

## OPERATOR CONVEXITY OF AN INTEGRAL TRANSFORM WITH APPLICATIONS

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### ABSTRACT

For a continuous and positive function  $w(\lambda)$ ,  $\lambda > 0$  and  $\mu$  a positive measure on  $(0, \infty)$  we consider the following *integral transform*

$$D(w, \mu)(t) := \int_0^\infty w(\lambda)(\lambda + t)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for  $t > 0$ .

We show among others that  $D(w, \mu)$  is operator convex on  $(0, \infty)$ . From this we derive that, if  $f : [0, \infty) \rightarrow \mathbb{R}$  is an operator monotone function on  $[0, \infty)$ , then the function  $[f(0) - f(t)]t^{-1}$  is operator convex on  $(0, \infty)$ . Also, if  $f : [0, \infty) \rightarrow \mathbb{R}$  is an operator convex function on  $[0, \infty)$ , then the function  $[f(0) + f'_+(0)t - f(t)]t^{-2}$  is operator convex on  $(0, \infty)$ . Some lower and upper bounds for the Jensen's difference

$$\frac{D(w, \mu)(A) + D(w, \mu)(B)}{2} - D(w, \mu)\left(\frac{A+B}{2}\right)$$

under some natural assumptions for the positive operators  $A$  and  $B$  are given. Examples for power, exponential and logarithmic functions are also provided.

### KEYWORDS

Operator monotone functions, operator convex functions, operator inequalities, Löwner–Heinz inequality, logarithmic operator inequalities

### MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 47A63; Secondary 47A60

## 1. INTRODUCTION

Consider a complex Hilbert space  $(H, \langle \cdot, \cdot \rangle)$ . An operator  $T$  is said to be positive (denoted by  $T \geq 0$ ) if  $\langle Tx, x \rangle \geq 0$  for all  $x \in H$  and also an operator  $T$  is said to be *strictly positive* (denoted by  $T > 0$ ) if  $T$  is positive and invertible. A real valued continuous function  $f$  on  $(0, \infty)$  is said to be operator

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monotone if  $f(A) \geq f(B)$  holds for any  $A \geq B > 0$ . We have the following representation of operator monotone functions [8], see for instance [1, p. 144-145]:

**THEOREM 1.1.** A function  $f : [0, \infty) \rightarrow \mathbb{R}$  is operator monotone in  $[0, \infty)$  if and only if it has the representation

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \quad (1.1)$$

where  $b \geq 0$  and a positive measure  $\mu$  on  $[0, \infty)$  such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \quad (1.2)$$

A real valued continuous function  $f$  on an interval  $I$  is said to be *operator convex (operator concave)* on  $I$  if

$$f((1-\lambda)A + \lambda B) \leq (\geq) (1-\lambda)f(A) + \lambda f(B) \quad (\text{OC})$$

in the operator order, for all  $\lambda \in [0, 1]$  and for every selfadjoint operator  $A$  and  $B$  on a Hilbert space  $H$  whose spectra are contained in  $I$ . Notice that a function  $f$  is operator concave if  $-f$  is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

**THEOREM 1.2.** A function  $f : [0, \infty) \rightarrow \mathbb{R}$  is operator convex in  $[0, \infty)$  with  $f'_+(0) \in \mathbb{R}$  if and only if it has the representation

$$f(t) = f(0) + f'_+(0)t + ct^2 + \int_0^\infty \frac{t^2\lambda}{t+\lambda} d\mu(\lambda), \quad (1.3)$$

where  $c \geq 0$  and a positive measure  $\mu$  on  $[0, \infty)$  such that (1.2) holds.

We have the following integral representation for the power function when  $t > 0$ ,  $r \in (0, 1]$ , see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda+t} d\lambda.$$

Observe that for  $t > 0$ ,  $t \neq 1$ , we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)} = \frac{\ln t}{t-1} + \frac{1}{1-t} \ln \left( \frac{u+t}{u+1} \right)$$

for all  $u > 0$ .

By taking the limit over  $u \rightarrow \infty$  in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1) \int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)}$$

for all  $t > 0$ .

Motivated by these representations, we introduce, for a continuous and positive function  $w(\lambda)$ ,  $\lambda > 0$ , the following *integral transform*

$$D(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\mu(\lambda), \quad t > 0, \quad (1.4)$$

where  $\mu$  is a positive measure on  $(0, \infty)$  and the integral (1.4) exists for all  $t > 0$ .

For  $\mu$  the Lebesgue usual measure, we put

$$D(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\lambda, \quad t > 0. \quad (1.5)$$

If we take  $\mu$  to be the usual Lebesgue measure and the kernel  $w_r(\lambda) = \lambda^{r-1}$ ,  $r \in (0, 1]$ , then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} D(w_r)(t), \quad t > 0. \quad (1.6)$$



For the same measure, if we take the kernel  $w_{\ln}(\lambda) = (\lambda + 1)^{-1}$ ,  $t > 0$ , we have the representation

$$\ln t = (t - 1)D(w_{\ln})(t), \quad t > 0. \tag{1.7}$$

Assume that  $T > 0$ , then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$D(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

where  $w$  and  $\mu$  are as above. Also, when  $\mu$  is the usual Lebesgue measure, then

$$D(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda, \tag{1.8}$$

for  $T > 0$ .

From (1.6) we have the representation

$$T^{r-1} = \frac{\sin(r\pi)}{\pi} D(w_r)(T) \tag{1.9}$$

where  $T > 0$  and from (1.7)

$$(T - 1)^{-1} \ln T = D(w_{\ln})(T) \tag{1.10}$$

provided  $T > 0$  and  $T - 1$  is invertible.

In this paper, we show among others that  $D(w, \mu)$  is operator convex on  $(0, \infty)$ . From this we derive that, if  $f : [0, \infty) \rightarrow \mathbb{R}$  is an operator monotone function on  $[0, \infty)$ , then the function  $[f(0) - f(t)]t^{-1}$  is operator convex on  $(0, \infty)$ . Also, if  $f : [0, \infty) \rightarrow \mathbb{R}$  is an operator convex function on  $[0, \infty)$ , then the function  $[f(0) + f'_+(0)t - f(t)]t^{-2}$  is operator convex on  $(0, \infty)$ . Some lower and upper bounds for the Jensen's difference

$$\frac{D(w, \mu)(A) + D(w, \mu)(B)}{2} - D(w, \mu)\left(\frac{A + B}{2}\right)$$

under some natural assumptions for the positive operators  $A$  and  $B$  are given. Examples for power, exponential and logarithmic functions are also provided.

## 2. PRELIMINARY RESULTS

We start with the following elementary identity that give a simple proof for the fact that the function  $f(t) = t^{-1}$  is operator convex on  $(0, \infty)$ , see for instance [6, p. 8]:

**LEMMA 2.1.** For any  $A, B > 0$  we have

$$\frac{A^{-1} + B^{-1}}{2} - \left(\frac{A + B}{2}\right)^{-1} = \frac{(A^{-1} - B^{-1})(A^{-1} + B^{-1})^{-1}(A^{-1} - B^{-1})}{2} \geq 0. \tag{2.1}$$

If more assumptions are made for the operators  $A$  and  $B$ , then one can obtain the following lower and upper bounds:

**COROLLARY 2.2.** Assume that  $0 < \alpha \leq A \leq \beta$  and  $0 < \gamma \leq B \leq \delta$  for some constants  $\alpha, \beta, \gamma, \delta$ . Then

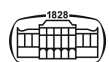
$$\begin{aligned} \frac{1}{2}(\alpha^{-1} + \gamma^{-1})^{-1}(A^{-1} - B^{-1})^2 &\leq \frac{A^{-1} + B^{-1}}{2} - \left(\frac{A + B}{2}\right)^{-1} \\ &\leq \frac{1}{2}(\beta^{-1} + \delta^{-1})^{-1}(A^{-1} - B^{-1})^2. \end{aligned} \tag{2.2}$$

**Proof.** We have  $\beta^{-1} \leq A^{-1} \leq \alpha^{-1}$  and  $\delta^{-1} \leq B^{-1} \leq \gamma^{-1}$ , which gives

$$\beta^{-1} + \delta^{-1} \leq A^{-1} + B^{-1} \leq \alpha^{-1} + \gamma^{-1},$$

namely

$$(\alpha^{-1} + \gamma^{-1})^{-1} \leq (A^{-1} + B^{-1})^{-1} \leq (\beta^{-1} + \delta^{-1})^{-1}.$$



By multiplying both sides by  $(A^{-1} - B^{-1})$  and dividing by 2, we get

$$\begin{aligned} \frac{1}{2} (\alpha^{-1} + \gamma^{-1})^{-1} (A^{-1} - B^{-1})^2 &\leq \frac{(A^{-1} - B^{-1}) (A^{-1} + B^{-1})^{-1} (A^{-1} - B^{-1})}{2} \\ &\leq \frac{1}{2} (\beta^{-1} + \delta^{-1})^{-1} (A^{-1} - B^{-1})^2. \end{aligned} \quad \square$$

We know that for  $T > 0$ , we have the operator inequalities

$$0 < \|T^{-1}\|^{-1} \leq T \leq \|T\|. \quad (2.3)$$

Indeed, it is well known that, if  $P \geq 0$ , then

$$|\langle Px, y \rangle|^2 \leq \langle Px, x \rangle \langle Py, y \rangle$$

for all  $x, y \in H$ .

Therefore, if  $T > 0$ , then

$$\begin{aligned} 0 \leq \langle x, x \rangle^2 &= \langle T^{-1}Tx, x \rangle^2 = \langle Tx, T^{-1}x \rangle^2 \\ &\leq \langle Tx, x \rangle \langle TT^{-1}x, T^{-1}x \rangle = \langle Tx, x \rangle \langle x, T^{-1}x \rangle \end{aligned}$$

for all  $x \in H$ .

If  $x \in H$ ,  $\|x\| = 1$ , then

$$1 \leq \langle Tx, x \rangle \langle x, T^{-1}x \rangle \leq \langle Tx, x \rangle \sup_{\|x\|=1} \langle x, T^{-1}x \rangle = \langle Tx, x \rangle \|T^{-1}\|,$$

which implies the following operator inequality

$$\|T^{-1}\|^{-1} \leq T.$$

The second inequality in (2.3) is obvious.

**REMARK 2.3.** If  $A, B > 0$  and  $B - A > 0$ , then by taking  $\alpha = \|A^{-1}\|^{-1}$ ,  $\beta = \|A\|$ ,  $\gamma = \|B^{-1}\|^{-1}$  and  $\delta = \|B\|$  in (2.2), we get

$$\begin{aligned} \frac{1}{2} (\|A^{-1}\| + \|B^{-1}\|)^{-1} (A^{-1} - B^{-1})^2 &\leq \frac{A^{-1} + B^{-1}}{2} - \left( \frac{A + B}{2} \right)^{-1} \\ &\leq \frac{1}{2} (\|A\|^{-1} + \|B\|^{-1})^{-1} (A^{-1} - B^{-1})^2. \end{aligned} \quad (2.4)$$

A continuous function  $g : \mathcal{SA}_I(H) \rightarrow \mathcal{B}(H)$  is said to be *Gâteaux differentiable* in  $A \in \mathcal{SA}_I(H)$ , the class of selfadjoint operators on  $I$ , along the direction  $B \in \mathcal{B}(H)$  if the following limit exists in the strong topology of  $\mathcal{B}(H)$

$$\nabla g_A(B) := \lim_{s \rightarrow 0} \frac{g(A + sB) - g(A)}{s} \in \mathcal{B}(H). \quad (2.5)$$

If the limit (2.5) exists for all  $B \in \mathcal{B}(H)$ , then we say that  $g$  is *Gâteaux differentiable* in  $A$  and we can write  $g \in \mathcal{G}(A)$ . If this is true for any  $A$  in an open set  $\mathcal{S}$  from  $\mathcal{SA}_I(H)$  we write that  $g \in \mathcal{G}(\mathcal{S})$ .

If  $g$  is a continuous function on  $I$ , by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators  $A, B \in \mathcal{SA}_I(H)$  we consider the segment of selfadjoint operators

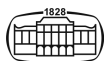
$$[A, B] := \{(1-t)A + tB \mid t \in [0, 1]\}.$$

We observe that  $A, B \in [A, B]$  and  $[A, B] \subset \mathcal{SA}_I(H)$ .

We have the following gradient inequalities, see for instance:

**LEMMA 2.4.** Let  $f$  be an operator convex function on  $I$  and  $A, B \in \mathcal{SA}_I(H)$ , with  $A \neq B$ . If  $f \in \mathcal{G}([A, B])$ , then

$$\nabla_B f(B - A) \geq f(B) - f(A) \geq \nabla_A f(B - A). \quad (2.6)$$



Let  $T, S > 0$ . The function  $f(t) = t^{-1}$  is operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S) := \lim_{t \rightarrow 0} \left[ \frac{f(T + tS) - f(T)}{t} \right] = -T^{-1}ST^{-1} \tag{2.7}$$

for  $T, S > 0$ .

Using (2.7) for the operator convex function  $f(t) = t^{-1}$ , we get

$$-D^{-1}(D - C)D^{-1} \geq D^{-1} - C^{-1} \geq -C^{-1}(D - C)C^{-1}$$

that is equivalent to

$$D^{-1}(D - C)D^{-1} \leq C^{-1} - D^{-1} \leq C^{-1}(D - C)C^{-1} \tag{2.8}$$

for all  $C, D > 0$ .

If

$$m \leq D - C \leq M$$

for some constants  $m, M$ , then

$$mD^{-2} \leq D^{-1}(D - C)D^{-1}$$

and

$$C^{-1}(D - C)C^{-1} \leq MC^{-2}$$

and by (2.8) we derive

$$mD^{-2} \leq C^{-1} - D^{-1} \leq MC^{-2}. \tag{2.9}$$

Moreover, if  $C \geq \alpha > 0$  and  $D \leq \delta$ , then we get

$$C^{-2} \leq \alpha^{-2} \text{ and } D^{-2} \geq \delta^{-2},$$

which implies that

$$\frac{m}{\delta^2} \leq C^{-1} - D^{-1} \leq \frac{M}{\alpha^2} \tag{2.10}$$

**COROLLARY 2.5.** Assume that  $0 < \alpha \leq A \leq \beta$ ,  $0 < \gamma \leq B \leq \delta$  and  $0 < m \leq B - A \leq M$  for some constants  $\alpha, \beta, \gamma, \delta, m, M$ . Then

$$\begin{aligned} 0 &< \frac{1}{2} (\alpha^{-1} + \gamma^{-1})^{-1} \frac{m^2}{\delta^4} \\ &\leq \frac{1}{2} (\alpha^{-1} + \gamma^{-1})^{-1} (A^{-1} - B^{-1})^2 \leq \frac{A^{-1} + B^{-1}}{2} - \left( \frac{A + B}{2} \right)^{-1} \\ &\leq \frac{1}{2} (\beta^{-1} + \delta^{-1})^{-1} (A^{-1} - B^{-1})^2 \leq \frac{1}{2} (\beta^{-1} + \delta^{-1})^{-1} \frac{M^2}{\alpha^4}. \end{aligned} \tag{2.11}$$

**Proof.** From (2.10) we have

$$0 < \frac{m}{\delta^2} \leq A^{-1} - B^{-1} \leq \frac{M}{\alpha^2},$$

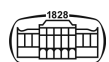
which implies that

$$0 < \frac{m^2}{\delta^4} \leq (A^{-1} - B^{-1})^2 \leq \frac{M^2}{\alpha^4}$$

and by (2.2) we get (2.11). □

**REMARK 2.6.** If the positive operators  $A, B$  are separated, namely  $0 < \alpha \leq A \leq \beta < \gamma \leq B \leq \delta$  for some constants  $\alpha, \beta, \gamma, \delta$ , then obviously  $0 < \gamma - \beta \leq B - A \leq \delta - \alpha$  and by (2.11) for  $m = \gamma - \beta$  and  $M = \delta - \alpha$ , we get

$$\begin{aligned} 0 &< \frac{1}{2} (\alpha^{-1} + \gamma^{-1})^{-1} \frac{(\gamma - \beta)^2}{\delta^4} \leq \frac{1}{2} (\alpha^{-1} + \gamma^{-1})^{-1} (A^{-1} - B^{-1})^2 \\ &\leq \frac{A^{-1} + B^{-1}}{2} - \left( \frac{A + B}{2} \right)^{-1} \\ &\leq \frac{1}{2} (\beta^{-1} + \delta^{-1})^{-1} (A^{-1} - B^{-1})^2 \leq \frac{1}{2} (\beta^{-1} + \delta^{-1})^{-1} \frac{(\delta - \alpha)^2}{\alpha^4}. \end{aligned} \tag{2.12}$$



If  $0 < \|A\| \|B^{-1}\| < 1$ , then

$$0 < \|A^{-1}\|^{-1} \leq A \leq \|A\| < \|B^{-1}\|^{-1} \leq B \leq \|B\|$$

and by (2.12) we get

$$\begin{aligned} 0 &< \frac{1}{2} (\|A^{-1}\| + \|B^{-1}\|)^{-1} \frac{(\|B^{-1}\|^{-1} - \|A\|)^2}{\|B\|^4} \\ &\leq \frac{1}{2} (\|A^{-1}\| + \|B^{-1}\|)^{-1} (A^{-1} - B^{-1})^2 \\ &\leq \frac{A^{-1} + B^{-1}}{2} - \left(\frac{A+B}{2}\right)^{-1} \\ &\leq \frac{1}{2} (\|A\|^{-1} + \|B\|^{-1})^{-1} (A^{-1} - B^{-1})^2 \\ &\leq \frac{1}{2} (\beta^{-1} + \delta^{-1})^{-1} (\|B\| - \|A^{-1}\|^{-1})^2 \|A^{-1}\|^4. \end{aligned} \quad (2.13)$$

We can present now our main results.

### 3. MAIN RESULTS

We have

**THEOREM 3.1.** For all  $A, B > 0$  we have

$$\begin{aligned} &\frac{D(w, \mu)(A) + D(w, \mu)(B)}{2} - D(w, \mu) \left(\frac{A+B}{2}\right) \\ &= \frac{1}{2} \int_0^\infty ((\lambda + A)^{-1} - (\lambda + B)^{-1}) ((\lambda + A)^{-1} + (\lambda + B)^{-1})^{-1} \times ((\lambda + A)^{-1} - (\lambda + B)^{-1}) w(\lambda) d\mu(\lambda) \\ &\geq 0. \end{aligned} \quad (3.1)$$

The function  $D(w, \mu)$  is an operator convex function on  $(0, \infty)$

**Proof.** We have for all  $A, B > 0$

$$\begin{aligned} &\frac{D(w, \mu)(A) + D(w, \mu)(B)}{2} - D(w, \mu) \left(\frac{A+B}{2}\right) \\ &= \int_0^\infty w(\lambda) \left[ \frac{(\lambda + A)^{-1} + (\lambda + B)^{-1}}{2} - \left(\lambda + \frac{A+B}{2}\right)^{-1} \right] d\mu(\lambda). \end{aligned} \quad (3.2)$$

Since, by (2.1)

$$\begin{aligned} &\frac{(\lambda + A)^{-1} + (\lambda + B)^{-1}}{2} - \left(\lambda + \frac{A+B}{2}\right)^{-1} \\ &= \frac{1}{2} ((\lambda + A)^{-1} - (\lambda + B)^{-1}) ((\lambda + A)^{-1} + (\lambda + B)^{-1})^{-1} \times ((\lambda + A)^{-1} - (\lambda + B)^{-1}) \\ &\geq 0 \end{aligned}$$

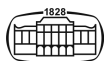
for all  $\lambda \geq 0$ , then by (3.2) we obtain the representation (3.1).

Since  $D(w, \mu)$  is continuous in  $\mathcal{B}(H)$  and satisfies Jensen's inequality (3.1), it follows that  $D(w, \mu)$  is an operator convex function on  $(0, \infty)$ .  $\square$

The case of operator monotone functions is as follows:

**COROLLARY 3.2.** Assume that  $f : [0, \infty) \rightarrow \mathbb{R}$  is an operator monotone function on  $[0, \infty)$ . Then the function  $[f(t) - f(0)]t^{-1}$  is operator convex on  $(0, \infty)$ . For all  $A, B > 0$  we have

$$\frac{f(A)A^{-1} + f(B)B^{-1}}{2} - f\left(\frac{A+B}{2}\right) \left(\frac{A+B}{2}\right)^{-1} \geq f(0) \left[ \frac{(A^{-1} - B^{-1})(A^{-1} + B^{-1})^{-1}(A^{-1} - B^{-1})}{2} \right]. \quad (3.3)$$



If  $f(0) = 0$ , then  $f(t)t^{-1}$  is operator convex on  $(0, \infty)$  and

$$\frac{f(A)A^{-1} + f(B)B^{-1}}{2} \geq f\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^{-1}$$

for all  $A, B > 0$ .

**Proof.** From (1.1) we have

$$\frac{f(t) - f(0)}{t} - b = \mathcal{D}(\ell, \mu)(t), \tag{3.4}$$

for some  $\mu$ , a positive measure on  $(0, \infty)$ , where  $\ell(\lambda) = \lambda, \lambda \geq 0$ . By utilising Theorem 3.1 and Lemma 2.1 we deduce the desired results.  $\square$

**COROLLARY 3.3.** Assume that  $f : [0, \infty) \rightarrow \mathbb{R}$  is an operator convex function on  $[0, \infty)$ . Then the function  $[f(t) - f(0) - f'_+(0)t]t^{-2}$  is operator convex on  $(0, \infty)$ . For all  $A, B > 0$  we have

$$\begin{aligned} & \frac{f(A)A^{-2} + f(B)B^{-2}}{2} - f\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^{-2} \\ & \geq f(0)\left[\frac{A^{-2} + B^{-2}}{2} - \left(\frac{A+B}{2}\right)^{-2}\right] + f'_+(0)\left[\frac{(A^{-1} - B^{-1})(A^{-1} + B^{-1})^{-1}(A^{-1} - B^{-1})}{2}\right]. \end{aligned} \tag{3.5}$$

If  $f(0) = 0$ , then  $[f(t) - f'_+(0)t]t^{-2}$  is operator convex on  $(0, \infty)$  and

$$\frac{f(A)A^{-2} + f(B)B^{-2}}{2} - f\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^{-2} \geq f'_+(0)\left[\frac{(A^{-1} - B^{-1})(A^{-1} + B^{-1})^{-1}(A^{-1} - B^{-1})}{2}\right] \tag{3.6}$$

for all  $A, B > 0$ .

**Proof.** From (1.3) we have

$$[f(t) - f(0) - f'_+(0)t]t^{-2} - c = \mathcal{D}(\ell, \mu)(t),$$

for some  $\mu$ , a positive measure on  $(0, \infty)$ , where  $\ell(\lambda) = \lambda, \lambda \geq 0$ . By utilising Theorem 3.1 and Lemma 2.1 we deduce the desired results.  $\square$

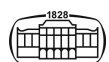
When more assumptions are imposed on the operators  $A$  and  $B$ , then the following improvement and refinement of Jensen's inequality hold:

**THEOREM 3.4.** Assume that  $0 < \alpha \leq A \leq \beta, 0 < \gamma \leq B \leq \delta$  and  $0 < m \leq B - A \leq M$  for some constants  $\alpha, \beta, \gamma, \delta, m, M$ . Then

$$\begin{aligned} 0 & < -\frac{m^2\gamma\alpha}{12(\alpha + \gamma)}\mathcal{D}'''(w, \mu)(\delta) \\ & \leq \frac{\mathcal{D}(w, \mu)(A) + \mathcal{D}(w, \mu)(B)}{2} - \mathcal{D}(w, \mu)\left(\frac{A+B}{2}\right) \\ & \leq \frac{M^2}{2(\beta + \delta)}\left[-\mathcal{D}'(w, \mu)(\alpha) + \left(\frac{\delta + \beta}{2} - \alpha\right)\mathcal{D}''(w, \mu)(\alpha) - \frac{1}{6}(\beta - \alpha)(\delta - \alpha)\mathcal{D}'''(w, \mu)(\alpha)\right]. \end{aligned} \tag{3.7}$$

**Proof.** We have  $0 < \alpha + \lambda \leq A + \lambda \leq \beta + \lambda, 0 < \gamma + \lambda \leq B + \lambda \leq \delta + \lambda$  and  $0 < m \leq B + \lambda - A - \lambda = B - A \leq M$  for all  $\lambda \geq 0$ . By (2.11) we get

$$\begin{aligned} 0 & < \frac{1}{2}\left(\frac{1}{\alpha + \lambda} + \frac{1}{\gamma + \lambda}\right)^{-1}\frac{m^2}{(\delta + \lambda)^4} \\ & \leq \frac{(A + \lambda)^{-1} + (B + \lambda)^{-1}}{2} - \left(\lambda + \frac{A+B}{2}\right)^{-1} \\ & \leq \frac{1}{2}\left(\frac{1}{\beta + \lambda} + \frac{1}{\delta + \lambda}\right)^{-1}\frac{M^2}{(\alpha + \lambda)^4}. \end{aligned} \tag{3.8}$$



We have that

$$\left( \frac{1}{\beta + \lambda} + \frac{1}{\delta + \lambda} \right)^{-1} = \frac{(\beta + \lambda)(\delta + \lambda)}{\beta + \delta + 2\lambda} \leq \frac{(\beta + \lambda)(\delta + \lambda)}{\beta + \delta} \quad (3.9)$$

and

$$\left( \frac{1}{\alpha + \lambda} + \frac{1}{\gamma + \lambda} \right)^{-1} = \frac{(\gamma + \lambda)(\alpha + \lambda)}{\alpha + \gamma + 2\lambda} = g(\lambda).$$

We have

$$g'(\lambda) = \frac{(\alpha + \gamma + 2\lambda)^2 - 2(\gamma + \lambda)(\alpha + \lambda)}{(\alpha + \gamma + 2\lambda)^2} = \frac{(\alpha + \lambda)^2 + (\gamma + \lambda)^2}{(\alpha + \gamma + 2\lambda)^2} > 0,$$

which shows that  $g$  is increasing on  $[0, \infty)$ .

Therefore

$$g(\lambda) \geq g(0) = \frac{\gamma\alpha}{\alpha + \gamma} \text{ for all } \lambda \geq 0. \quad (3.10)$$

By (3.8)–(3.10) we derive that

$$\begin{aligned} 0 &< \frac{1}{2} \frac{\gamma\alpha}{\alpha + \gamma} \frac{m^2}{(\delta + \lambda)^4} \\ &\leq \frac{(A + \lambda)^{-1} + (B + \lambda)^{-1}}{2} - \left( \lambda + \frac{A + B}{2} \right)^{-1} \\ &\leq \frac{1}{2} \frac{(\beta + \lambda)(\delta + \lambda)}{\beta + \delta} \frac{M^2}{(\alpha + \lambda)^4}, \end{aligned}$$

which implies that

$$\begin{aligned} 0 &< \frac{1}{2} m^2 \frac{\gamma\alpha}{\alpha + \gamma} \int_0^\infty \frac{w(\lambda)}{(\delta + \lambda)^4} d\mu(\lambda) \\ &\leq \int_0^\infty \left[ \frac{(A + \lambda)^{-1} + (B + \lambda)^{-1}}{2} - \left( \lambda + \frac{A + B}{2} \right)^{-1} \right] w(\lambda) d\mu(\lambda) \\ &\leq \frac{1}{2} \frac{M^2}{\beta + \delta} \int_0^\infty \frac{(\beta + \lambda)(\delta + \lambda)}{(\alpha + \lambda)^4} w(\lambda) d\mu(\lambda). \end{aligned} \quad (3.11)$$

We observe that, by the definition of  $\mathcal{D}(w, \mu)(t)$ , and the properties of the derivatives of integrals with a parameter, we have

$$\mathcal{D}'(w, \mu)(t) := - \int_0^\infty \frac{w(\lambda)}{(\lambda + t)^2} d\mu(\lambda),$$

$$\mathcal{D}''(w, \mu)(t) := 2 \int_0^\infty \frac{w(\lambda)}{(\lambda + t)^3} d\mu(\lambda),$$

and

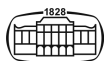
$$\mathcal{D}'''(w, \mu)(t) := -6 \int_0^\infty \frac{w(\lambda)}{(\lambda + t)^4} d\mu(\lambda),$$

which gives that

$$\int_0^\infty \frac{w(\lambda)}{(\lambda + \delta)^4} d\mu(\lambda) = -\frac{1}{6} \mathcal{D}'''(w, \mu)(\delta). \quad (3.12)$$

Also, we observe that

$$\begin{aligned} \frac{(\beta + \lambda)(\delta + \lambda)}{(\alpha + \lambda)^4} &= \frac{(\beta - \alpha + \lambda + \alpha)(\delta - \alpha + \lambda + \alpha)}{(\alpha + \lambda)^4} \\ &= (\beta - \alpha)(\delta - \alpha) \frac{1}{(\alpha + \lambda)^4} + (\delta + \beta - 2\alpha) \frac{1}{(\alpha + \lambda)^3} + \frac{1}{(\alpha + \lambda)^2}. \end{aligned}$$





Therefore,

$$\begin{aligned} & \int_0^\infty \frac{(\beta + \lambda)(\delta + \lambda)}{(\alpha + \lambda)^4} w(\lambda) d\mu(\lambda) \\ &= \int_0^\infty \frac{w(\lambda) d\mu(\lambda)}{(\alpha + \lambda)^2} + (\delta + \beta - 2\alpha) \int_0^\infty \frac{w(\lambda) d\mu(\lambda)}{(\alpha + \lambda)^3} + (\beta - \alpha)(\delta - \alpha) \int_0^\infty \frac{w(\lambda) d\mu(\lambda)}{(\alpha + \lambda)^4} \quad (3.13) \\ &= -D'(w, \mu)(\alpha) + \left(\frac{\delta + \beta}{2} - \alpha\right) D''(w, \mu)(\alpha) - \frac{1}{6}(\beta - \alpha)(\delta - \alpha) D'''(w, \mu)(\alpha). \end{aligned}$$

By making use of (3.11)-(3.13), we deduce (3.7). □

**COROLLARY 3.5.** If the positive operators  $A, B$  are separated, namely  $0 < \alpha \leq A \leq \beta < \gamma \leq B \leq \delta$  for some constants  $\alpha, \beta, \gamma, \delta$ , then

$$\begin{aligned} 0 &< -\frac{(\gamma - \beta)^2 \gamma \alpha}{12(\alpha + \gamma)} D'''(w, \mu)(\delta) \\ &\leq \frac{D(w, \mu)(A) + D(w, \mu)(B)}{2} - D(w, \mu)\left(\frac{A + B}{2}\right) \\ &\leq \frac{(\delta - \alpha)^2}{2(\beta + \delta)} \left[ -D'(w, \mu)(\alpha) + \left(\frac{\delta + \beta}{2} - \alpha\right) D''(w, \mu)(\alpha) - \frac{1}{6}(\beta - \alpha)(\delta - \alpha) D'''(w, \mu)(\alpha) \right]. \quad (3.14) \end{aligned}$$

We have:

**COROLLARY 3.6.** Assume that  $f : [0, \infty) \rightarrow \mathbb{R}$  is an operator monotone function on  $[0, \infty)$  with  $f(0) = 0, 0 < \alpha \leq A, 0 < \gamma \leq B \leq \delta$  and  $0 < m \leq B - A$  for some constants  $\alpha, \gamma, \delta, m$ . Then we have the refinement of Jensen's inequality

$$\begin{aligned} 0 &< -\frac{m^2 \gamma \alpha}{12(\alpha + \gamma)} \left[ \frac{f'''(\delta)\delta^3 - 3f''(\delta)\delta^2 + 6f'(\delta)\delta - 6f(\delta)}{\delta^4} \right] \\ &\leq \frac{D(w, \mu)(A) + D(w, \mu)(B)}{2} - D(w, \mu)\left(\frac{A + B}{2}\right). \quad (3.15) \end{aligned}$$

**Proof.** From (3.4) for  $f(0) = 0$  we have

$$\begin{aligned} D'(\ell, \mu)(t) &= \frac{f'(t)t - f(t)}{t^2}, \\ D''(\ell, \mu)(t) &= \frac{f''(t)t^2 - 2f'(t)t + 2f(t)}{t^3} \end{aligned}$$

and

$$D'''(\ell, \mu)(t) = \frac{f'''(t)t^3 - 3f''(t)t^2 + 6f'(t)t - 6f(t)}{t^4}.$$

Employing the first part of (3.14) we derive (3.15). □

### 4. SOME EXAMPLES

By employing the first inequality in Theorem 3.4, we derive (3.15). If  $g(t) = t^{r-1}$  for  $t > 0, r \in (0, 1)$ , then

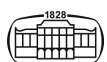
$$g'(t) = (r - 1)t^{r-2}, \quad g''(t) = (r - 1)(r - 2)t^{r-3},$$

and

$$g'''(t) = (r - 1)(r - 2)(r - 3)t^{r-4}.$$

From (1.6) we get

$$D(w_r)(t) = \frac{\pi}{\sin(r\pi)} t^{r-1}, \quad t > 0.$$



Then by (3.7) we get

$$\begin{aligned} 0 &< \frac{(1-r)(2-r)(3-r)m^2\gamma\alpha}{12(\alpha+\gamma)\delta^{4-r}} \\ &\leq \frac{A^{r-1} + B^{r-1}}{2} - \left(\frac{A+B}{2}\right)^{r-1} \\ &\leq \frac{M^2}{2(\beta+\delta)\alpha^{4-r}} \left[ (1-r)\alpha^2 + \left(\frac{\delta+\beta}{2} - \alpha\right) \alpha(1-r)(2-r) + \frac{1}{6}(\beta-\alpha)(\delta-\alpha)(1-r)(2-r)(3-r) \right] \end{aligned} \quad (4.1)$$

provided that  $0 < \alpha \leq A \leq \beta$ ,  $0 < \gamma \leq B \leq \delta$  and  $0 < m \leq B - A \leq M$  for some constants  $\alpha, \beta, \gamma, \delta, m, M$ .

If we take  $r \rightarrow 0+$  in (4.1), then we get

$$0 < \frac{m^2\gamma\alpha}{2(\alpha+\gamma)\delta^4} \leq \frac{A^{-1} + B^{-1}}{2} - \left(\frac{A+B}{2}\right)^{-1} \leq \frac{M^2\delta\beta}{2(\beta+\delta)\alpha^4} \quad (4.2)$$

which is the same as (2.11).

If  $0 < \alpha \leq A \leq \beta < \gamma \leq B \leq \delta$  for some constants  $\alpha, \beta, \gamma, \delta$ , then

$$\begin{aligned} 0 &< \frac{(1-r)(2-r)(3-r)(\gamma-\beta)^2\gamma\alpha}{12(\alpha+\gamma)\delta^{4-r}} \\ &\leq \frac{A^{r-1} + B^{r-1}}{2} - \left(\frac{A+B}{2}\right)^{r-1} \\ &\leq \frac{(\delta-\alpha)^2}{2(\beta+\delta)\alpha^{4-r}} \left[ (1-r)\alpha^2 + \left(\frac{\delta+\beta}{2} - \alpha\right) \alpha(1-r)(2-r) + \frac{1}{6}(\beta-\alpha)(\delta-\alpha)(1-r)(2-r)(3-r) \right], \end{aligned} \quad (4.3)$$

where  $r \in (0, 1)$ .

If we take  $r \rightarrow 0+$  in (4.3), then we get, see also (2.12),

$$0 < \frac{(\gamma-\beta)^2\gamma\alpha}{2(\alpha+\gamma)\delta^4} \leq \frac{A^{-1} + B^{-1}}{2} - \left(\frac{A+B}{2}\right)^{-1} \leq \frac{(\delta-\alpha)^2\delta\beta}{2(\beta+\delta)\alpha^4}. \quad (4.4)$$

We define the *upper incomplete Gamma function* as [12]

$$\Gamma(a, z) := \int_z^\infty t^{a-1} e^{-t} dt,$$

which for  $z = 0$  gives *Gamma function*

$$\Gamma(a) := \int_0^\infty t^{a-1} e^{-t} dt \text{ for } \operatorname{Re} a > 0.$$

We have the integral representation [13]

$$\Gamma(a, z) = \frac{z^a e^{-z}}{\Gamma(1-a)} \int_0^\infty \frac{t^{-a} e^{-t}}{z+t} dt \quad (4.5)$$

for  $\operatorname{Re} a < 1$  and  $|\operatorname{ph} z| < \pi$ .

Now, we consider the weight  $w_{-a_e^-}(\lambda) := \lambda^{-a} e^{-\lambda}$  for  $\lambda > 0$ . Then by (4.5) we have

$$\mathcal{D}(w_{-a_e^-})(t) = \int_0^\infty \frac{\lambda^{-a} e^{-\lambda}}{t+\lambda} d\lambda = \Gamma(1-a) t^{-a} e^t \Gamma(a, t) \quad (4.6)$$

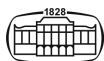
for  $a < 1$  and  $t > 0$ .

For  $a = 0$  in (4.6) we get

$$\mathcal{D}(w_{e^-})(t) = \int_0^\infty \frac{e^{-\lambda}}{t+\lambda} d\lambda = \Gamma(1) e^t \Gamma(0, t) = e^t E_1(t) \quad (4.7)$$

for  $t > 0$ , where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du. \quad (4.8)$$



Let  $a = 1 - n$ , with  $n$  a natural number with  $n \geq 0$ , then by (4.6) we have

$$\begin{aligned} \mathcal{D}(w_{n-1}e^{-}) (t) &= \int_0^\infty \frac{\lambda^{n-1}e^{-\lambda}}{t + \lambda} d\lambda = \Gamma(n)t^{n-1}e^t\Gamma(1 - n, t) \\ &= (n - 1)!t^{n-1}e^t\Gamma(1 - n, t). \end{aligned} \tag{4.9}$$

If we define the generalized exponential integral [14] by

$$E_p(z) := z^{p-1}\Gamma(1 - p, z) = z^{p-1} \int_z^\infty \frac{e^{-t}}{t^p} dt$$

then

$$t^{n-1}\Gamma(1 - n, t) = E_n(t)$$

for  $n \geq 1$  and  $t > 0$ .

Using the identity [14, Eq 8.19.7], for  $n \geq 2$

$$E_n(z) = \frac{(-z)^{n-1}}{(n - 1)!}E_1(z) + \frac{e^{-z}}{(n - 1)!} \sum_{k=0}^{n-2} (n - k - 2)!(-z)^k,$$

we get

$$\begin{aligned} \mathcal{D}(w_{n-1}e^{-}) (t) &= (n - 1)!e^tE_n(t) \\ &= (n - 1)!e^t \left[ \frac{(-t)^{n-1}}{(n - 1)!}E_1(t) + \frac{e^{-t}}{(n - 1)!} \sum_{k=0}^{n-2} (n - k - 2)!(-t)^k \right] \\ &= \sum_{k=0}^{n-2} (-1)^k (n - k - 2)!t^k + (-1)^{n-1}t^{n-1}e^tE_1(t) \end{aligned} \tag{4.10}$$

for  $n \geq 2$  and  $t > 0$ .

For  $n = 2$ , we also get

$$\mathcal{D}(w_{e^{-}}) (t) = \int_0^\infty \lambda e^{-\lambda} (t + \lambda)^{-1} d\lambda = 1 - t \exp(t)E_1(t) \tag{4.11}$$

for  $t > 0$ .

**PROPOSITION 4.1.** For all  $a < 1$ , the function  $t^{-a}e^t\Gamma(a, t)$  is operator convex on  $(0, \infty)$ . In particular,  $e^tE_n(t)$  is operator convex on  $(0, \infty)$ . As a consequence  $e^tE_1(t)$  is operator convex and  $te^tE_1(t)$  is operator concave on  $(0, \infty)$ .

We can also consider the weight  $w_{(t^2+a^2)^{-1}}(\lambda) := \frac{1}{\lambda^2+a^2}$  for  $\lambda > 0$  and  $a > 0$ . Then, by simple calculations, we get

$$\begin{aligned} \mathcal{D}(w_{(t^2+a^2)^{-1}}) (t) &:= \int_0^\infty \frac{1}{(\lambda + t)(\lambda^2 + a^2)} d\lambda \\ &= \frac{1}{(t^2 + a^2)} \left( \frac{\pi t}{2a} - \ln t + \ln a \right) \end{aligned}$$

for  $t > 0$  and  $a > 0$ .

For  $a = 1$  we also have

$$\begin{aligned} \mathcal{D}(w_{(t^2+1)^{-1}}) (t) &:= \int_0^\infty \frac{1}{(\lambda + t)(\lambda^2 + 1)} d\lambda \\ &= \frac{1}{t^2 + 1} \left( \frac{\pi}{2}t - \ln t \right) \end{aligned}$$

for  $t > 0$ .

**PROPOSITION 4.2.** For all  $a > 0$ , the functions

$$\frac{1}{(t^2 + a^2)} \left( \frac{\pi t}{2a} - \ln t + \ln a \right)$$



are operator convex on  $(0, \infty)$ . In particular,

$$\frac{1}{t^2 + 1} \left( \frac{\pi}{2} t - \ln t \right)$$

is operator convex on  $(0, \infty)$ .

The interested reader may state other similar results by employing the examples of monotone operator functions provided in [3], [4], [5], [10] and [11].

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