Mathematica Pannonica ***New Series*** **29 /NS 3/ (2023) 2, 188–194**
DOI: 10.1556/314.2023.00018

AKADÉMIAI KIADÓ

A NOTE ON THE PINELIS EXTENSION OF STOLARSKY'S INEQUALITY

Sanja Varošanec[1,*]

[1] Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička 30, Zagreb, Croatia

Communicated by László Tóth

Original Research Paper
Received: May 24, 2023 • Accepted: Aug 22, 2023
First published online: Sep 12, 2023

ABSTRACT

We present generalizations of the Pinelis extension of Stolarsky's inequality and its reverse. In particular, a new Stolarsky-type inequality is obtained. We study the properties of the linear functional related to the new Stolarsky-type inequality, and finally apply these new results in the theory of fractional integrals.

KEYWORDS

Stolarsky inequality, monotone function, superadditivity, fractional integral

MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 26D10; Secondary 26D15, 26A33

1. PRELIMINARIES

This paper is motivated by some results related to the Stolarsky inequality. Let us recall that in 1991, K. B. Stolarsky proved the following inequality, ([9]):

If a function $f:\ [0,1]\to[0,1]$ is decreasing, then for any $a,b>0$:

$$\int_0^1 f\left(x^{\frac{1}{a+b}}\right)dx \geq \int_0^1 f\left(x^{\frac{1}{a}}\right)dx \int_0^1 f\left(x^{\frac{1}{b}}\right)dx. \tag{1.1}$$

In its original form, this inequality can be considered as the inequality for the moments of a probability function supported on $[0,1]$. In this sense, generalizations were made in papers [5] and [6], where the reverse version of (1.1) was obtained and extension of (1.1) was given according to a new, different method of proof. The reverse version of the Stolarsky inequality reads: if $f:\ [0,1]\to\mathbb{R}$ is a non-negative increasing function, then

$$f(0)\int_0^1 f\left(x^{\frac{1}{a+b}}\right)dx \leq \int_0^1 f\left(x^{\frac{1}{a}}\right)dx \int_0^1 f\left(x^{\frac{1}{b}}\right)dx.$$

Results where the power functions are replaced by a more general functions are studied in [1, 4, 7, 10, 11] while results in which several functions appear instead of a one function f are discussed in [8]. Also, inequality (1.1) has been studied in other settings, see [2].

* Corresponding author. E-mail: varosans@math.hr

The main goal of this article is to provide a generalization of (1.1) that will allow us to obtain different inequalities not only for probability moments, but also for different fractional integrals.

Let us pay attention to Pinelis' result from [8] which reads:

THEOREM A. Let α, β, γ be non-negative numbers.

(i) If f and g are non-negative, decreasing functions on $[0, 1]$ such that they are left-continuous on $(0, 1]$ and right-continuous at 0, then

$$\begin{aligned} &Q(f,\gamma)Q(g,\alpha+\beta+\gamma)+Q(g,\gamma)Q(f,\alpha+\beta+\gamma) \\ &\quad \geq Q(f,\alpha+\gamma)Q(g,\beta+\gamma)+Q(g,\alpha+\gamma)Q(f,\beta+\gamma), \end{aligned} \tag{1.2}$$

where

$$Q(f,\alpha) := \int_0^1 f\left(x^{\frac{1}{\alpha}}\right)dx = \alpha \int_0^1 x^{\alpha-1} f(x)\,dx. \tag{1.3}$$

(ii) If f and g are non-negative, increasing functions on $[0, 1]$ such that they are right-continuous on $[0, 1)$ and left-continuous at 1, then the reverse sign in (1.2) holds.

The expression $x^{\alpha-1}$ in formula (1.3) can be considered as a particular case of the expression $g^{\alpha-1}(x)$, where g is a monotone function. Guided by this idea, we define the expression $Q_g(f, \alpha)$ as follows:

$$Q_g(f,\alpha) := \alpha \int_a^b g^{\alpha-1}(x) f(x)\,dx, \quad \alpha > 0.$$

In this paper, we develop Pinelis' idea and study inequalities involving expressions $Q_{g_i}(f, \alpha)$, where g_i, $i = 1, 2$ are monotone functions. That is the content of the second section. In the same section, we obtain the extended Stolarsky inequality and its reverse. The third section is devoted to a positive linear functional associated with the extended Stolarsky inequality. In the final section, we give an application of these new results in the theory of fractional integrals.

2. EXTENSIONS OF THE PINELIS INEQUALITIES INVOLVING MONOTONE FUNCTIONS

The following results have a very similar form to the results of I. Pinelis from [8], but here we include monotone functions under the sign of power. This is why we call these results the extended Pinelis inequalities.

THEOREM 2.1. Let α, β, γ be positive real numbers. Let g_1, g_2, f_1, f_2 be non-negative functions on $[a, b]$ such that g_1 and g_2 have a continuous derivative on $[a, b]$ and $\frac{f_i}{g_i'}$, $i = 1, 2$ are decreasing on $[a, b]$.

If g_1 and g_2 are strictly decreasing with $g_1(b) = g_2(b) = 0$, or if g_1 and g_2 are strictly increasing with $g_1(a) = g_2(a) = 0$, then

$$\begin{aligned} &Q_{g_1}(f_1,\gamma)Q_{g_2}(f_2,\alpha+\beta+\gamma)+Q_{g_1}(f_1,\alpha+\beta+\gamma)Q_{g_2}(f_2,\gamma) \\ &\quad \geq Q_{g_1}(f_1,\alpha+\gamma)Q_{g_2}(f_2,\beta+\gamma)+Q_{g_1}(f_1,\beta+\gamma)Q_{g_2}(f_2,\alpha+\gamma). \end{aligned} \tag{2.1}$$

Proof. Let us suppose that g_1 and g_2 are strictly decreasing with $g_1(b) = g_2(b) = 0$. Using integration by parts and abbreviations:

$$\varphi_i(t) := \frac{f_i(t)}{-g'(t)}, \quad G_i := g_i(a), \quad F_i := \varphi_i(a),$$

we get the following:

$$\begin{aligned} Q_{g_i}(f_i,\alpha) &= -\int_a^b \left(g_i^{\alpha}(t)\right)' \frac{f_i(t)}{-g_i'(t)}\,dt = g_i^{\alpha}(a)\varphi_i(a) + \int_a^b g_i^{\alpha}(t)d\varphi_i(t) \\ &= G_i^{\alpha}F_i + \int_a^b g_i^{\alpha}(t)d\varphi_i(t), \qquad i = 1, 2. \end{aligned} \tag{2.2}$$

We emphasize that $G_i > 0$, $F_i \geq 0$ and φ_i is increasing, $i = 1,2$. Let us simplify the difference D between the left-hand side and the right-hand side of (2.1) using (2.2):

$$\begin{aligned}
D &= \left(G_1^{\gamma}F_1 + \int_a^b g_1^{\gamma}(t)d\varphi_1(t)\right)\left(G_2^{\alpha+\beta+\gamma}F_2 + \int_a^b g_2^{\alpha+\beta+\gamma}(t)d\varphi_2(t)\right)\\
&\quad + \left(G_1^{\alpha+\beta+\gamma}F_1 + \int_a^b g_1^{\alpha+\beta+\gamma}(t)d\varphi_1(t)\right)\left(G_2^{\gamma}F_2 + \int_a^b g_2^{\gamma}(t)d\varphi_2(t)\right)\\
&\quad - \left(G_1^{\alpha+\gamma}F_1 + \int_a^b g_1^{\alpha+\gamma}(t)d\varphi_1(t)\right)\left(G_2^{\beta+\gamma}F_2 + \int_a^b g_2^{\beta+\gamma}(t)d\varphi_2(t)\right)\\
&\quad - \left(G_1^{\beta+\gamma}F_1 + \int_a^b g_1^{\beta+\gamma}(t)d\varphi_1(t)\right)\left(G_2^{\alpha+\gamma}F_2 + \int_a^b g_2^{\alpha+\gamma}(t)d\varphi_2(t)\right)\\
&= G_1^{\gamma}G_2^{\gamma}F_1F_2(G_2^{\alpha}-G_1^{\alpha})(G_2^{\beta}-G_1^{\beta}) + G_1^{\gamma}F_1\int_a^b g_2^{\gamma}(t)\big(g_2^{\alpha}(t)-G_1^{\alpha}\big)\left(g_2^{\beta}(t)-G_1^{\beta}\right)d\varphi_2(t)\\
&\quad + G_2^{\gamma}F_2\int_a^b g_1^{\gamma}(t)\big(g_1^{\alpha}(t)-G_2^{\alpha}\big)\left(g_1^{\beta}(t)-G_2^{\beta}\right)d\varphi_1(t)\\
&\quad + \int_a^b\int_a^b g_1^{\gamma}(t)g_2^{\gamma}(s)\left(g_2^{\alpha}(s)-g_1^{\alpha}(t)\right)\left(g_2^{\beta}(s)-g_1^{\beta}(t)\right)d\varphi_1(t)\,d\varphi_2(s)\\
&\geq 0
\end{aligned}$$

and the proof for this case is complete. If g_1, g_2 are strictly increasing, then the proof is done in a similar way. □

THEOREM 2.2. Let α, β, γ be positive real numbers. Let g_1, g_2, f_1, f_2 be non-negative functions on $[a,b]$ such that g_1 and g_2 have a continuous derivative on $[a,b]$ and $\frac{f_i}{g_i'}$, $i = 1,2$ are increasing on $[a,b]$.

If g_1 and g_2 are:

(i) strictly decreasing with $g_1(b) = g_2(b) = 0$, $g_1(a) = g_2(a)$, or
(ii) strictly increasing with $g_1(a) = g_2(a) = 0$, $g_1(b) = g_2(b)$,

then

$$\begin{aligned}
&Q_{g_1}(f_1,\gamma)Q_{g_2}(f_2,\alpha+\beta+\gamma) + Q_{g_1}(f_1,\alpha+\beta+\gamma)Q_{g_2}(f_2,\gamma)\\
&\qquad \leq Q_{g_1}(f_1,\alpha+\gamma)Q_{g_2}(f_2,\beta+\gamma) + Q_{g_1}(f_1,\beta+\gamma)Q_{g_2}(f_2,\alpha+\gamma).
\end{aligned}$$

Proof. Without loss of generality, we assume that $\frac{f_i}{g_i}$ is not a constant function, $i = 1,2$. Let us suppose that g_1 and g_2 are strictly decreasing with $g_1(b) = g_2(b) = 0$, $g_1(a) = g_2(a)$. Let us denote:

$$\bar{\varphi}_i := \frac{f_i}{g_i'}, \quad G := g_i(a), \quad \bar{F}_i := \frac{\bar{\varphi}_i(a)}{\bar{\varphi}_i(a) - \bar{\varphi}_i(b)},$$

for $i = 1,2$. It is obvious that $\bar{\varphi}_i$ is increasing, G is a positive number, and $\bar{F}_1, \bar{F}_2$ are non-negative numbers. Using integration by parts, we get the following:

$$\begin{aligned}
Q_{g_i}(f_i,\alpha) &= \int_a^b \left(g_i^{\alpha}(t)\right)'\frac{f_i(t)}{g_i'(t)}\,dt = \int_a^b \left(g_i^{\alpha}(t)\right)'\bar{\varphi}(t)\,dt\\
&= -g_i^{\alpha}(a)\bar{\varphi}_i(a) - \int_a^b g_i^{\alpha}(t)\,d\bar{\varphi}_i(t) = -g_i^{\alpha}(a)\bar{\varphi}_i(a)\frac{\int_a^b d\bar{\varphi}_i(t)}{\int_a^b d\bar{\varphi}_i(t)} - \int_a^b g_i^{\alpha}(t)\,d\bar{\varphi}_i(t) \qquad (2.3)\\
&= \int_a^b \left(G^{\alpha}\bar{F}_i - g_i^{\alpha}(t)\right)d\bar{\varphi}_i(t), \qquad i = 1,2.
\end{aligned}$$

Like in the proof of the previous Theorem, we consider a difference D.

$$\begin{aligned} D = &\int_a^b \left(G^\gamma \bar{F}_1 - g_1^\gamma(t)\right) d\bar{\varphi}_1(t) \int_a^b \left(G^{\alpha+\beta+\gamma} \bar{F}_2 - g_2^{\alpha+\beta+\gamma}(s)\right) d\bar{\varphi}_2(s) \\ &+ \int_a^b \left(G^{\alpha+\beta+\gamma} \bar{F}_1 - g_1^{\alpha+\beta+\gamma}(t)\right) d\bar{\varphi}_1(t) \int_a^b \left(G^\gamma \bar{F}_2 - g_2^\gamma(s)\right) d\bar{\varphi}_2(s) \\ &- \int_a^b \left(G^{\alpha+\gamma} \bar{F}_1 - g_1^{\alpha+\gamma}(t)\right) d\bar{\varphi}_1(t) \int_a^b \left(G^{\beta+\gamma} \bar{F}_2 - g_2^{\beta+\gamma}(s)\right) d\bar{\varphi}_2(s) \\ &- \int_a^b \left(G^{\beta+\gamma} \bar{F}_1 - g_1^{\beta+\gamma}(t)\right) d\bar{\varphi}_1(t) \int_a^b \left(G^{\alpha+\gamma} \bar{F}_2 - g_2^{\alpha+\gamma}(s)\right) d\bar{\varphi}_2(s) \\ = &- \int_a^b \int_a^b G^\gamma \bar{F}_1 g_2^\gamma(s) \left(g_2^\beta(s) - G^\beta\right)\left(g_2^\alpha(s) - G^\alpha\right) d\bar{\varphi}_1(t) d\bar{\varphi}_2(s) \\ &- \int_a^b \int_a^b G^\gamma \bar{F}_2 g_1^\gamma(t) \left(g_1^\beta(t) - G^\beta\right)\left(g_1^\alpha(t) - G^\alpha\right) d\bar{\varphi}_1(t) d\bar{\varphi}_2(s) \\ &+ \int_a^b \int_a^b g_1^\gamma(t) g_2^\gamma(s) \left(g_2^\beta(s) - g_1^\beta(t)\right)\left(g_2^\alpha(s) - g_1^\alpha(t)\right) d\bar{\varphi}_1(t) d\bar{\varphi}_2(s). \end{aligned}$$

Since $Q_{g_i}(f_i, \gamma)$ is non-negative, then multiplying (2.3) with a non-negative expression $g_2^\gamma\left(g_2^\beta - G^\beta\right)\left(g_2^\alpha - G^\alpha\right)$ and integrating over $[a, b]$, we get:

$$\int_a^b \int_a^b \left(G^\gamma \bar{F}_1 - g_1^\gamma(t)\right) g_2^\gamma(s) \left(g_2^\beta(s) - G^\beta\right)\left(g_2^\alpha(s) - G^\alpha\right) d\bar{\varphi}_1(t) d\bar{\varphi}_2(s) \geq 0,$$

i.e.

$$\begin{aligned} &- \int_a^b \int_a^b g_1^\gamma(t) g_2^\gamma(s) \left(g_2^\beta(s) - G^\beta\right)\left(g_2^\alpha(s) - G^\alpha\right) d\bar{\varphi}_1(t) d\bar{\varphi}_2(s) \\ &\quad \geq - \int_a^b \int_a^b G^\gamma \bar{F}_1 g_2^\gamma(s) \left(g_2^\beta(s) - G^\beta\right)\left(g_2^\alpha(s) - G^\alpha\right) d\bar{\varphi}_1(t) d\bar{\varphi}_2(s). \end{aligned} \tag{2.4}$$

Similarly, we obtain

$$\begin{aligned} &- \int_a^b \int_a^b g_1^\gamma(t) g_2^\gamma(s) \left(g_1^\beta(t) - G^\beta\right)\left(g_1^\alpha(t) - G^\alpha\right) d\bar{\varphi}_1(t) d\bar{\varphi}_2(s) \\ &\quad \geq - \int_a^b \int_a^b G^\gamma \bar{F}_2 g_1^\gamma(t) \left(g_1^\beta(t) - G^\beta\right)\left(g_1^\alpha(t) - G^\alpha\right) d\bar{\varphi}_1(t) d\bar{\varphi}_2(s). \end{aligned} \tag{2.5}$$

Taking into account (2.4) and (2.5), we have:

$$\begin{aligned} D \leq &- \int_a^b \int_a^b g_1^\gamma(t) g_2^\gamma(s) \left(g_2^\beta(s) - G^\beta\right)\left(g_2^\alpha(s) - G^\alpha\right) d\bar{\varphi}_1(t) d\bar{\varphi}_2(s) \\ &- \int_a^b \int_a^b g_1^\gamma(t) g_2^\gamma(s) \left(g_1^\beta(t) - G^\beta\right)\left(g_1^\alpha(t) - G^\alpha\right) d\bar{\varphi}_1(t) d\bar{\varphi}_2(s) \\ &+ \int_a^b \int_a^b g_1^\gamma(t) g_2^\gamma(s) \left(g_2^\beta(s) - g_1^\beta(t)\right)\left(g_2^\alpha(s) - g_1^\alpha(t)\right) d\bar{\varphi}_1(t) d\bar{\varphi}_2(s) \\ = &- \int_a^b \int_a^b \left[\left(g_1^\beta(t) - G^\beta\right)\left(g_2^\alpha(s) - G^\alpha\right) + \left(g_1^\alpha(t) - G^\alpha\right)\left(g_2^\beta(t) - G^\beta\right)\right] \times g_1^\gamma(t) g_2^\gamma(s) d\bar{\varphi}_1(t) d\bar{\varphi}_2(s) \\ \leq &\ 0. \end{aligned}$$

The second case is proved similarly. □

REMARK 2.3. If $\frac{f}{g'}$ is finite on $[a, b]$, then we can redefined $Q_g(f, \alpha)$ for $\alpha = 0$ as follows:

$$Q_g(f, 0) := \begin{cases} -\frac{f(b)}{g'(b)}, & g \text{ is strictly decreasing on } [a, b],\ g(b) = 0 \\ \frac{f(a)}{g'(a)}, & g \text{ is strictly increasing on } [a, b],\ g(a) = 0. \end{cases}$$

With this addition to the definition of $Q_g(f, \alpha)$, it can be proved that Theorems 2.1 and 2.2 are valid for non-negative real numbers α, β and γ.

REMARK 2.4. Putting in Theorems 2.2 and 2.1 $g_1 = g_2 = id$, $[a, b] = [0, 1]$, we get Pinelis' results i.e. Theorem A.

As a consequence of previously proven theorems we get inequalities related to the Stolarsky inequality and its reverse version. These new inequalities have a very similar form to the classical Stolarsky inequality and its reverse, but now a monotone function g has appeared in new inequalities. We call them the extended Stolarsky inequalities.

THEOREM 2.5. Let α, β, γ be non-negative real numbers. Let g and f be non-negative functions on $[a, b]$ such that g has a continuous derivative on $[a, b]$, and g is strictly decreasing with $g(b) = 0$ or g is strictly increasing with $g(a) = 0$.

(i) If $\frac{f}{g'}$ is decreasing on $[a, b]$, then

$$Q_g(f, \gamma)Q_g(f, \alpha + \beta + \gamma) \geq Q_g(f, \alpha + \gamma)Q_g(f, \beta + \gamma). \tag{2.6}$$

(ii) If $\frac{f}{g'}$ is increasing on $[a, b]$, then the reversed (2.6) holds.

Proof. Putting in Theorem 2.1 and Theorem 2.2 $g_1 = g_2 = g$ and $f_1 = f_2 = f$, we get the statement of this Theorem. □

REMARK 2.6. It is obvious that for $[a, b] = [0, 1]$, $g = id$, $\gamma = 0$, inequality (2.6) becomes the classical Stolarsky inequality.

Inequality (2.6) and its reverse version in which increasing function g appears are obtained in [10, Thm 5] as a consequence of the log-convexity or log-concavity of function $\alpha \mapsto Q_g(f, \alpha + 1)$.

3. A FUNCTIONAL RELATED TO THE EXTENDED STOLARSKY INEQUALITY

In this section we fix interval $[a, b]$, non-negative numbers α, β, γ, and a strictly monotone function $g : [a, b] \to [0, \infty)$ with the properties: g has a continuous derivative on $[a, b]$ and if g is strictly decreasing, then $g(b) = 0$, and if g is strictly increasing, then $g(a) = 0$. Let us define sets $E_\downarrow$ and $E_\uparrow$ on the following way:

$$E_\downarrow := \left\{ f : [a, b] \to [0, \infty) : \frac{f}{g'} \text{ is decreasing on } [a, b] \right\}$$
$$E_\uparrow := \left\{ f : [a, b] \to [0, \infty) : \frac{f}{g'} \text{ is increasing on } [a, b] \right\}.$$

We define a functional R on sets $E_\downarrow$ and $E_\uparrow$:

$$R(f) := Q_g(f, \gamma)Q_g(f, \alpha + \beta + \gamma) - Q_g(f, \alpha + \gamma)Q_g(f, \beta + \gamma).$$

Since $Q_g(tf, \alpha) = tQ_g(f, \alpha)$ for $t \geq 0$, R is positive homogeneous of order 2. Furthermore, from Theorem 2.5, we conclude that R is non-negative on $E_\downarrow$ and non-positive on $E_\uparrow$.

More properties of a functional R are given in the following theorem. As we will see, the extended Pinelis results given in Theorems 2.1 and 2.2 will play a crucial role in the proof of the following theorem.

THEOREM 3.1. (i) Functional R is superadditive on $E_\downarrow$ and subadditive on $E_\uparrow$.

(ii) If f_1 and f_2 are functions from $E_\downarrow$ such that there exist numbers m and M such that $0 \leq m \leq M$, $Mf_2 - f_1, f_1 - mf_2 \in E_\downarrow$, then

$$m^2 R(f_2) \leq R(f_1) \leq M^2 R(f_2). \tag{3.1}$$

Furthermore, if $f_2 \geq f_1$ with $f_2 - f_1 \in E_\downarrow$, then

$$R(f_1) \leq R(f_2). \tag{3.2}$$

(iii) If f_1 and f_2 belong to $E_\uparrow$ such that there exist numbers m and M such that $0 \le m \le M$, $Mf_2 - f_1, f_1 - mf_2 \in E_\uparrow$, then the reverse signs in (3.1) are valid. Furthermore, if $f_2 \ge f_1$ with $f_2 - f_1 \in E_\uparrow$, then the reverse sign in (3.2) holds.

Proof. We prove the properties of R on $E_\downarrow$ in detail, while leaving the second case to the reader.

(i) We have to prove that for $f_1, f_2 \in E_\downarrow$ the following inequality holds:

$$R(f_1 + f_2) \ge R(f_1) + R(f_2).$$

Since $Q_g(f_1 + f_2) = Q_g(f_1) + Q_g(f_2)$, we get:

$$\begin{aligned} R(f_1+f_2) - R(f_1) - R(f_2) &= Q_g(f_1+f_2,\gamma)Q_g(f_1+f_2,\alpha+\beta+\gamma) - Q_g(f_1+f_2,\alpha+\gamma)Q_g(f_1+f_2,\beta+\gamma) \\ &\quad - Q_g(f_1,\gamma)Q_g(f_1,\alpha+\beta+\gamma) + Q_g(f_1,\alpha+\gamma)Q_g(f_1,\beta+\gamma) \\ &\quad - Q_g(f_2,\gamma)Q_g(f_2,\alpha+\beta+\gamma) + Q_g(f_2,\alpha+\gamma)Q_g(f_2,\beta+\gamma) \\ &= Q_g(f_1,\gamma)Q_g(f_2,\alpha+\beta+\gamma) + Q_g(f_2,\gamma)Q_g(f_1,\alpha+\beta+\gamma) \\ &\quad - Q_g(f_1,\alpha+\gamma)Q_g(f_2,\beta+\gamma) - Q_g(f_2,\alpha+\gamma)Q_g(f_1,\beta+\gamma) \\ &\ge 0, \end{aligned}$$

where in the last line we use inequality (2.2). So, we prove the superadditivity of R.

(ii) Using homogenity and superadditivity of R, and since $R(Mf_2 - f_1) \ge 0$, we get:

$$M^2R(f_2) = R(Mf_2) = R((Mf_2 - f_1) + f_1) \ge R(Mf_2 - f_1) + R(f_1) \ge R(f_1).$$

So, the upper boundedness is proved. The second inequality in (3.1) is proved in a similar way. If $f_2 \ge f_1$, then $M = 1$ and (3.2) follows from (3.1). □

4. APPLICATIONS

In this section we want to show how the results of the previous text can be applied to fractional integrals. The classical Stolarsky inequality, its reverse and the Pinelis results give some useful information about the moments of the probability function. The above-proved extended Pinelis and extended Stolarsky inequalities can be used in calculus with fractional integrals. Let us mention the so-called Katugampola fractional integral which is defined in [3].

DEFINITION 4.1. Let $\alpha > 0$, $\rho \ne -1$ and f be integrable on $[a,b]$, $a > 0$. The Katugampola fractional integral of order α is defined by:

$${}^{\rho}_{a}I^{\alpha}_{x} f(x) := \frac{(\rho+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x^{\rho+1} - t^{\rho+1})^{\alpha-1} t^{\rho} f(t)\,dt, \quad x > a.$$

For $\rho = 0$, the Katugampola fractional integral becomes the classical Riemann-Liouville fractional integral, while if $\rho \to -1$, then it tends to the Hadamard fractional integral, see [3].

Considering the Katugampola fractional integral, using notations: $g(t) := \frac{1}{\rho+1}(x^{\rho+1} - t^{\rho+1})$ and $h(t) := -g'(t)f(t)$, we get:

$$Q_g(h,\alpha) = \Gamma(\alpha+1)\, {}^{\rho}_{a}I^{\alpha}_{x} f(x).$$

The above-defined function g is a strictly decreasing function on $[a,x]$ which satisfies the assumptions of Theorems 2.1 and 2.2. After simple calculations, the extended Pinelis inequality for the Katugampola fractional integrals collapses into the following inequality:

$$\begin{aligned} &{}^{\rho}_{a}I^{\gamma}_{x} f_1(x)\, {}^{\rho}_{a}I^{\alpha+\beta+\gamma}_{x} f_2(x) + {}^{\rho}_{a}I^{\alpha+\beta+\gamma}_{x} f_1(x)\, {}^{\rho}_{a}I^{\gamma}_{x} f_2(x) \\ &\quad \ge \frac{\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)} \left({}^{\rho}_{a}I^{\alpha+\gamma}_{x} f_1(x)\, {}^{\rho}_{a}I^{\beta+\gamma}_{x} f_2(x) + {}^{\rho}_{a}I^{\beta+\gamma}_{x} f_1(x)\, {}^{\rho}_{a}I^{\alpha+\gamma}_{x} f_2(x) \right), \quad x > a, \end{aligned} \tag{4.1}$$

where functions f_1 and f_2 are increasing. If f_1 and f_2 are decreasing, then by Theorem 2.2, the reverse in the above inequality holds. Similarly, we get the inequality of Stolarsky-type using a particular case: $f_1 = f_2 = f$. Namely, we get the following Stolarsky-type inequality:

$${}^{\rho}_{a}I^{\gamma}_{x} f(x)\, {}^{\rho}_{a}I^{\alpha+\beta+\gamma}_{x} f(x) \ge \frac{\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)}\, {}^{\rho}_{a}I^{\alpha+\gamma}_{x} f(x)\, {}^{\rho}_{a}I^{\beta+\gamma}_{x} f(x),$$

where f is increasing, α, β, γ are positive numbers. The corresponding functional R defined as

$$R(f) = {}^{\rho}_{a}I^{\gamma}_{x} f(x)\, {}^{\rho}_{a}I^{\alpha+\beta+\gamma}_{x} f(x) - \frac{\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)}\, {}^{\rho}_{a}I^{\alpha+\gamma}_{x} f(x)\, {}^{\rho}_{a}I^{\beta+\gamma}_{x} f(x),$$

is superadditive on the set of increasing non-negative functions on $[a, b]$, homogeneous of order 2, and satisfies the boundedness and monotone properties given in Theorem 3.1.

Obviously, these results can be extended to other classes of fractional integrals and we leave that to the interested reader.

REFERENCES

[1] Abramovich, S. Pečarić, J., Varošanec, S. Further extension of Stolarsky's inequality with general weights. *Math. Pannon.* 10 (1999), 249–256.

[2] Daraby, B. Investigation of a Stolarsky type inequality for integrals in pseudo-analysis. *Fract. Calc. Appl. Anal.*, 13(5) (2010), 467–473.

[3] Katugampola, U. N. New approach to a generalized fractional integral. *Appl. Math. Comput.*, 218 (2011), 860–865.

[4] Maligranda, L., Pečarić, J., Persson, L. E. Stolarsky's inequality with general weights. *Proc. Amer. Math. Soc.* 123(7) (1995), 2113–2117.

[5] Pečarić, J. On Stolarsky's quotient. *Makedon. Akad. Nauk. Umet. Oddel. Mat.-Tehn. Nauk.* 14(2) (1993), 55–60.

[6] Pečarić, J. A reverse Stolarsky's inequality. *Amer. Math. Monthly,* 101(6) (1994), 566–568.

[7] Pečarić, J., Varošanec, S. Remarks on Gauss–Winckler's and Stolarsky's inequality. *Utilitas Math.* 48(1995), 233–241.

[8] Pinelis, I. Multilinear direct and reverse Stolarsky inequalities. *Math. Inequal. Appl.* 5(4) (2002), 671–691.

[9] Stolarsky, K. From Wythoff's Nim to Chebyshev's inequality. *Amer. Math. Monthly,* 98(10) (1991), 889–900.

[10] Varošanec, S., Pečarić, J. Gauss' and related inequalities. *Z. Anal. Anwend.* 14(1) (1995) 175–183.

[11] Varošanec, S., Pečarić, J., Šunde, J. Some discrete inequalities. *Z. Anal. Anwend.* 15(4) (1996), 1025–1032.