lim((x_\theta)_{\theta \in \Theta}) = \bigcup_{x \in X} \bigcap_{G \in \mathcal{T}_X(x)} \bigcup_{\gamma \in \Theta} \bigcap_{\theta \widetilde{>} \gamma} (G^2)^{x_\theta};$$

- iii) X is T_0 , and for every sequence $(x_n)_{n \in \mathbb{N}}$ in X we have

$$\lim((x_n)_{n \in \mathbb{N}}) = \bigcup_{x \in X} \bigcap_{G \in \mathcal{T}_X(x)} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} (G^2)^{x_n}.$$

Proof. That i) implies ii) follows directly from Theorem 2; that ii) implies iii) is trivial.

To prove that iii) implies i), suppose X is T_0 but not T_1 . Since X is not T_1 , we can choose from definition some distinct $x, y \in X$ such that, without loss of generality, every neighborhood of x contains y . Then y lies in the right-hand-side set associated with the constant sequence $(x)_{n \in \mathbb{N}}$ that has x as one of its limits. But as X is T_0 , at least one of x and y has some neighborhood not containing the other, which in this case is necessarily y ; it follows that $y \notin \lim((x)_{n \in \mathbb{N}})$, and so the equality fails for the sequence $(x)_{n \in \mathbb{N}}$. The proof is complete. \square

The following is a simple example of a T_0 non- T_1 space that, as ensured by Theorem 5, does not satisfy the statement iii) in Theorem 5:

EXAMPLE 6. Give the set $X := \{0, 1, 2\}$ the topology $\{\emptyset, \{0\}, \{0, 1\}, X\}$. Then X is T_0 ; but since $\{0\}$ is not closed, the space X is not T_1 .

The constant sequence (x_n) with $x_n := 1$ for all n has 1 and 2 as its only limits. But X is the only closed set containing 0, and so

$$0 \in \bigcup_{x \in X} \bigcap_{G \in \mathcal{T}_X(x)} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} (G^2)^{x_n}.$$

Therefore, we have

$$\{1, 2\} = \lim((x_n)) \subseteq \bigcup_{x \in X} \bigcap_{G \in \mathcal{T}_X(x)} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} (G^2)^{x_n} = X.$$

Moreover, it would be conceptually worthwhile to remark that, in a T_0 space, a set of the form of the right-hand-side set occasionally equals the whole space.

In general, as immediately verifiable, every partially ordered set $(X, <')$, with a least element x_0 and of cardinal at least two, receiving the topology generated by the collection $\{\{y \in X \mid x_0 <' y <' x\} \mid x \in X\}$, in particular the Sierpiński space, is such an example; every topological space of cardinal at least two and with a dense open singleton is such an example. \square

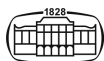
The T_0 -ness assumption in Theorem 5 is essential; the other property in the statement ii) or iii) in Theorem 5 alone does not imply T_1 -ness:

EXAMPLE 7. Let $X := \{0, 1, 2, 3\}$ be topologized with $\{\emptyset, \{0, 1\}, \{2, 3\}, X\}$. By considering the distinct elements 0, 1 of X , the space X is not T_0 .

Let (x_θ) be a net in X . Suppose first that $\lim((x_\theta)) = \emptyset$. If y belongs to the right-hand-side set associated with (x_θ) of the equality in the statement iii) in Theorem 5, then $\text{cl}(\{y\})$ contains some limit of (x_θ) , a contradiction; so the equality holds for all nonconvergent nets in X .

If (x_θ) is convergent, then $\lim((x_\theta)) = \{0, 1\}$ or $\lim((x_\theta)) = \{2, 3\}$ by construction. We claim that in either case the equality in the statement iii) in Theorem 5 always holds. Without loss of generality, suppose $\lim((x_\theta)) = \{0, 1\}$. If y belongs to the right-hand-side set associated with (x_θ) in the statement iii), then $0 \in \text{cl}(\{y\})$ [without loss of generality]. Since $\text{cl}(\{2\})$ and $\text{cl}(\{3\})$ are both equal to $\{2, 3\}$ by construction, it follows that $y \in \{0, 1\}$. This proves the claim.

Thus the non- T_0 , and hence non- T_1 , space X satisfies the statement iii) in Theorem 5. \square



In view of Theorem 2 and Theorem 4.7 in [1], we record

PROPOSITION 8. If X is a T_1 space, and if $(x_\theta)_{\theta \in \Theta}$ is a net in X , then

$$\bigcap_{\varphi : \Theta \rightarrow \Theta \text{ cofinal}} \text{cl}(\{x_{\varphi(\theta)} \mid \theta \in \Theta\}) = \bigcup_{x \in X} \bigcap_{G \in \mathcal{T}_X(x)} \bigcup_{\gamma \in \Theta} \bigcap_{\theta \succ \gamma} (G^2)^{x_\theta}$$

and, in particular, the right-hand-side set is closed in X . \square

For reference, by a G_δ set in a topological space is meant precisely a countable intersection of open sets in the space. A countable union of G_δ sets is called a $G_{\delta\sigma}$ set, and so forth. A net is called *countable* if and only if its index set is countable; thus every sequence is a countable net.

THEOREM 9.

- i) If X is a first countable T_1 space of countable cardinal, then the limit set of any given countable net in X is a $G_{\delta\sigma\delta\sigma}$ set.
- ii) If X is a first countable Hausdorff space, then every singleton in X is a union of $G_{\delta\sigma\delta}$ sets.

Proof. i) Let $(x_\theta)_{\theta \in \Theta}$ be a countable net in X ; define

$$A_G((x_\theta)) := \bigcup_{\gamma \in \Theta} \bigcap_{\theta \succ \gamma} (G^2)^{x_\theta}$$

for all open G in X . Since X is first countable by assumption, for every $x \in X$ we can, by acknowledging the Axiom of Choice, choose some countable neighborhood basis $\widehat{\mathcal{T}}_X(x)$ at x ; then

$$\bigcap_{G \in \mathcal{T}_X(x)} A_G((x_\theta)) = \bigcap_{G \in \widehat{\mathcal{T}}_X(x)} A_G((x_\theta)).$$

The space X is assumed to be of countable cardinal, and so the openness of every $(G^2)^{x_\theta}$ and Theorem 2 now jointly imply that $\lim((x_\theta))$ is a $G_{\delta\sigma\delta\sigma}$ set.

ii) By the proof of the statement i), the desired result follows from considering the constant sequences in X . This completes the proof. \square

REMARK 10.

- i) A T_1 (even normal Hausdorff) space of countable cardinal need not be first countable; one may recall the Arens-Fort topologization (see, e.g. [5]): The set \mathbb{N}^2 is topologized such that every point other than $(1, 1)$ is isolated and $(1, 1)$ receives as a basic neighborhood every subset G of \mathbb{N}^2 for which $G \ni (1, 1)$ and $G \ni (n, m)$ for some N , all $n \geq N$, some \widehat{N} , and all $m \geq \widehat{N}$.
- ii) A non-Hausdorff first countable T_1 space of countable cardinal is readily available; any countably infinite set receiving the cofinite topology is a handy example. \square

For reference, a *Fréchet–Urysohn space* is by definition precisely a topological space where a point lies in the closure of any given set A in the space if and only if there is some sequence in A converging to the point. Thus every first countable space is Fréchet–Urysohn.

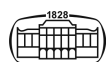
PROPOSITION 11. If X is a T_1 Fréchet–Urysohn space, then

$$\text{cl}(A) = \bigcup_{(x_n) \in A^{\mathbb{N}}} \bigcup_{x \in X} \bigcap_{G \in \mathcal{T}_X(x)} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} (G^2)^{x_n}$$

for all $A \subset X$.

Proof. For all $A \subset X$, the right-hand-side set is by Theorem 2 the union of every limit set of a sequence in A ; the result then follows immediately from the definition of a Fréchet–Urysohn space. The proof is complete. \square

REMARK 12. We draw a generic remark. One way to go about the intersections present in the formula of limit sets is to consider topological spaces such as P -spaces (i.e. spaces where every countable intersection of open sets is still open) or Alexandroff spaces (i.e. spaces where every intersection of open sets is still open).



Although the former class of spaces is well-studied (in particular those Tychonoff ones) in general topology and is even natural ([3]) in some areas of analysis, and although the latter class plays a role in category theory and admits applications in digital topology ([2]), these spaces would not sit well simultaneously with T_1 -ness and first countableness in terms of curiousness. Every T_1 Alexandroff space is simply a discrete space; and, since the intersection of all neighborhoods of a point equals the intersection of any given countable neighborhood basis at the point, so is every T_1 first countable P -space.

Therefore, while there is the fact that the limit set of a net in either a T_1 Alexandroff space or a T_1 first countable P -space is always clopen, it is merely due to the fact that these spaces are discrete spaces indeed. \square

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