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THE NORMING SETS OF $\mathcal{L}(^2 d_*(1,w)^2)$

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ABSTRACT

Let $n \in \mathbb{N}$. An element $(x_1, \ldots, x_n) \in E^n$ is called a *norming point* of $T \in \mathcal{L}(^n E)$ if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1, \ldots, x_n)| = \|T\|$, where $\mathcal{L}(^n E)$ denotes the space of all continuous n-linear forms on E. For $T \in \mathcal{L}(^n E)$, we define

$$\text{Norm}(T) = \{(x_1, \ldots, x_n) \in E^n : (x_1, \ldots, x_n) \text{ is a norming point of } T\}.$$

Norm(T) is called the *norming set* of T. We classify Norm(T) for every $T \in \mathcal{L}(^2 d_*(1,w)^2)$, where $d_*(1,w)^2 = \mathbb{R}^2$ with the octagonal norm of weight $0 < w < 1$ endowed with $\|(x,y)\|_{d_*(1,w)} = \max\{|x|, |y|, \frac{|x|+|y|}{1+w}\}$.

KEYWORDS

Norming points, bilinear forms, the plane with the octagonal norm

MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 46A22; Secondary 46G25

1. INTRODUCTION

In 1961 Bishop and Phelps [2] showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, especially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon–Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon–Nikodym property is also sufficient for the denseness of norm attaining polynomials. Jiménez-Sevilla and Payá [5] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

Let $n \in \mathbb{N}$, $n \geq 2$. We write S_E for the unit sphere of a Banach space E. We denote by $\mathcal{L}(^n E)$ the Banach space of all continuous n-linear forms on E endowed with the norm

$$\|T\| = \sup_{(x_1,\ldots,x_n) \in S_E \times \cdots \times S_E} |T(x_1, \ldots, x_n)|.$$

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$\mathcal{L}_s(^nE)$ denote the closed subspace of all continuous symmetric n-linear forms on E. An element $(x_1, \ldots, x_n) \in E^n$ is called a *norming point* of T if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1, \ldots, x_n)| = \|T\|$.

For $T \in \mathcal{L}(^nE)$, we define

$$\text{Norm}(T) = \{(x_1, \ldots, x_n) \in E^n : (x_1, \ldots, x_n) \text{ is a norming point of } T\}.$$

Norm(T) is called the *norming set* of T. Notice that $(x_1, \ldots, x_n) \in \text{Norm}(T)$ if and only if

$$(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \text{Norm}(T)$$

for some $\epsilon_k = \pm 1$ $(k = 1, \ldots, n)$. Indeed, if $(x_1, \ldots, x_n) \in \text{Norm}(T)$, then

$$|T(\epsilon_1 x_1, \ldots, \epsilon_n x_n)| = |\epsilon_1 \cdots \epsilon_n T(x_1, \ldots, x_n)| = |T(x_1, \ldots, x_n)| = \|T\|,$$

which shows that $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \text{Norm}(T)$. If $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \text{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1, \ldots, n)$, then

$$(x_1, \ldots, x_n) = (\epsilon_1(\epsilon_1 x_1), \ldots, \epsilon_n(\epsilon_n x_n)) \in \text{Norm}(T).$$

The following examples show that it is possible that Norm(T) be empty or an infinite set.

EXAMPLES . (a) Let

$$T((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) = \sum_{i=1}^{\infty} \frac{1}{2^i} x_i y_i \in \mathcal{L}_s(^2c_0).$$

We claim that Norm(T) $= \emptyset$. Obviously, $\|T\| = 1$. Assume that Norm(T) $\neq \emptyset$. Let $((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) \in \text{Norm}(T)$. Then,

$$1 = |T((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}})| \leq \sum_{i=1}^{\infty} \frac{1}{2^i} |x_i||y_i| \leq \sum_{i=1}^{\infty} \frac{1}{2^i} = 1,$$

which shows that $|x_i| = |y_i| = 1$ for all $i \in \mathbb{N}$. Hence, $(x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}} \notin c_0$. This is a contradiction. Therefore, Norm(T) $= \emptyset$.

(b) Let

$$T((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) = x_1 y_1 \in \mathcal{L}_s(^2c_0).$$

Then,

$$\text{Norm}(T) = \{((\pm 1, x_2, x_3, \ldots), (\pm 1, y_2, y_3, \ldots)) \in c_0 \times c_0 : |x_j| \leq 1, |y_j| \leq 1 \text{ for } j \geq 2\}.$$

A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form L on the product $E \times \cdots \times E$ such that $P(x) = L(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$.

An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^nE)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. For $P \in \mathcal{P}(^nE)$, we define

$$\text{Norm}(P) = \{x \in E : x \text{ is a norming point of } P\}.$$

Norm(P) is called the *norming set* of P. Notice that Norm(P) $= \emptyset$ or a finite set or an infinite set.

Kim [7] classified Norm(P) for every $P \in \mathcal{P}(^2l_\infty^2)$, where $l_\infty^2 = \mathbb{R}^2$ with the supremum norm.

If Norm(T) $\neq \emptyset$, $T \in \mathcal{L}(^nE)$ is called a *norm attaining* n-linear form and if Norm(P) $\neq \emptyset$, $P \in \mathcal{P}(^nE)$ is called a *norm attaining* n-homogeneous polynomial. (See [3].)

For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [4].

It seems to be natural and interesting to study Norm(T) for $T \in \mathcal{L}(^nE)$. For $m \in \mathbb{N}$, let $l_1^m := \mathbb{R}^m$ with the the l_1-norm and $l_\infty^2 = \mathbb{R}^2$ with the supremum norm. Notice that if $E = l_1^m$ or l_∞^2 and $T \in \mathcal{L}(^nE)$, Norm(T) $\neq \emptyset$ since S_E is compact. Kim [6, 8, 9] classified Norm(T) for every $T \in \mathcal{L}_s(^2l_\infty^2), \mathcal{L}(^2l_\infty^2), \mathcal{L}(^2l_1^2)$ or $\mathcal{L}_s(^3l_1^2)$. Recently, Kim [10] classified Norm(T) for every $T \in \mathcal{L}(^2\mathbb{R}^2_{h(w)})$, where $\mathbb{R}^2_{h(w)}$ denotes the plane with the hexagonal norm with weight $0 < w < 1$ $\|(x, y)\|_{h(w)} = \max\{|y|, |x| + (1 - w)|y|\}$.

Let $d_*(1, w)^2 = \mathbb{R}^2$ with the octagonal norm of weight $0 < w < 1$ endowed with $\|(x, y)\|_{d_*(1,w)} = \max\{|x|, |y|, \frac{|x|+|y|}{1+w}\}$.

In this paper, we classify Norm(T) for every $T \in \mathcal{L}(^2d_*(1, w)^2)$.

2. MAIN RESULTS

Throughout this paper we let $0 < w < 1$.

THEOREM A. Let $0 < w < 1$ and $T \in \mathcal{L}(^2d_*(1,w)^2)$ with $T((x_1,y_1),(x_2,y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1$ for some $a,b,c,d \in \mathbb{R}$. Then, there exists $T' = (a^*,b^*,c^*,d^*) \in \mathcal{L}(^2d_*(1,w)^2)$ such that $a^*,b^*,c^*,d^* \in \{\pm a, \pm b, \pm c, \pm d\}$ with $a^* \geq |b^*|$ and $a^* \geq c^* \geq d^* \geq 0$ and $\|T\| = \|T'\|$.

Proof. If $a < 0$, taking $-T$, we assume that $a \geq 0$. If $|b| > a$, we define $T_1 = (|b|, \text{sing}(b)a, \text{sing}(b)d, \text{sing}(b)c)$. Then, $\|T_1\| = \|T\|$. Hence, we may assume that $a \geq |b|$. If $c < 0$, we define $T_2 = (a,-b,-c,d)$. Then, $\|T_2\| = \|T\|$. Hence, we may assume that $a \geq |b|, c \geq 0$. If $d < 0$, we define $T_3 = (a,-b,c,-d)$. Then, $\|T_3\| = \|T\|$. Hence, we may assume that $a \geq |b|, c \geq 0, d \geq 0$. If $c < d$, we define $T_4 = (a,b,d,c)$. Then, $\|T_4\| = \|T\|$. Hence, we may assume that $a \geq |b|, c \geq d \geq 0$. If $a < c, b \geq 0$, we define $T_5 = (c,d,a,b)$. Then, $\|T_5\| = \|T\|$. Hence, we may assume that $a \geq |b|, c \geq d$. If $a < c, b < 0$, we define $T_6 = (c,-d,a,-b)$. Then, $\|T_6\| = \|T\|$. Hence, we may assume that $a \geq |b|$ and $a \geq c \geq d \geq 0$. Therefore, we can find a bilinear form T' which satisfies the conditions of the theorem. □

THEOREM B. Let $0 < w < 1$ and $T \in \mathcal{L}(^2d_*(1,w)^2)$ with $T((x_1,y_1),(x_2,y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1$ for some $a \geq |b|, a \geq c \geq d \geq 0$. Then

$$\begin{aligned}\|T\| &= \max\{a + bw^2 + (c+d)w, a - bw^2 + (c-d)w, (a+b)w + c + dw^2, (a-b)w + c - dw^2\} \\ &= \max\{a + cw + |bw+d|w, c + aw + |dw+b|w\}.\end{aligned}$$

Proof. Notice that the set of extreme points of $B_{d_*(1,w)^2}$ is $\{\pm(1,\pm w), \pm(w,\pm 1)\}$. By the Krein-Milman theorem, $B_{d_*(1,w)^2}$ is the closed convex hull of $\{\pm(1,\pm w), \pm(w,\pm 1)\}$. By bilinearity of T, it follows that

$$\begin{aligned}\|T\| &= \max\{|T(\pm(1,\pm w),\pm(1,\pm w))|, |T(\pm(w,\pm 1),\pm(w,\pm 1))|, \\ &\qquad |T(\pm(1,\pm w),\pm(w,\pm 1))|, |T(\pm(w,\pm 1),\pm(1,\pm w))|\} \\ &= \max\{a + bw^2 + (c+d)w, a - bw^2 + (c-d)w, (a+b)w + c + dw^2, (a-b)w + c - dw^2\} \\ &= \max\{a + cw + |bw+d|w, c + aw + |dw+b|w\}.\end{aligned}$$ □

Let $T((x_1,y_1),(x_2,y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}(^2d_*(1,w)^2)$ for some $a \geq |b|, a \geq c \geq d \geq 0$. For simplicity we denote $T = (a,b,c,d)$. By Theorem B, if $\|T\| = 1$, then $a \leq 1, |b| \leq 1, c \leq 1$ and $d \leq 1$.

LEMMA C. Let $0 < w < 1$ and $a \geq |b| = -b > 0, a \geq c \geq d \geq 0$. Then the following situations can not happen:

$$1 = a - bw^2 + (c-d)w = (a+b)w + c + dw^2 = (a-b)w + c - dw^2 > a + bw^2 + (c+d)w, \tag{2.1}$$

$$1 = a + bw^2 + (c+d)w = a - bw^2 + (c-d)w = (a+b)w + c + dw^2 > (a-b)w + c - dw^2, \tag{2.2}$$

$$1 = a + bw^2 + (c+d)w = (a+b)w + c + dw^2 = (a-b)w + c - dw^2 = a - bw^2 + (c-d)w. \tag{2.3}$$

Proof. Suppose not.

Suppose that (2.1) holds. Then $|b| = dw, d < |b|w$. Thus, $d < |b|w = dw^2 < d$, which is impossible.

Suppose that (2.2) holds. Then $|b|w = d, |b| < dw$. Thus, $|b| < dw^2 = |b|w^2 < |b|$, which is impossible.

Suppose that (2.3) holds. Then $|b| = dw, d = |b|w$. Thus, $d = |b|w = dw^2 < d$, which is impossible.

Therefore, this complete the proof. □

Let $\mathcal{W} \subseteq S_{d_*(1,w)^2} \times S_{d_*(1,w)^2}$. We denote

$$\text{Sym}(\mathcal{W}) := \{(X,Y),(Y,X) : (X,Y) \in \mathcal{W}\}.$$

Let

$$B_1 = \{t(0,1) + (1-t)(w,1) :\ 0 \le t \le 1\},$$
$$B_2 = \{t(w,1) + (1-t)(1,w) :\ 0 \le t \le 1\},$$
$$B_3 = \{t(1,w) + (1-t)(1,-w) :\ 0 \le t \le 1\},$$
$$B_4 = \{t(1,-w) + (1-t)(w,-1) :\ 0 \le t \le 1\},$$
$$B_5 = \{t(w,-1) + (1-t)(0,-1) :\ 0 \le t \le 1\}.$$

Let $A_{ij} = B_i \times B_j$ for $i, j = 1, \dots, 5$.

We let

$$\Omega = \{(0,1), (w,1), (1,w), (1,-w), (w,-1), (0,-1)\}.$$

THEOREM D. Let $0 < w < 1$ and $T \in \mathcal{L}({}^2 d_*(1,w)^2)$. Then

(1)

$$\mathrm{Norm}(T) \subseteq \bigcup_{\substack{(X,Y)\in \mathrm{Norm}(T)\cap(\Omega\times\Omega)\\ X\in B_i, Y\in B_j}} (A_{ij} \cup A_{ji}).$$

(2) Let $(X,Y) \in \mathrm{Norm}(T)$ be such that

$X = tW_1 + (1-t)W_2, Y = sW_1' + (1-s)W_2'$ for some $W_i, W_j' \in \Omega \quad (i, j = 1, 2, 0 \le t, s \le 1)$.

If there are W_{i_0}, W_{j_0}' such that $(W_{i_0}, W_{j_0}') \notin \mathrm{Norm}(T)$, then ($t = 0$ or 1) or ($s = 0$ or 1).

Proof. (1) Let $S := \{(x,y) \in S_{d_*(1,w)^2} :\ x \ge 0\}$ and

$$\mathcal{F} := \bigcup_{\substack{(X,Y)\in \mathrm{Norm}(T)\cap(\Omega\times\Omega)\\ X\in B_i, Y\in B_j}} (A_{ij} \cup A_{ji}).$$

It suffices to show that if $(U,V) \in (S \times S) \setminus \mathcal{F}$, then $(U,V) \notin \mathrm{Norm}(T)$.

Let $(U,V) \in (S \times S) \setminus \mathcal{F}$. Then there are $W_1, W_2, W_1', W_2' \in \Omega$ and $0 \le t, s \le 1$ such that

$$U = tW_1 + (1-t)W_2, V = sW_1' + (1-s)W_2'.$$

CASE 1. ($t = 0$ or 1) and ($s = 0$ or 1).

Thus $(U,V) = (W_{i_1}, W_{j_1}')$ for some $1 \le i_1, j_1 \le 2$. Thus $(U,V) \notin \mathrm{Norm}(T)$. Indeed, assume that $(U,V) \in \mathrm{Norm}(T)$. Let $W_{i_1} \in B_{i_0}, W_{j_1}' \in B_{j_0}$ for some $1 \le i_0, j_0 \le 4$. Thus $(U,V) = (W_{i_1}, W_{j_1}') \in \mathrm{Norm}(T) \cap (\Omega \times \Omega)$ and hence $(U,V) \in (A_{i_0 j_0} \cup A_{j_0 i_0}) \subseteq \mathcal{F}$. This is a contradiction.

CASE 2. ($0 < t < 1$) and ($s = 0$ or 1).

For each $j = 1, 2$, $W_j \in B_{i_0}$ for some $1 \le i_0 \le 4$ because $\|U\|_{d_*(1,w)} = 1$ and $V = W_1'$ or W_2'. Without loss of generality we may assume that $V = W_1'$. Let $W_1' \in B_{j_0}$ for some $1 \le j_0 \le 4$. We claim that $(W_j, W_1') \notin \mathrm{Norm}(T)$ for $j = 1, 2$. Suppose not. We may assume that $(W_1, W_1') \in \mathrm{Norm}(T)$. Thus $(W_1, W_1') \in \mathrm{Norm}(T) \cap (\Omega \times \Omega)$. Hence, $(U,V) \in (A_{i_0 j_0} \cup A_{j_0 i_0}) \subseteq \mathcal{F}$. This is a contradiction.

CLAIM . $(U,V) \notin \mathrm{Norm}(T)$.

Suppose not. It follows that

$$\|T\| = |T(U,V)| \le t|T(W_1, W_1')| + (1-t)|T(W_2, W_1')| < t\|T\| + (1-t)\|T\| = \|T\|,$$

which is a contradiction. Hence, the claim holds.

CASE 3. ($0 < s < 1$) and ($t = 0$ or 1).

For each $j = 1, 2$, $W_j' \in B_{i_0}$ for some $1 \le i_0 \le 4$ because $\|V\|_{d_*(1,w)} = 1$ and $U = W_1$ or W_2. Without loss of generality we may assume that $U = W_1$. Let $W_1 \in B_{j_0}$ for some $1 \le j_0 \le 4$. We claim that $(W_1, W_j') \notin \mathrm{Norm}(T)$ for $j = 1, 2$. Suppose not. We may assume that $(W_1, W_2') \in \mathrm{Norm}(T)$. Thus $(W_1, W_2') \in \mathrm{Norm}(T) \cap (\Omega \times \Omega)$. Hence, $(U,V) \in (A_{j_0 i_0} \cup A_{i_0 j_0}) \subseteq \mathcal{F}$. This is a contradiction.

By an analogous argument as in the Case 2, $(U,V) \notin \mathrm{Norm}(T)$.

CASE 4. ($0 < t < 1$) and ($0 < s < 1$).

For each $j = 1, 2$, $W_j \in B_{i_0}$ and $W'_j \in B_{j_0}$ for some $1 \le i_0, j_0 \le 4$ because $\|U\|_{d_*(1,w)} = \|V\|_{d_*(1,w)} = 1$. We claim that $(W_i, W'_j) \notin \text{Norm}(T)$ for $i, j = 1, 2$. Suppose not. We may assume that $(W_2, W'_2) \in \text{Norm}(T)$. Thus $(W_2, W'_2) \in \text{Norm}(T) \cap (\Omega \times \Omega)$. Hence, $(U, V) \in (A_{i_0 j_0} \cup A_{j_0 i_0}) \subseteq \mathcal{F}$. This is a contradiction.

We will show that $(U, V) \notin \text{Norm}(T)$. Suppose not.

$$\begin{aligned}\|T\| &= |T(U,V)|\\ &\le ts|T(W_1,W'_1)| + t(1-s)|T(W_1,W'_2)| + (1-t)s|T(W_2,W'_1)| + (1-t)(1-s)|T(W_2,W'_2)|\\ &< ts|T(W_1,W'_1)| + t(1-s)\|T\| + (1-t)s|T(W_2,W'_1)| + (1-t)(1-s)|T(W_2,W'_2)|\\ &\le (ts + t(1-s) + (1-t)s + (1-t)(1-s))\|T\| = \|T\|,\end{aligned}$$

which is a contradiction. This completes the proof of (1).

(2) Assume the contrary. Then $0 < t < 1$ and $0 < s < 1$. Without loss of generality we may assume that $i_0 = 1$ and $j_0 = 2$.

CLAIM . $t(1-s) = 0$.

Assume that $t(1-s) > 0$. Then

$$\begin{aligned}\|T\| &= |T(X,Y)|\\ &\le ts|T(W_1,W'_1)| + t(1-s)|T(W_1,W'_2)| + (1-t)s|T(W_2,W'_1)| + (1-t)(1-s)|T(W_2,W'_2)|\\ &< ts|T(W_1,W'_1)| + t(1-s)\|T\| + (1-t)s|T(W_2,W'_1)| + (1-t)(1-s)|T(W_2,W'_2)|\\ &\le (ts + t(1-s) + (1-t)s + (1-t)(1-s))\|T\| = \|T\|,\end{aligned}$$

which is a contradiction. Hence, the claim holds.

Since $0 < t < 1$ and $0 < s < 1$, $0 < t(1-s) = 0$, which is impossible. Therefore, this completes the proof of (2). □

LEMMA E. Suppose that $0 < w < 1$ and $T = (a, b, c, d) \in \mathcal{L}(^2 d_*(1, w)^2)$ be such that $a \ge |b|$, $a \ge c \ge d \ge 0$ and $\|T\| = 1$. Let $((x_1, y_1), (x_2, y_2)) \in \text{Norm}(T)$. Then the following statements hold: let $0 \le t, s \le 1$.

(1) If $((x_1, y_1), (x_2, y_2)) \in A_{22}$, then

$$\begin{aligned}|T((x_1,y_1),(x_2,y_2))| &= |T(t(w,1) + (1-t)(1,w), s(w,1) + (1-s)(1,w))|\\ &= ts|T((w,1),(w,1))| + t(1-s)|T((w,1),(1,w))| + s(1-t)|T((1,w),(w,1))|\\ &\quad + (1-t)(1-s)|T((1,w),(1,w))|\\ &= ts|aw^2 + b + (c+d)w| + t(1-s)((a+b)w + cw^2 + d)\\ &\quad + (1-t)s((a+b)w + c + dw^2) + (1-t)(1-s)(a + bw^2 + (c+d)w) = 1.\end{aligned}$$

(2) If $((x_1, y_1), (x_2, y_2)) \in A_{23}$, then

$$\begin{aligned}|T((x_1,y_1),(x_2,y_2))| &= |T(t(w,1) + (1-t)(1,w), s(1,w) + (1-s)(1,-w))|\\ &= ts|T((w,1),(1,w))| + t(1-s)|T((w,1),(1,-w))| + s(1-t)|T((1,w),(1,w))|\\ &\quad + (1-t)(1-s)|T((1,w),(1,-w))|\\ &= ts((a+b)w + cw^2 + d) + t(1-s)|(a-b)w - cw^2 + d|\\ &\quad + (1-t)s(a + bw^2 + (c+d)w) + (1-t)(1-s)|a - bw^2 - (c-d)w| = 1.\end{aligned}$$

(3) If $((x_1, y_1), (x_2, y_2)) \in A_{32}$, then

$$\begin{aligned}|T((x_1, y_1), (x_2, y_2))| &= |T(t(1, w) + (1 - t)(1, -w), s(w, 1) + (1 - s)(1, w))| \\ &= ts|T((1, w), (w, 1))| + t(1 - s)|T((1, w), (1, w))| + s(1 - t)|T((1, -w), (w, 1))| \\ &\quad + (1 - t)(1 - s)|T((1, -w), (1, w))| \\ &= ts((a + b)w + c + dw^2) + t(1 - s)(a + bw^2 + (c + d)w) \\ &\quad + (1 - t)s((a - b)w + c - dw^2) + (1 - t)(1 - s)(a - bw^2 + (c - d)w) = 1.\end{aligned}$$

(4) If $((x_1, y_1), (x_2, y_2)) \in A_{33}$, then

$$\begin{aligned}|T((x_1, y_1), (x_2, y_2))| &= |T(t(1, w) + (1 - t)(1, -w), s(1, w) + (1 - s)(1, -w))| \\ &= ts|T((1, w), (1, w))| + t(1 - s)|T((1, w), (1, -w))| + s(1 - t)|T((1, -w), (1, w))| \\ &\quad + (1 - t)(1 - s)|T((1, -w), (1, -w))| \\ &= ts(a + bw^2 + (c + d)w) + t(1 - s)|a - bw^2 - (c - d)w| \\ &\quad + (1 - t)s(a - bw^2 + (c - d)w) + (1 - t)(1 - s)|a + bw^2 - (c + d)w| = 1.\end{aligned}$$

Proof. This is obvious. □

We are in position to prove the main result of this paper.

THEOREM F. Let $0 < w < 1$ and $T = (a, b, c, d) \in \mathcal{L}(^2d_*(1, w)^2)$ be such that $\|T\| = 1$ with $a \geq |b|$ and $a \geq c \geq d \geq 0$. Then we have the following:

CASE 1. $b \geq 0$.

If $b = 0$ and ($d > 0$ or $a > c$), then

$$\mathrm{Norm}(T) = \{(\pm(1, w), \pm(1, w))\}.$$

If $b = d = 0$ and $a = c$, then

$$\begin{aligned}\mathrm{Norm}(T) = \{&(\pm(t(1, w) + (1 - t)(1, -w)), \\ &\pm (s(w, 1) + (1 - s)(1, w))) : \ 0 \leq t, s \leq 1\}.\end{aligned}$$

If $b = d = 0$ and $a > c > 0$, then

$$\mathrm{Norm}(T) = \{(\pm(t(1, w) + (1 - t)(1, -w)), \pm(1, w)) : \ 0 \leq t \leq 1\}.$$

If $b = d = c = 0$, then

$$\mathrm{Norm}(T) = \{(\pm(1, \pm u), \pm(1, \pm u)) : \ 0 \leq u, v \leq w\}.$$

Suppose that $b > 0$.
Let $1 = a + bw^2 + (c + d)w = (a + b)w^2 + c + dw^2$.
If $T \notin \mathcal{L}_s(^2d_*(1, w)^2)$, then

$$\mathrm{Norm}(T) = \{(\pm(1, w), \pm(t(1, w) + (1 - t)(w, 1)))\}.$$

If $T \in \mathcal{L}_s(^2d_*(1, w)^2)$, then

$$\mathrm{Norm}(T) = \mathrm{Sym}(\{(\pm(1, w), \pm(t(1, w) + (1 - t)(w, 1)))\}).$$

If $1 = a + bw^2 + (c + d)w > (a + b)w^2 + c + dw^2$, then

$$\mathrm{Norm}(T) = \{(\pm(1, w), \pm(1, w))\}.$$

Let $1 = (a + b)w^2 + c + dw^2 > a + bw^2 + (c + d)w$.
If $T \notin \mathcal{L}_s(^2d_*(1, w)^2)$, then

$$\mathrm{Norm}(T) = \{(\pm(1, w), \pm(w, 1))\}.$$

If $T \in \mathcal{L}_s(^2d_*(1, w)^2)$, then

$$\mathrm{Norm}(T) = \mathrm{Sym}(\{(\pm(1, w), \pm(w, 1))\}).$$

CASE 2. $b < 0$.

SUBCASE 2.1. $1 = a + cw + |bw + d|w > c + aw + |dw + b|w$.

If $|b|w = d$ and $T \notin \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\mathrm{Norm}(T) = \{(\pm(t(1,w) + (1-t)(1,-w)), \pm(1,w)) : 0 \le t \le 1\}.$$

If $|b|w = d$ and $T \in \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\mathrm{Norm}(T) = \mathrm{Sym}(\{(\pm(t(1,w) + (1-t)(1,-w)), \pm(1,w)) : 0 \le t \le 1\}).$$

If $|b|w < d$ and $T \notin \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\mathrm{Norm}(T) = \{(\pm(1,w), \pm(1,w))\}.$$

If $|b|w < d$ and $T \in \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\mathrm{Norm}(T) = \mathrm{Sym}(\{(\pm(1,w), \pm(1,w))\}).$$

If $|b|w > d$ and $T \notin \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\mathrm{Norm}(T) = \{(\pm(1,-w), \pm(1,w))\}.$$

If $|b|w > d$ and $T \in \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\mathrm{Norm}(T) = \mathrm{Sym}(\{(\pm(1,-w), \pm(1,w))\}).$$

SUBCASE 2.2. $1 = c + aw + |dw + b|w > a + cw + |bw + d|w$.

If $|b| = dw$ and $T \notin \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\mathrm{Norm}(T) = \{(\pm(t(1,w) + (1-t)(1,-w)), \pm(w,1)) : 0 \le t \le 1\}.$$

If $|b| = dw$ and $T \in \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\mathrm{Norm}(T) = \mathrm{Sym}(\{(\pm(t(1,w) + (1-t)(1,-w)), \pm(w,1)) : 0 \le t \le 1\}).$$

If $|b| > dw$ and $T \notin \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\mathrm{Norm}(T) = \{(\pm(1,-w), \pm(w,1))\}.$$

If $|b| > dw$ and $T \in \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\mathrm{Norm}(T) = \mathrm{Sym}(\{(\pm(1,-w), \pm(w,1))\}).$$

If $|b| < dw$ and $T \notin \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\mathrm{Norm}(T) = \{(\pm(1,w), \pm(w,1))\}.$$

If $|b| < dw$ and $T \in \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\mathrm{Norm}(T) = \mathrm{Sym}(\{(\pm(1,w), \pm(w,1))\}).$$

SUBCASE 2.3. $1 = a + cw + |bw + d|w = c + aw + |dw + b|w$.

Suppose that

$$\begin{aligned} 1 &= a + bw^2 + (c+d)w = (a+b)w + c + dw^2 \\ &> \max\{a - bw^2 + (c-d)w, (a-b)w + c - dw^2\}. \end{aligned}$$

If $T \notin \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\mathrm{Norm}(T) = \{(\pm(1,w), \pm(t(w,1) + (1-t)(1,w))) : 0 \le t \le 1\}.$$

If $T \in \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\mathrm{Norm}(T) = \mathrm{Sym}(\{(\pm(1,w), \pm(t(w,1) + (1-t)(1,w))) : 0 \le t \le 1\}).$$

Suppose that

$$\begin{aligned} 1 &= a + bw^2 + (c+d)w = (a-b)w + c - dw^2 \\ &> \max\{a - bw^2 + (c-d)w, (a+b)w + c + dw^2\}. \end{aligned}$$

If $T \notin \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\mathrm{Norm}(T) = \{(\pm(1,w), \pm(1,w)), (\pm(1,-w), \pm(w,1))\}.$$

If $T \in \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\mathrm{Norm}(T) = \mathrm{Sym}(\{(\pm(1,w), \pm(1,w)), (\pm(1,-w), \pm(w,1))\}).$$

Suppose that

$$\begin{aligned} 1 &= a - bw^2 + (c-d)w = (a-b)w + c - dw^2 \\ &> \max\{a + bw^2 + (c+d)w, (a+b)w + c + dw^2\}. \end{aligned}$$

If $T \notin \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\mathrm{Norm}(T) = \{(\pm(1,-w), \pm(t(1,w) + (1-t)(w,1))) : \ 0 \le t \le 1\}.$$

If $T \in \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\mathrm{Norm}(T) = \mathrm{Sym}(\{(\pm(1,-w), \pm(t(1,w) + (1-t)(w,1))) : \ 0 \le t \le 1\}).$$

Suppose that

$$\begin{aligned} 1 &= a + bw^2 + (c+d)w = a - bw^2 + (c-d)w \\ &= (a-b)w + c - dw^2 > (a+b)w + c + dw^2. \end{aligned}$$

If $T \notin \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\begin{aligned} \mathrm{Norm}(T) = \{&(\pm(t(1,w) + (1-t)(1,-w)), \pm(1,w)), \\ &(\pm(\pm(1,-w), t(1,w) + (1-t)(w,1))) : \ 0 \le t \le 1\}. \end{aligned}$$

If $T \in \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\begin{aligned} \mathrm{Norm}(T) = \mathrm{Sym}(\{&(\pm(t(1,w) + (1-t)(1,-w)), \pm(1,w)), \\ &(\pm(\pm(1,-w), t(1,w) + (1-t)(w,1))) : \ 0 \le t \le 1\}). \end{aligned}$$

Suppose that

$$\begin{aligned} 1 &= a + bw^2 + (c+d)w = (a+b)w + c + dw^2 \\ &= (a-b)w + c - dw^2 > a - bw^2 + (c-d)w. \end{aligned}$$

If $T \notin \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\begin{aligned} \mathrm{Norm}(T) = \{&(\pm(t(1,w) + (1-t)(1,-w)), \pm(w,1)), \\ &(\pm(\pm(1,w), t(1,w) + (1-t)(w,1))) : \ 0 \le t \le 1\}. \end{aligned}$$

If $T \in \mathcal{L}_s({}^2d_*(1,w)^2)$, then

$$\begin{aligned} \mathrm{Norm}(T) = \mathrm{Sym}(\{&(\pm(t(1,w) + (1-t)(1,-w)), \pm(w,1)), \\ &(\pm(\pm(1,w), t(1,w) + (1-t)(w,1))) : \ 0 \le t \le 1\}). \end{aligned}$$

Proof. Notice that

$$\begin{aligned} T((1,w),(1,w)) &= a + bw^2 + (c+d)w \ge 0, \\ T((1,-w),(1,w)) &= a - bw^2 + (c-d)w \ge 0, \\ T((1,w),(w,1)) &= (a+b)w + c + dw^2 \ge 0, \\ T((1,-w),(w,1)) &= (a-b)w + c - dw^2 \ge 0. \end{aligned}$$

By Theorem D, $\mathrm{Norm}(T) \subseteq A_{22} \cup A_{23} \cup A_{32} \cup A_{33}$.

Let $((x_1,y_1),(x_2,y_2)) \in \mathrm{Norm}(T)$. Without loss of generality we may assume that $x_j \ge 0$ for all $j = 1, 2$.

CASE 1. $b \geq 0$.

Notice that if $b = 0$ and ($d > 0$ or $a > c$), then, by Lemma E, $((x_1, y_1), (x_2, y_2)) \in A_{22}$. Thus

$$\mathrm{Norm}(T) = \{(\pm(1, w), \pm(1, w))\}.$$

If $b = d = 0$ and $a = c$, then $T = \frac{1}{1+w}(1, 0, 1, 0)$. By Lemma E, $((x_1, y_1), (x_2, y_2)) \in A_{32}$. Thus

$$\mathrm{Norm}(T) = \{(\pm(t(1, w) + (1 - t)(1, -w)), \pm (s(w, 1) + (1 - s)(1, w))) : 0 \leq t, s \leq 1\}.$$

If $b = d = 0$ and $a > c > 0$, then $T = (1 - cw, 0, c, 0)$ for $0 < c < \frac{1}{1+w}$. By Lemma E, $((x_1, y_1), (x_2, y_2)) \in A_{32}$. Thus

$$\mathrm{Norm}(T) = \{(\pm(t(1, w) + (1 - t)(1, -w)), \pm(1, w)) : 0 \leq t \leq 1\}.$$

If $b = d = c = 0$, then $T = (1, 0, c, 0)$. By Lemma E, $((x_1, y_1), (x_2, y_2)) \in A_{33}$. Thus

$$\mathrm{Norm}(T) = \{(\pm(1, \pm u), \pm(1, \pm u)) : 0 \leq u, v \leq w\} = A_{33}.$$

Suppose that $b > 0$.

Let $1 = a + bw^2 + (c + d)w = (a + b)w^2 + c + dw^2$. By Lemma E, $((x_1, y_1), (x_2, y_2)) \in A_{22}$. Thus

$$1 = T((1, w), (1, w)) = T((1, w), (w, 1)).$$

Notice that if $T \notin \mathcal{L}_s(^2d_*(1, w)^2)$, then $T = \frac{1}{1+w}(1, \frac{1}{1+w}, \frac{1}{1+w}, 0)$. By Lemma E, $((x_1, y_1), (x_2, y_2)) \in A_{22}$. Thus

$$\mathrm{Norm}(T) = \{(\pm(1, w), \pm(t(1, w) + (1 - t)(w, 1))) : 0 \leq t \leq 1\}$$

and that if $T \in \mathcal{L}_s(^2d_*(1, w)^2)$, then $T = \frac{1}{(1+w)^2}(1, 1, 1, 1, 1)$ and so

$$\mathrm{Norm}(T) = \mathrm{Sym}(\{(\pm(1, w), \pm(t(1, w) + (1 - t)(w, 1))) : 0 \leq t \leq 1\}).$$

Let $1 = a + bw^2 + (c + d)w > (a + b)w^2 + c + dw^2$. By Lemma E, $((x_1, y_1), (x_2, y_2)) \in A_{22}$. Thus

$$\mathrm{Norm}(T) = \{(\pm(1, w), \pm(1, w))\}.$$

Let $1 = (a + b)w^2 + c + dw^2 > a + bw^2 + (c + d)w$.

Notice that if $T \notin \mathcal{L}_s(^2d_*(1, w)^2)$, then, by Lemma E, $((x_1, y_1), (x_2, y_2)) \in A_{22}$. Thus

$$\mathrm{Norm}(T) = \{(\pm(1, w), \pm(w, 1))\}.$$

if $T \in \mathcal{L}_s(^2d_*(1, w)^2)$, then

$$\mathrm{Norm}(T) = \mathrm{Sym}(\{(\pm(1, w), \pm(w, 1))\}).$$

CASE 2. $b < 0$.

SUBCASE 2.1. $1 = a + cw + |bw + d|w > c + aw + |dw + b|w$.

Note the following equivalences:

(1) $|b|w \geq d$ if and only if $a - bw^2 + (c - d)w \geq a + bw^2 + (c + d)w$;
(2) $|b|w \leq d$ if and only if $a + bw^2 + (c + d)w \geq a - bw^2 + (c - d)w$.

From these equivalences, $|b|w = d$ if and only if $a - bw^2 + (c - d)w = a + bw^2 + (c + d)w$.

Let $|b|w = d$. If $0 < w \leq \sqrt{2} - 1$, then $T \in \mathcal{L}_s(^2d_*(1, w)^2)$, then

$$T = (1 - |b|w^2, b, |b|w, |b|w) \text{ for } 0 < |b| \leq \frac{1}{1 + w^2}.$$

If $\sqrt{2} - 1 < w < 1$, then $T \in \mathcal{L}_s(^2d_*(1, w)^2)$, then

$$T = (1 - |b|w^2, b, |b|w, |b|w) \text{ for } 0 < |b| \leq \frac{1}{2w(1 + w)}.$$

By Lemma E, $((x_1, y_1), (x_2, y_2)) \in A_{32} \cup A_{23}$. Thus

$$\mathrm{Norm}(T) = \mathrm{Sym}(\{(\pm(t(1, w) + (1 - t)(1, -w)), \pm(1, w)) : 0 \leq t \leq 1\}).$$

If $|b|w = d$ and $T \notin \mathcal{L}_s(^2d_*(1, w)^2)$, then by Lemma E, $((x_1, y_1), (x_2, y_2)) \in A_{32}$. Thus

$$\mathrm{Norm}(T) = \{(\pm(t(1, w) + (1 - t)(1, -w)), \pm(1, w)) : 0 \leq t \leq 1\}.$$

Since the proofs are analogous as in those of the above, we omit them.

SUBCASE 2.2. $1 = c + aw + |dw + b|w > a + cw + |bw + d|w$.

Note the following equivalences:

(1) $|b| \leq dw$ if and only if $(a+b)w + c + dw^2 \geq (a-b)w + c - dw^2$;
(2) $|b| \geq dw$ if and only if $(a-b)w + c - dw^2 \geq (a+b)w + c + dw^2$.

From these equivalences, $|b| = dw$ if and only if $(a+b)w + c + dw^2 = (a-b)w + c - dw^2$.
If $|b| = dw$ and $T \in \mathcal{L}_s(^2d_*(1,w)^2)$, then

$$T = \left(\frac{1}{w} - \frac{|b|}{w^2}, b, \frac{|b|}{w}, \frac{|b|}{w^2}\right) \text{ for } \frac{w}{(1-w)(1+w)^2} < |b| \leq w.$$

By Lemma E, $((x_1,y_1),(x_2,y_2)) \in A_{32} \cup A_{23}$. Thus

$$\text{Norm}(T) = \text{Sym}(\{(\pm(t(1,w) + (1-t)(1,-w)), \pm(w,1)) : \ 0 \leq t \leq 1\}).$$

If $|b| = dw$ and $T \notin \mathcal{L}_s(^2d_*(1,w)^2)$, by Lemma E, $((x_1,y_1),(x_2,y_2)) \in A_{32}$. Thus

$$\text{Norm}(T) = \{(\pm(t(1,w) + (1-t)(1,-w)), \pm(w,1)) : \ 0 \leq t \leq 1\}.$$

Since the proofs are analogous as in those of the above, we omit them.

SUBCASE 2.3. $1 = a + cw + |bw + d|w = c + aw + |dw + b|w$.

By Lemma C, we have five possibilities:

$$1 = a+bw^2+(c+d)w = (a+b)w+c+dw^2 > \max\{a - bw^2+(c-d)w, (a-b)w+c-dw^2\}, \tag{2.4}$$

$$1 = a+bw^2+(c+d)w = (a-b)w+c-dw^2 > \max\{a - bw^2+(c-d)w, (a+b)w+c+dw^2\}, \tag{2.5}$$

$$1 = a - bw^2+(c-d)w = (a-b)w+c-dw^2 > \max\{a+bw^2+(c+d)w, (a+b)w+c+dw^2\}, \tag{2.6}$$

$$1 = a+bw^2+(c+d)w = a - bw^2+(c-d)w = (a-b)w+c-dw^2 > (a+b)w+c+dw^2, \tag{2.7}$$

$$1 = a+bw^2+(c+d)w = (a+b)w+c+dw^2 = (a-b)w+c-dw^2 > a - bw^2+(c-d)w. \tag{2.8}$$

We only prove the case (2.7) because the proof of the other cases are analogous. Suppose that (2.7) holds.

Notice that

$$T = \left(\frac{1}{1+w} + |b|w^2, b, \frac{1}{1+w} - |b|w, |b|w\right) \text{ for } 0 < |b| \leq \frac{1}{2w(1+w)}$$

and that if $T \in \mathcal{L}_s(^2d_*(1,w)^2)$, then $|b| = \frac{1}{2w(1+w)}$ and

$$T = \frac{1}{2(1+w)}\left(2+w, \frac{-1}{w}, 1, 1\right).$$

If $T \notin \mathcal{L}_s(^2d_*(1,w)^2)$, then by Lemma E, $((x_1,y_1),(x_2,y_2)) \in A_{32}$. Thus

$$\begin{aligned}\text{Norm}(T) = \{&(\pm(t(1,w) + (1-t)(1,-w)), \pm(1,w)),\\ &(\pm(1,-w), \pm(t(1,w) + (1-t)(w,1))) : \ 0 \leq t \leq 1\}.\end{aligned}$$

If $T \in \mathcal{L}_s(^2d_*(1,w)^2)$, then by Lemma E, $((x_1,y_1),(x_2,y_2)) \in A_{32}$. Thus

$$\begin{aligned}\text{Norm}(T) = \text{Sym}(\{&(\pm(t(1,w) + (1-t)(1,-w)), \pm(1,w)),\\ &(\pm(1,-w), \pm(t(1,w) + (1-t)(w,1))) : \ 0 \leq t \leq 1\}).\end{aligned}$$

This completes the proof. □

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