Mathematica Pannonica ***New Series*** **29 /NS 3/ (2023) 2, 244–257**
DOI: 10.1556/314.2023.00026

AKADÉMIAI KIADÓ

ON THE SIMULTANEOUS SIGN CHANGES OF COEFFICIENTS OF RANKIN–SELBERG L-FUNCTIONS OVER A CERTAIN INTEGRAL BINARY QUADRATIC FORM

Guodong Hua[1,2,3,*]

[1] School of Mathematics and Statistics, Weinan Normal University, Weinan 714099, China
[2] Qindong Mathematical Research Institute, Weinan Normal University, Weinan 714099, China
[3] School of Mathematics, Shandong University, Shandong, Jinan 250100, China

Communicated by László Tóth

Original Research Paper
Received: Jul 19, 2023 • Accepted: Nov 4, 2023
First published online: Nov 17, 2023

ABSTRACT

In this paper, we consider the simultaneous sign changes of coefficients of Rankin–Selberg L-functions associated to two distinct Hecke eigenforms supported at positive integers represented by some certain primitive reduced integral binary quadratic form with negative discriminant D. We provide a quantitative result for the number of sign changes of such sequence in the interval $(x, 2x]$ for sufficiently large x.

KEYWORDS

Hecke eigenforms, Fourier coefficients, simultaneous sign changes

MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 11F11; Secondary 11F30, 11N37

1. INTRODUCTION

The Fourier coefficients of modular forms are important and interesting objects in number theory. In recent times, there is much a trend to study the sign changes of Fourier coefficients of mudular forms. Denote by $H_k^*(N)$ the set of all normalized primitive holomorphic cusp forms of even integral weight k for the Hecke congruence group $\Gamma_0(N)$ with trivial nebentypus, which consists of eigenfunctions of all Hecke operators T_n. In the case $N = 1$, we abbreviate by H_k^* the set of normalized Hecke eigenforms of integral weight k for the full modular group $\Gamma = SL(2, \mathbb{Z})$. It is well-known that $f \in H_k^*(N)$ admits Fourier expansion at the cusp ∞:

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz), \quad \Im(z) > 0,$$

where $e(z) = e^{2\pi i z}$, and $\lambda_f(n)$ is the n-th normalized Fourier coefficient (Hecke eigenvalue) with $\lambda_f(1) = 1$. It is known from the theory of Hecke operator $\lambda_f(n)$ is real and satisfies the multiplicative

* Corresponding author. E-mail: gdhuanumb@yeah.net

property

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right), \quad (mn, N) = 1, \tag{1.1}$$

where $m \geq 1$ and $n \geq 1$ are integers. In 1974, Deligne [6] proved the celebrated Ramanujan–Petersson conjecture

$$|\lambda_f(n)| \leq d(n),$$

where $d(n)$ is the divisor function. In fact, for each prime number p with $(p, N) = 1$, there exist two complex numbers $\alpha_f(p)$ and $\beta_f(p)$ such that

$$\alpha_f(p) + \beta_f(p) = \lambda_f(p), \quad |\alpha_f(p)| = |\beta_f(p)| = \alpha_f(p)\beta_f(p) = 1. \tag{1.2}$$

The sign change problem of Fourier coefficients attached to modular forms has a long history which can go back to Siegel [34]. Let $f \in H_k^*$ be a Hecke eigenform. By using a classical theorem of Landau and some analytic properties of the associated L-function $L(f, s)$, one can prove that $\{\lambda_f(n)\}_{n\in\mathbb{N}}$ has infinitely many sign changes (e.g.[19]). A number of authors considered the sign changes of Fourier coefficients for the subsequences. In the early 1980s, R. Murty [27] considered the sign changes of the sequence of Fourier coefficients at the prime number. Later, Meher et al. [28] established the lower bounds for the number of sign changes for the sequences $\{\lambda_f(n^j)\}_{n\geq1}$ in the interval $(x, 2x]$ with $j = 2, 3, 4$. More recently, Lao et al. [25] considered the number of sign changes of the sequences $\{\lambda_f(n^j)\}$ for $j \geq 3$ in the interval $(x, 2x]$, which improved and generalized the results in [28]. In [21], Kohnen and Martin proved that the sequence $\{\lambda_f(p^{jn})\}_{n\geq1}$ have infinitely many sign changes for almost all primes p and $j \in \mathbb{N}$.

In 2020, Banerjee and Pandey [1] considered the sign change of normalized Fourier coefficients supported at sums of two squares and they proved that the sequence $\{\lambda_f(c^2 + d^2)\}_{c,d\geq1}$ has at least $x^{\frac{1}{8}-2\varepsilon}$ sign changes in the interval $(x, 2x]$ for any $\varepsilon > 0$ for sufficiently large x. Very recently, Vaishya [37] extended the method of Banerjee and Pandey and they proved a more general result. Before proceeding to the result of Vaishya, we need to introduce some notations and definitions.

Let $S_k(l, \psi)$ denote the space of cusp forms of integral weight k for the congruence subgroup $\Gamma_0(l)$ with Dirichlet character ψ. Then a normalized Hecke eigenform $f \in S_k(l, \psi)$ has the Fourier expansion at the cusp ∞, given by

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz), \tag{1.3}$$

where $\lambda_f(n)$ is the n-th normalized Fourier coefficient (Hecke eigenvalue) of f with the normalization $\lambda_f(1) = 1$. Let $B(x_1, x_2) := ax_1^2 + bx_1x_2 + cx_2^2$ be an integral positive definite binary quadratic form with fixed discriminant $D = b^2 - 4ac$, where $(x_1, x_2) \in \mathbb{Z}^2, a, b, c \in \mathbb{Z}$. An integral binary quadratic form $B(x_1, x_2)$ is called a primitive form if $\gcd(a, b, c) = 1$, and reduced form if $|b| \leq a \leq c$. Two forms $B_1(x_1, x_2)$ and $B_2(x_1, x_2)$ are called to be equivalent if there are integers a, b, r, s with $as - br = \pm1$ such that $B_1(x_1, x_2) = B_2(ax_1 + bx_2.rx_1 + sx_2)$. For any given $D < 0$, we denote by $H(D)$ the set of equivalence classes of primitive integral binary quadratic reduced forms of discriminant D. It is well-known that $h(D) := \#H(D)$ is finite for any fixed discriminant $D < 0$. Here we call $h(D)$ the class number of the integral positive definite binary quadratic forms $B(x_1, x_2)$ with discriminant D.

Let $Q(X)$ denote a primitive integral positive definite binary quadratic (reduced) form given by $Q(X) = ax_1^2 + bx_1x_2 + cx_2^2$, where $(x_1, x_2) \in \mathbb{Z}^2, a, b, c \in \mathbb{Z}$, and $(a, b, c) = 1$ and with fixed discriminant $D = b^2 - 4ac < 0$.

Now we state the result given by Vaishya.

THEOREM A ([37, Theorem 1.1]). Let $f \in S_k(q, \psi)$ be a normalized Hecke newform with normalized Fourier coefficients $\lambda_f(n)$ given by (1.3). Let $Q(X)$ be a primitive integral positive definite binary quadratic reduced form with fixed discriminant $D < 0$ and $\gcd(l, |D|) = 1$. In addition, we assume the sequence $\{\lambda_f(n)\}_{n\geq1}$ are real and the class number $h(D) = 1$. Then the sequence $\{\lambda_f(Q(X))\}_{X=(x_1,x_2)\in\mathbb{Z}^2}$ has at least $x^{\frac{1}{5}-\varepsilon}$ sign changes in the interval $(x, 2x]$ for sufficiently large x.

In recent times, the study of simultaneous sign changes of Fourier coefficients of two cusp forms have also been considered by a number of authors (see e.g. [9, 20, 22, 25]). Let $f \in H^*_{k_1}$ and $g \in H^*_{k_2}$ be two distinct Hecke eigenforms with the n-th normalized Fourier coefficients denote by $\lambda_f(n)$ and $\lambda_g(n)$, respectively. In particular, Lao et al. [25] proved that the sequence $\{\lambda_f(n^i)\lambda_g(n^j)\}$ has at least one sign change for $n \in (x, x+x^r]$ with $1 - \frac{42}{21(i+1)^2(j+1)^2-29} < r < 1$ for sufficiently large x. Recently, the author considered the simultaneous sign changes of Fourier coefficients of two distinct Hecke eigenforms supported on certain binary quadratic form. More accurately, the author established the following theorem.

THEOREM 1.1 ([10, Thoerem 1.1]). Let $i \geq 1$ and $j \geq 1$ be two positive integers. Let $f \in H^*_{k_1}$ and $g \in H^*_{k_2}$ be two distinct normalized Hecke eigenforms. Let $Q(X)$ be a primitive integral positive definite binary quadratic reduced form with fixed discriminant $D < 0$. In addition, we assume the class number $h(D) = 1$. Then the sequence $\{\lambda_f(n^i)\lambda_g(n^j)\}_{\substack{n=Q(X)\\ X\in\mathbb{Z}^2}}$ has at least $\gg x^{1-\delta}$ sign changes in the interval $(x, 2x]$ with

$$1 - \frac{210}{210(i+1)^2(j+1)^2 - 103} < \delta < 1$$

for sufficiently large x. In particular, the sequence $\{\lambda_f(n^i)\lambda_g(n^j)\}_{\substack{n=Q(X)\\ X\in\mathbb{Z}^2}}$ has infinitely many sign changes.

REMARK 1.1. let $\lambda_{\mathrm{sym}^i f}(n)$ and $\lambda_{\mathrm{sym}^j g}(n)$ denote the n-th normalized coefficients of the Dirichlet expansion of the symmetric power L-functions $L(\mathrm{sym}^i f, s)$ and $L(\mathrm{sym}^j g, s)$, respectively. In fact, using the similar techniques in establishing Theorem 1.1, one can easily prove the same result for the simultaneous sign changes for the sequence $\{\lambda_{\mathrm{sym}^i f}(n)\lambda_{\mathrm{sym}^j g}(n)\}_{\substack{n=Q(X)\\ X\in\mathbb{Z}^2}}$ in the interval $(x, 2x]$.

For simplicity, in this paper we only consider the situation that $f \in H^*_{k_1}$ and $g \in H^*_{k_2}$ be two distinct normalized cuspidal Hecke eigenforms. Let $1 \leq i < j$ be any fixed positive integers, and denote by $\lambda_{\mathrm{sym}^i f\times\mathrm{sym}^j f}(n)$ and $\lambda_{\mathrm{sym}^i g\times\mathrm{sym}^j g}(n)$ the n-th normalized coefficients of the Dirichlet expansion of Rankin–Selberg L-functions $L(\mathrm{sym}^i f \times \mathrm{sym}^j f, s)$ and $L(\mathrm{sym}^i g \times \mathrm{sym}^j g, s)$, respectively. Inspired by the above results, in this paper we consider the analogous problem concerning the simultaneous sign changes of normalized coefficients of Rankin–Selberg L-functions attached to two distinct Hecke eigenforms on sequence of positive integers represented by a primitive integral binary quadratic form with negative discriminant $D < 0$. More precisely, we prove the following theorem.

THEOREM 1.2. Let $1 \leq i < j$ be any two given positive integers. Let $f \in H^*_{k_1}$ and $g \in H^*_{k_2}$ be two distinct normalized Hecke eigenforms. Let $Q(X)$ be a certain primitive integral positive definite binary quadratic reduced form with fixed discriminant $D < 0$. Then the sequence

$$\{\lambda_{\mathrm{sym}^i f\times\mathrm{sym}^j f}(n)\lambda_{\mathrm{sym}^i g\times\mathrm{sym}^j g}(n)\}_{\substack{n=Q(X)\\ X\in\mathbb{Z}^2}}$$

has at least $\gg x^{1-\theta}$ sign changes in the interval $(x, 2x]$ with

$$1 - \frac{42}{42(i+1)^4(j+1)^4 - 16(i+1)^2 - 3} < \theta < 1$$

for sufficiently large x. In particular, the sequence $\{\lambda_{\mathrm{sym}^i f\times\mathrm{sym}^j f}(n)\lambda_{\mathrm{sym}^i g\times\mathrm{sym}^j g}(n)\}_{\substack{n=Q(X)\\ X\in\mathbb{Z}^2}}$ has infinitely many sign changes.

REMARK 1.2. The result in Theorem 1.2 can be generalized to the occurrence for $f \in S_{k_1}(l_1, \psi_1)$ and $g \in S_{k_2}(l_2, \psi_2)$ that are not equivalently twist, by invoking the analytic properties of the associated symmetric power L-functions and Rankin–Selberg L-functions, in combination with the individual or averaged convexity bounds or sub-convexity bounds for these automorphic L-functions.

The proof of Theorem 1.2 is based on the celebrated works that Newton and Thorne [29, 30] proved that $\mathrm{sym}^j f$ corresponds with a cuspidal automorphic representation of $GL_{j+1}(\mathbb{A}_{\mathbb{Q}})$ for all $j \geq 1$, together with the nice analytic properties of the associated automorphic L-functions.

Throughout the paper, for the sake of simplicity, we always work on the finite dimensional vector space H^*_k. And we also assume that $f \in H^*_{k_1}$ and $g \in H^*_{k_2}$ are two distinct normalized cuspidal Hecke eigenforms. And let $\varepsilon > 0$ denotes an arbitrarily small positive constant which may vary in different occurrence. The absolute constant in O terms and $\ll$ terms depend at most on f, g, ε, D.

2. PRELIMINARIES

In this section, we briefly review some properties of automorphic L-functions and give some lemmas which plays an important role in the establishment of the main results in this paper.

Let $f \in H_{k_1}^*$ be a normalized cuspidal Hecke eigenform. Then we can define the L-function $L(f, s)$ attached to f by

$$L(f,s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1}$$
$$= \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1}, \quad \Re(s) > 1,$$

where $\alpha_f(p), \beta_f(p)$ are the local parameters given by (1.2). The j-th symmetric power L-function $L(\mathrm{sym}^j f, s)$ attached to $\mathrm{sym}^j f$ is defined by

$$L(\mathrm{sym}^j f, s) := \prod_p \prod_{m=0}^{j} (1 - \alpha_f(p)^{j-m} \beta_f(p)^m p^{-s})^{-1}, \quad \Re(s) > 1. \tag{2.1}$$

We may expand it into a Dirichlet series

$$L(\mathrm{sym}^j f, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s}$$
$$= \prod_p \left(1 + \frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s} + \cdots + \frac{\lambda_{\mathrm{sym}^j f}(p^k)}{p^{ks}} + \cdots\right), \quad \Re(s) > 1. \tag{2.2}$$

One sees directly that $\lambda_{\mathrm{sym}^j f}(n)$ is a real multiplicative function. Let $g \in H_{k_2}^*$ be another normalized cuspidal Hecke eigenform. We can also define the Rankin–Selberg function $L(s, \mathrm{sym}^i f \times \mathrm{sym}^j g)$ attached to $\mathrm{sym}^i f$ and $\mathrm{sym}^j g$ as

$$L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s) := \prod_p \prod_{m=0}^{i} \prod_{m'=0}^{j} \left(1 - \frac{\alpha_f(p)^{i-m} \beta_f(p)^i \alpha_g(p)^{j-m'} \beta_g(p)^{m'}}{p^s}\right)^{-1}, \tag{2.3}$$

for $\Re(s) > 1$, where $\alpha_g(p)$ and $\beta_g(p)$ are local parameters of g defined similarly as that of f. Here, the Hecke eigenforms f and g are not necessarily different. Similarly, for $\Re(s) > 1$, we have

$$L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(n)}{n^s}$$
$$= \prod_p \left(1 + \sum_{r \ge 1} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(p^r)}{p^{rs}}\right). \tag{2.4}$$

It is well-known that $\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(n)$ is also a real multiplicative function.

From (1.2), (2.1), (2.2) and the Hecke operator theory, we have

$$\lambda_f(p^j) = \sum_{m=0}^{j} \alpha_f(p)^{j-2m} = \lambda_{\mathrm{sym}^j f}(p), \quad j \ge 1. \tag{2.5}$$

From (2.3)–(2.5), we derive that for $i, j \ge 1$,

$$\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(p) = \sum_{m=0}^{i} \sum_{m'=0}^{j} \alpha_f(p)^{i-2m} \alpha_g(p)^{j-2m'} = \lambda_{\mathrm{sym}^i f}(p) \lambda_{\mathrm{sym}^j g}(p). \tag{2.6}$$

As is well-known, every primitive form $f(z)$ is associated to an automorphic cuspidal representation $\pi_f \in GL_2(\mathbb{A}_{\mathbb{Q}})$, then there is an automorphic L-function $L(\pi_f, s)$ which coincide with $L(f, s)$. It is predicted that π_f gives rise to a symmetric power lift–an automorphic representation whose

L-function is the symmetric power L-function attached to f. The celebrated works of Gelbart and Jacquet [8], Kim and Shahidi [16, 17], and Kim [18], Dieulefait [7] and Clozel and Thorne [3, 4, 5] showed that $L(\mathrm{sym}^j f, s), (1 \leq j \leq 8)$ is a general L-function, which has an analytic continuation as an entire function in the whole complex plane $\mathbb{C}$ and satisfies a certain Riemann-type functional equation of degree $j+1$. Very recently, Newton and Thorne [29, 30] proved that $\mathrm{sym}^j f$ corresponds with a cuspidal automorphic representation of $GL_{j+1}(\mathbb{A}_{\mathbb{Q}})$ for all $j \geq 1$. Due to the work of Jacquet and Shalika [14, 15], Shahidi [35, 36], and the reformulation of Rudnick and Sarnak [33], we know that the symmetric power L-function $L(\mathrm{sym}^j f, s), (j \geq 1)$ and the Rankin–Selberg L-function $L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s), (i, j \geq 1)$ can be extended to the whole complex plane as an entire function (except for the case $\mathrm{sym}^j \pi_f \cong \mathrm{sym}^j \pi_g$ which has simple poles at $s = 0, 1$) and satisfies a certain functional equation of Riemann-type.

We firstly state some basic definitions and analytic properties of general L-functions. Let $L(\phi, s)$ be a Dirichlet series (associated with the object ϕ) that admits an Euler product of degree $m \geq 1$, namely

$$L(\phi, s) = \sum_{n=1}^{\infty} \frac{\lambda_\phi(n)}{n^s} = \prod_{p<\infty} \prod_{j=1}^{m} \left(1 - \frac{\alpha_\phi(p, j)}{p^s}\right)^{-1},$$

where $\alpha_\phi(p, j), j = 1, 2, \ldots, m$ are the local parameters of $L(\phi, s)$ at a finite prime p. Suppose that this series and its Euler product are absolutely convergent for $\Re(s) > 1$. We denote the gamma factor by

$$L_\infty(\phi, s) = \prod_{j=1}^{m} \pi^{-\frac{s+\mu_\phi(j)}{2}} \Gamma\left(\frac{s + \mu_\phi(j)}{2}\right)$$

with local parameters $\mu_\phi(j), j = 1, 2, \ldots, m$ of $L(\phi, s)$ at ∞. The complete L-function $\Lambda(\phi, s)$ is defined by

$$\Lambda(\phi, s) = q(\phi)^{\frac{s}{2}} L_\infty(\phi, s) L(\phi, s),$$

where $q(\phi)$ is the conductor of $L(\phi, s)$. We assume that $\Lambda(\phi, s)$ admits an analytic continuation to the the whole complex plane $\mathbb{C}$ and is holomorphic everywhere except for possible poles of finite order at $s = 0, 1$. Furthermore, it satisfies a functional equation of the Riemann-type

$$\Lambda(\phi, s) = \epsilon_\phi \Lambda(\tilde{\phi}, 1 - s)$$

where ϵ_ϕ is the root number with $|\epsilon_\phi| = 1$ and $\tilde{\phi}$ is dual of ϕ such that $\lambda_{\tilde{\phi}}(n) = \overline{\lambda_\phi(n)}, L_\infty(\tilde{\phi}, s) = L_\infty(\phi, s)$ and $q(\tilde{\phi}) = q(\phi)$. We say that $\phi \in S_e^\#$ if it endowed with the above conditions. We call the L-function $L(\phi, s)$ satisfies the Ramanujan conjecture if $\lambda_\phi(n) \ll n^\varepsilon$ for any ε.

Here we state a very general theorem due to Lau and Lü [24].

LEMMA 2.1 ([24, Lemma 2.4]). Suppose that $L(f, s)$ is a product of two L-functions $L_1, L_2 \in S_e^\#$ with both $\deg L_i \geq 2, i = 1, 2$ and $L(f, s)$ satisfies the Ramanujan conjecture. Then for any $\varepsilon > 0$, we have

$$\sum_{n \leq x} \lambda_f(n) = M(x) + O\left(x^{1-\frac{2}{m}+\varepsilon}\right),$$

where $M(x) = \mathrm{Res}_{s=1}\{L(f, s)x^s/s\}$ and $m = \deg L$.

Let $1 \leq i < j$ be any given positive integers. To study the simultaneous sign changes of the normalized Fourier coefficients of the cusp forms f, g at integers represented by an integral binary quadratic form $Q(X)$ with the fixed discriminant $D < 0$, one needs to consider the following two summatory functions:

$$S_{1,i,j}(x) := \sum_{\substack{n=Q(X)\leq x \\ X\in\mathbb{Z}^2}} \lambda_{\mathrm{sym}^i f\times\mathrm{sym}^j f}(n)\lambda_{\mathrm{sym}^i g\times\mathrm{sym}^j g}(n), \tag{2.7}$$

and

$$S_{2,i,j}(x) := \sum_{\substack{n=Q(X)\leq x \\ X\in\mathbb{Z}^2}} \lambda^2_{\mathrm{sym}^i f\times\mathrm{sym}^j f}(n)\lambda^2_{\mathrm{sym}^i g\times\mathrm{sym}^j g}(n). \tag{2.8}$$

The sums defined in (2.7) and (2.8) can be rewritten as follows:

$$S_{1,i,j}(x) := \sum_{n\le x} \lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(n)\lambda_{\mathrm{sym}^i g\times \mathrm{sym}^j g}(n) r_Q(n),$$
$$S_{2,i,j}(x) := \sum_{n\le x} \lambda^2_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(n)\lambda^2_{\mathrm{sym}^i g\times \mathrm{sym}^j g}(n) r_Q(n).$$

In order to detect the sign changes of the sequence $\{\lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(n)\lambda_{\mathrm{sym}^i g\times \mathrm{sym}^j g}(n)\}_{\substack{n=Q(X)\\ X\in\mathbb{Z}^2}}$ in the interval $(x, 2x]$, we need to find an upper bound for $S_{1,i,j}(x)$ and obtain the asymptotic behaviour for $S_{2,i,j}(x)$.

For $n > 0$ with $\gcd(n, D) = 1$, the character sum $r(n; D)$ (see [12, (11.9)], [37, (5)]) defined in terms of Kronecker symbol $\chi_D(d) := \left(\frac{D}{d}\right)$ is given by

$$r(n; D) = w_D \sum_{d\mid n} \chi_D(d), \qquad \text{where} \qquad w_D = \begin{cases} 6, & \text{if } D = -3,\\ 4, & \text{if } D = -4,\\ 2, & \text{if } D < -4.\end{cases}$$

which gives the number of all representatives of n by representative of forms of all classes of discriminant D. Therefore, for some certain primitive reduced quadratic form Q of negative discriminant D, we have

$$r_Q(n) = r(n; D) = w_D \sum_{d\mid n} \chi_D(d).$$

In particular, $r_Q(p) = w_D(1 + \chi_D(p))$. From [12, Theorem 10.9], the generating function $\theta_Q(\tau)$ for $r_Q(n)$ is a modular form of weight 1 for the congruence subgroup $\Gamma_0(|D|)$ with character χ_D, i.e.

$$\theta_Q(\tau) := \sum_{n\ge 0} r_Q(n) q^n = \sum_{X\in\mathbb{Z}^2} q^{Q(X)} \in M_1(\Gamma_0(|D|), \chi_D),$$

where $q = e^{2\pi i\tau}$ and $\mathfrak{I}(\tau) > 0$. By Weil's bound, we have $r_Q(n) \ll n^{\varepsilon}$.

Let $i \ge 1$ and $j \ge 1$ be positive integers. We define the automorphic L-functions related to $S_{1,i,j}(x)$ and $S_{2,i,j}(x)$ by

$$\mathfrak{F}(f, g, i, j, \theta_Q; s) := \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(n)\lambda_{\mathrm{sym}^i g\times \mathrm{sym}^j g}(n) r_Q(n)}{n^s}, \quad \mathfrak{R}(s) > 1$$

and

$$\mathfrak{G}(f, g, i, j, \theta_Q; s) := \sum_{n=1}^{\infty} \frac{\lambda^2_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(n)\lambda^2_{\mathrm{sym}^i g\times \mathrm{sym}^j g}(n) r_Q(n)}{n^s}, \quad \mathfrak{R}(s) > 1$$

respectively. Now we give the decomposition of $\mathfrak{F}(f, g, i, j, \theta_Q; s)$ and $\mathfrak{G}(f, g, i, j, \theta_Q; s)$ in order to get the asymptotic behaviour of the sums $S_{1,i,j}(x)$ and $S_{2,i,j}(x)$.

LEMMA 2.2. Let $1 \le i < j$ be any fixed positive integers. For $\mathfrak{R}(s) > 1$, we have

$$\mathfrak{G}(f, g, i, j, \theta_Q; s) = \zeta(s)^{(i+1)^2} L(s, \chi_D)^{(i+1)^2} \prod_{l}\prod_{l'} \Big(L(\mathrm{sym}^{2l} f, s) L(\mathrm{sym}^{2l'} g, s)$$
$$L(\mathrm{sym}^{2l} f \times \mathrm{sym}^{2l'} g, s) L(\mathrm{sym}^{2l} f \times \chi_D, s) L(\mathrm{sym}^{2l'} g \times \chi_D, s) L(\mathrm{sym}^{2l} f \times \mathrm{sym}^{2l'} g \times \chi_D, s)\Big) U_{f,g,i,j,2}(s),$$

where l, l' ranges over suitable positive integers, and the function $U_{f,g,i,j,2}(s)$ admits the Dirichlet series which converges uniformly and absolutely in the half-plane $\mathfrak{R}(s) > \frac{1}{2}$ and $U_{f,g,i,j,2}(s) \neq 0$ for $\mathfrak{R}(s) = 1$.

Proof. Let $r_Q(n) = w_D r_Q^*(n)$. Then $r_Q^*(n) = \sum_{d|n} \chi_D(d)$. Since $\lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(n), \lambda_{\mathrm{sym}^i g\times \mathrm{sym}^j g}(n)$ and $r_Q^*(n)$ are multiplicative functions and also satisfies the trivial bound $O(n^\varepsilon)$ for any $\varepsilon > 0$, we get

$$\begin{aligned}\mathfrak{G}(f,g,i,j,\theta_Q;s) &= w_D \sum_{n=1}^{\infty} \frac{\lambda^2_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(n)\lambda^2_{\mathrm{sym}^i g\times \mathrm{sym}^j g}(n) r_Q^*(n)}{n^s} \\ &= w_D \prod_p \left(1 + \sum_{r\geq 1} \frac{\lambda^2_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(p^r)\lambda^2_{\mathrm{sym}^i g\times \mathrm{sym}^j g}(p^r) r_Q^*(p^r)}{p^{rs}}\right) \qquad (2.9)\end{aligned}$$

for $\Re(s) > 1$. By (2.5), (2.6) and Hecke relation (1.1), we have

$$\begin{aligned}\lambda^2_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(p) = \lambda_f^2(p^i)\lambda_f^2(p^j) &= \left(1 + \sum_{l_1=1}^{i} \lambda_f(p^{2i_1})\right)\left(1 + \sum_{l_2=1}^{j} \lambda_f(p^{2j_2})\right) \\ &= \left(1 + \sum_{l_1=1}^{i} \lambda_{\mathrm{sym}^{2l_1} f}(p)\right)\left(1 + \sum_{l_2=1}^{j} \lambda_{\mathrm{sym}^{2l_2} f}(p)\right) \\ &= \left(1 + \sum_{l_1=1}^{i} \lambda_{\mathrm{sym}^{2l_1} f}(p) + \sum_{l_2=1}^{j} \lambda_{\mathrm{sym}^{2l_2} f}(p) + \sum_{l_1=1}^{i}\sum_{l_2=1}^{j} \lambda_{\mathrm{sym}^{2l_1} f\times \mathrm{sym}^{2l_2} f}(p)\right). \qquad (2.10)\end{aligned}$$

By the Hecke relation (1.1), we also have

$$\lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(p) = \lambda_f(p^i)\lambda_f(p^j) = \sum_{m=0}^{i} \lambda_{\mathrm{sym}^{i+j-2m} f}(p),$$

which produces a constant term 1 if and only if $i = j$. Combining the relations (2.10) and (2), we derive that

$$\lambda^2_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(p) = \lambda_f^2(p^i)\lambda_f^2(p^j) = (i+1) + \sum_l \lambda_{\mathrm{sym}^{2l} f}(p),$$

where l ranges over certain positive integers. The analogous result also holds for $\lambda^2_{\mathrm{sym}^i g\times \mathrm{sym}^j g}(p)$, i.e.,

$$\lambda^2_{\mathrm{sym}^i g\times \mathrm{sym}^j g}(p) = \lambda_g^2(p^i)\lambda_g^2(p^j) = (i+1) + \sum_{l'} \lambda_{\mathrm{sym}^{2l'} g}(p),$$

where l' runs over certain positive integers.

Hence,

$$\begin{aligned}&\lambda^2_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(p)\lambda^2_{\mathrm{sym}^i g\times \mathrm{sym}^j g}(p) r_Q^*(p) \\ &\quad= \lambda_f^2(p^i)\lambda_f^2(p^j)\lambda_g^2(p^i)\lambda_g^2(p^j)(1+\chi_D(p)) \\ &\quad= \left((i+1) + \sum_l \lambda_{\mathrm{sym}^{2l} f}(p)\right)\left((i+1) + \sum_{l'} \lambda_{\mathrm{sym}^{2l'} g}(p)\right)(1+\chi_D(p)) \\ &\quad= (i+1)^2 + (i+1)\left(\sum_l \lambda_{\mathrm{sym}^{2l} f}(p) + \sum_{l'} \lambda_{\mathrm{sym}^{2l'} g}(p)\right) + \sum_l\sum_{l'} \lambda_{\mathrm{sym}^{2l} f\times \mathrm{sym}^{2l'} g}(p) \\ &\qquad+ \left((i+1)^2 + (i+1)\left(\sum_l \lambda_{\mathrm{sym}^{2l} f}(p) + \sum_{l'} \lambda_{\mathrm{sym}^{2l'} g}(p)\right)\right. \\ &\qquad\left.+ \sum_l\sum_{l'} \lambda_{\mathrm{sym}^{2l} f\times \mathrm{sym}^{2l'} g}(p)\right)\chi_D(p). \qquad (2.11)\end{aligned}$$

In the half-plane $\Re(s) > \frac{1}{2}$, the corresponding coefficients of the term p^{-s} determines the analytic properties of $\mathfrak{G}(f,g,i,j,\theta_Q;s)$. Therefore, using (2.9) and (2.11), we have

$$\mathfrak{G}(f,g,i,j,\theta_Q;s) = \zeta(s)^{(i+1)^2} L(s,\chi_D)^{(i+1)^2} \prod_l\prod_{l'} \Big(L(\mathrm{sym}^{2l} f, s) L(\mathrm{sym}^{2l'} g, s)$$

$$L(\mathrm{sym}^{2l} f\times \mathrm{sym}^{2l'} g, s) L(\mathrm{sym}^{2l} f\times\chi_D, s) L(\mathrm{sym}^{2l'} g\times\chi_D, s) L(\mathrm{sym}^{2l} f\times \mathrm{sym}^{2l'} g\times\chi_D, s)\Big) U_{f,g,i,j,2}(s),$$

where l, l' ranges over suitable positive integers, and the function $U_{f,g,i,j,2}(s)$ is a Dirichlet series converges uniformly and absolutely in the half-plane $\Re(s) \geq \frac{1}{2} + \varepsilon$. □

LEMMA 2.3. For $\Re(s) > 1$, we have

$$\mathfrak{F}(f, g, i, j, \theta_Q; s)$$
$$= \prod_{l_1=0}^{i} \prod_{l_2=0}^{i} \left(L(\mathrm{sym}^{i+j-2l_1} f \times \mathrm{sym}^{i+j-2l_2} g, s) L(\mathrm{sym}^{i+j-2l_1} f \times \mathrm{sym}^{i+j-2l_2} g \times \chi_D.s) \right) U_{f,g,i,j,1}(s),$$

where the function $U_{f,g,i,j,1}(s)$ admits the Dirichlet series which converges uniformly and absolutely in the half-plane $\Re(s) > \frac{1}{2}$ and $U_{f,g,i,j,1}(s) \neq 0$ for $\Re(s) = 1$.

Proof. Since $\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(n), \lambda_{\mathrm{sym}^i g \times \mathrm{sym}^j g}(n)$ and $r_Q^*(n)$ are multiplicative functions and also satisfies the trivial bound $O(n^\varepsilon)$ for any $\varepsilon > 0$, we have

$$\begin{aligned}\mathfrak{F}(f, g, i, j, \theta_Q; s) &= w_D \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(n) \lambda_{\mathrm{sym}^i g \times \mathrm{sym}^j g}(n) r_Q^*(n)}{n^s} \\ &= w_D \prod_p \left(1 + \sum_{r \geq 1} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(p^r) \lambda_{\mathrm{sym}^i g \times \mathrm{sym}^j g}(p^r) r_Q^*(p^r)}{p^{rs}} \right)\end{aligned}$$

for $\Re(s) > 1$. In the half-plane $\Re(s) > \frac{1}{2}$, the corresponding coefficients of the term p^{-s} determines the analytic properties of $\mathfrak{F}(f, g, i, j, \theta_Q; s)$. By the following relation

$$\begin{aligned}\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(p) \lambda_{\mathrm{sym}^i g \times \mathrm{sym}^j g}(p) r_Q^*(p) &= \lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(p) \lambda_{\mathrm{sym}^i g \times \mathrm{sym}^j g}(p)(1 + \chi_D(p)) \\ &= \lambda_f(p^i)\lambda_f(p^j) \times \lambda_g(p^i)\lambda_g(p^j)(1 + \chi_D(p)) \\ &= \sum_{l_1=0}^{i} \sum_{l_2=0}^{i} \lambda_{\mathrm{sym}^{i+j-2l_1} f \times \mathrm{sym}^{i+j-2l_2} g}(p)(1 + \chi_D(p)).\end{aligned}$$

Arguing similarly as that of Lemma 2.2, we can obtain the result. □

LEMMA 2.4. For any $\varepsilon > 0$, we have

$$\zeta(\sigma + it) \ll \left(1 + |t|\right)^{\max\{\frac{13}{42}(1-\sigma), 0\} + \varepsilon},$$
$$L(\sigma + it, \chi_D) \ll \left(1 + |t|\right)^{\max\{\frac{13}{42}(1-\sigma), 0\} + \varepsilon},$$

uniformly for $\frac{1}{2} \leq \sigma \leq 2$ and $|t| \geq 1$.

Proof. The first result follows from the Phragmén–Lindelöf principle for a strip [13, Theorem 5.53] and the works of Bourgain [2], and the second result can be obtained by following the similar argument in [26, Lemma 2.7]. □

LEMMA 2.5. For any $\varepsilon > 0$, one has

$$\int_1^T \left| \zeta\left(\frac{5}{7} + it \right) \right|^{12} dt \ll T^{1+\varepsilon}$$

uniformly for $T \geq 1$.

Proof. The result is established by Ivić [11]. □

LEMMA 2.6 ([32, Lemma 1]). For $\frac{1}{2} \leq \sigma \leq 2$, and T sufficiently large, there exists $T^* \in [T, T + T^{1/3}]$ such that the bound

$$\log \zeta(\sigma + iT^*) \ll (\log \log T^*)^2 \ll (\log \log T)^2$$

holds uniformly for $\frac{1}{2} \leq \sigma \leq 2$, and hence

$$\left|\zeta(\sigma + iT^*)\right| \ll \exp\left((\log \log T^*)^2\right) \ll T^\varepsilon \tag{2.12}$$

on the horizontal line with $t = T^*$ for $\frac{1}{2} \leq \sigma \leq 2$.

REMARK 2.7. Following the similar argument as in [26, Lemma 2.7], one can also obtain the same result for Dirichlet L-function $L(s, \chi_D)$.

Due to the recent work of Newton and Thorne [29, 30], we observe that the automorphic L-functions $L(\mathrm{sym}^j f, s), L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)$ $(i, j \geq 1)$ and its twisted L-functions are the general L-functions in the sense of Perelli [31]. For general L-functions, we have the following averaged and individual convexity bounds.

LEMMA 2.8. Suppose that $\mathfrak{L}(s)$ is a general L-function of degree m. Then we have

$$\int_T^{2T} |\mathfrak{L}(\sigma + it)|^2 dt \ll T^{\max\{m(1-\sigma),1\}+\epsilon}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \varepsilon$ and $T \geq 1$; and

$$\mathfrak{L}(\sigma + it) \ll \big(1 + |t|\big)^{\max\{\frac{m}{2}(1-\sigma),0\}+\varepsilon}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \varepsilon$ and $|t| \geq 1$.

Proof. This follows from the results of Perelli's mean value theorem and convexity bound for the general L-function in [31]. □

3. AUXILIARY RESULTS

In this section, we need to handle the two sums $S_{1,i,j}(x)$ and $S_{2,i,j}(x)$ in order to detect the sign changes of the subsequence given in Theorem 1.1. In the following, we give two important propositions.

PROPOSITION 3.1. We have

$$S_{1,i,j}(x) \ll_{f,g,\varepsilon,D} x^{1-\frac{1}{(i+1)^2(j+1)^2}+\varepsilon} \tag{3.1}$$

and

$$S_{2,i,j}(x) = xP_{(i+1)^2-1}(\log x) + O\big(x^{1-\frac{42}{42(i+1)^4(j+1)^4-16(i+1)^2-3}+\varepsilon}\big), \tag{3.2}$$

where $P_j(t)$ denotes a polynomial in t with degree j.

Proof. For $S_{1,i,j}(x)$, by Lemma 2.3 we have

$$\mathfrak{F}(f, g, i, j, \theta_Q; s) = \mathfrak{F}^*(f, g, i, j, \theta_Q; s)U_{f,g,i,j,1}(s) \tag{3.3}$$

with

$$\begin{aligned}\mathfrak{F}^*(f, g, i, j, \theta_Q; s) &= \prod_{l_1=0}^{i}\prod_{l_2=0}^{i} \big(L(\mathrm{sym}^{i+j-2l_1} f \times \mathrm{sym}^{i+j-2l_2} g, s)L(\mathrm{sym}^{i+j-2l_1} f \times \mathrm{sym}^{i+j-2l_2} g \times \chi_D, s)\big)\\ &=: \sum_{n=1}^{\infty} \frac{a(n)}{n^s},\end{aligned}$$

here the Dirichlet series $U_{f,g,i,j,1}(s) := \sum_{l=1}^{\infty} \frac{b(n)}{n^s}$ converges uniformly and absolutely in the half-plane $\Re(s) > \frac{1}{2}$.

It is not hard to find that $\mathfrak{F}^*(f, g, i, j, \theta_Q; s)$ is a general L-function with degree $2(i + 1)^2(j + 1)^2$ in the sense of Lemma 2.1. And from [29, 30] and the Rankin–Selberg theory for the convolution of two L-functions, the L-function $\mathfrak{F}^*(f, g, i, j, \theta_Q; s)$ can be extended to the whole plane as an entire function and satisfies certain Riemann zeta-type functional equation. Then

$$\sum_{n\leq x} a(n) \ll x^{1-\frac{1}{(i+1)^2(j+1)^2}+\varepsilon}.$$

It follows from (3.3) that

$$\lambda_{\mathrm{sym}^i f\times\mathrm{sym}^j f}(n)\lambda_{\mathrm{sym}^i g\times\mathrm{sym}^j g}(n)r_Q(n) = \sum_{n=ml} a(m)b(l),$$

and for $\Re(s) > \frac{1}{2}$ we have $\sum_{n=1}^{\infty} \frac{b(l)}{l^s} \ll 1$. Hence

$$\sum_{n\le x} \lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(n)\lambda_{\mathrm{sym}^i g\times \mathrm{sym}^j g}(n) r_Q(n) = \sum_{l\le x} b(l) \sum_{m\le \frac{x}{l}} a(m)$$
$$\ll x^{1-\frac{1}{(i+1)(j+1)}+\varepsilon} \sum_{l\le x} \frac{b(l)}{l^{1-\frac{1}{(i+1)^2(j+1)^2}+\varepsilon}}$$
$$\ll x^{1-\frac{1}{(i+1)^2(j+1)^2}+\varepsilon}.$$

This gives the upper bound for $S_{1,i,j}(x)$.

Now we turn to the asymptotic behaviour of $S_{2,i,j}(x)$. Applying Perron's formula (see e.g.[23, Theorem 2.1]) to the generating function $\mathfrak{G}(f, g, i, j, \theta_Q; s)$ in Lemma 2.2, we have

$$S_{2,i,j}(x) = \int_{\eta-iT}^{\eta+iT} \mathfrak{G}(f, g, i, j, \theta_Q; s)\frac{x^s}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T}\right), \tag{3.4}$$

where $\eta = 1 + \eta$. By shifting the line of integration in (3.4) to the parallel line $\Re(s) = \kappa := \frac{5}{7}$ and invoking the Cauchy residue theorem, then

$$\begin{aligned} S_{2,i,j}(x) &= \mathrm{Res}_{s=1}\left\{\mathfrak{G}(f, g, i, j, \theta_Q; s)\frac{x^s}{s}\right\} \\ &\quad + \frac{1}{2\pi i}\left\{\int_{\kappa-iT}^{\kappa+iT} + \int_{1+\varepsilon-iT}^{\kappa-iT} + \int_{\kappa+iT}^{1+\varepsilon+iT}\right\}\mathfrak{G}(f, g, i, j, \theta_Q; s)\frac{x^s}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T}\right) \\ &:= xP_{(i+1)^2-1}(\log x) + J_1 + J_2 + J_3 + O\left(\frac{x^{1+\varepsilon}}{T}\right), \end{aligned} \tag{3.5}$$

where $\eta := 1 + \varepsilon$, here $P_j(t)$ denotes a polynomial in t with degree j, and we make the special choice $T = T^*$ of Lemma 2.6 which satisfies (2.12), and $1 \le T \le x$ is some parameter to be chosen later. From [29, 30] and the Rankin–Selberg theory, we learn that $L(\mathrm{sym}^i f, s), L(\mathrm{sym}^j g, s)$ and $L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s), (i, j \ge 1)$ and its twisted L-functions are analytic functions in the range $\kappa \le \sigma \le \eta$ and $|t| \le T$. Hence $\mathfrak{G}(f, g, i, j, \theta_Q; s)$ has only a pole at $s = 1$ of order $(i + 1)^2$ coming from $\zeta(s)^{(i+1)^2}$.

For simplicity, we may write

$$\mathfrak{G}(f, g, i, j, \theta_Q; s) = \zeta(s)^{(i+1)^2} L(s, \chi_D)^{(i+1)^2} \mathfrak{G}^*(f, g, i, j, \theta_Q; s)\widetilde{\mathfrak{G}}(f, g, i, j, \theta_Q; s)U_{f,g,i,j,2}(s),$$

where

$$\mathfrak{G}^*(f, g, i, j, \theta_Q; s) = \prod_{l}\prod_{l'} \left(L(\mathrm{sym}^{2l} f, s)L(\mathrm{sym}^{2l'} g, s)L(\mathrm{sym}^{2l} f \times \mathrm{sym}^{2l'} g, s)\right),$$

and

$$\widetilde{\mathfrak{G}}(f, g, i, j, \theta_Q; s) = \prod_{l}\prod_{l'} \left(L(\mathrm{sym}^{2l} f \times \chi_D, s)L(\mathrm{sym}^{2l'} g \times \chi_D, s)L(\mathrm{sym}^{2l} f \times \mathrm{sym}^{2l'} g \times \chi_D, s)\right),$$

where l, l' ranges over suitable positive integers. It is clear from Lemma 2.2 that the L-function $\mathfrak{G}(f, g, i, j, \theta_Q; s)$ is of degree $2(i + 1)^4(j + 1)^4$, and the L-functions $\mathfrak{G}^*(f, g, i, j, \theta_Q; s)$ and $\widetilde{\mathfrak{G}}(f, g, i, j, \theta_Q; s)$ both have degree $(i + 1)^4(j + 1)^4 - (i + 1)^2$.

For J_1, by Hölder's inequality, we have

$$J_1 \ll x^{\kappa+\varepsilon} \sup_{1\le T_1\le T/2} \left\{T_1^{-1} \sup_{T_1\le t\le 2T_1} \left|\zeta(\kappa + it)\right|^{(i+1)^2-4} \left|L(f \times \chi_D, \kappa + it)\right|^{(i+1)^2} J_1(T_1)^{\frac{1}{3}} J_2(T_1)^{\frac{1}{2}} J_3(T_1)^{\frac{1}{6}}\right\} + x^{\kappa+\varepsilon}, \tag{3.6}$$

where

$$J_1(T_1) = \int_{T_1}^{2T_1} \left|\zeta\left(\frac{5}{7}+it\right)\right|^{12} dt,$$
$$J_2(T_1) = \int_{T_1}^{2T_1} \left|\mathfrak{G}^*\left(f,g,i,j,\theta_Q;\frac{5}{7}+it\right)\right|^{2} dt,$$
$$J_3(T_1) = \int_{T_1}^{2T_1} \left|\widetilde{\mathfrak{G}}\left(f,g,i,j,\theta_Q;\frac{5}{7}+it\right)\right|^{6} dt.$$

From Lemmas 2.4, 2.5 and 2.8, we have

$$J_1(T_1) \ll T_1^{1+\varepsilon},$$
$$J_2(T_1) \ll T_1^{((i+1)^4(j+1)^4-(i+1)^2)(1-\frac{5}{7})+\varepsilon} \ll T_1^{\frac{2}{7}((i+1)^4(j+1)^4-(i+1)^2)+\varepsilon},$$
$$J_3(T_1) \ll T_1^{3((i+1)^4(j+1)^4-(i+1)^2)(1-\frac{5}{7})+\varepsilon} \ll T_1^{\frac{6}{7}((i+1)^4(j+1)^4-(i+1)^2)+\varepsilon}.$$

Therefore, from (3.6), we get

$$\begin{aligned} J_1 &\ll x^{\kappa+\varepsilon} \sup_{1\le T_1\le T} T_1^{\frac{13}{42}\times\frac{2}{7}\times(2(i+1)^2-4)+\frac{1}{3}+\frac{1}{7}((i+1)^4(j+1)^4-(i+1)^2)+\frac{1}{7}((i+1)^4(j+1)^4-(i+1)^2)-1+\varepsilon} \\ &\ll x^{\frac{5}{7}+\varepsilon} T^{\frac{2}{7}(i+1)^4(j+1)^4-\frac{16}{147}(i+1)^2-\frac{50}{49}+\varepsilon}. \end{aligned} \tag{3.7}$$

The estimates for the integrals J_2 and J_3 over the horizontal segments are similar. By Lemma 2.4 and Lemma 2.6 (arguing similarly as that of e.g.[38, p.15, case 2]), we get

$$\begin{aligned} J_2+J_3 &\ll \sup_{\kappa\le\sigma\le\eta} x^{\sigma} T^{((i+1)^2\times 2\times\varepsilon+2((i+1)^4(j+1)^4-(i+1)^2)\times\frac{1}{2})(1-\sigma)+\varepsilon} T^{-1} \\ &\ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{5}{7}+\varepsilon} T^{\frac{2}{7}((i+1)^4(j+1)^4-(i+1)^2)-1+\varepsilon}. \end{aligned} \tag{3.8}$$

Combining (3.5)–(3.8), we obtain

$$S_{2,i,j}(x) = xP_{(i+1)^2-1}(\log x) + O\left(\frac{x^{1+\varepsilon}}{T}\right) + O\left(x^{\frac{5}{7}+\varepsilon} T^{\frac{2}{7}(i+1)^4(j+1)^4-\frac{16}{147}(i+1)^2-\frac{50}{49}+\varepsilon}\right). \tag{3.9}$$

On taking $T = x^{\frac{42}{42(i+1)^4(j+1)^4-16(i+1)^2-3}}$ in (3.9), we can obtain

$$S_{2,i,j}(x) = xP_{(i+1)^2-1}(\log x) + O\left(x^{1-\frac{42}{42(i+1)^4(j+1)^4-16(i+1)^2-3}+\varepsilon}\right).$$

This completes the proof of Proposition 3.1. □

4. PROOF OF THEOREM 1.2

In this section, we give the proof of Theorem 1.2 by the argument of contradiction by following the similar approach as that in [37].

Let

$$h := h(x) = x^{\theta} \qquad \text{with} \qquad 1 - \frac{42}{42(i+1)^4(j+1)^4 - 16(i+1)^2 - 3} < \theta < 1.$$

Suppose that the sequence $\{\lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(n)\lambda_{\mathrm{sym}^i g\times \mathrm{sym}^j g}(n)\}_{\substack{n=Q(X)\\ X\in\mathbb{Z}^2}}$ has the same sign (say positive) in the interval $(x, x+h]$, here $Q(X)$ is an integral binary quadratic form as in Theorem 1.2. Then by (3.1) and Deligne's bound (1.2), we have

$$\begin{aligned}
&\sum_{x<n\leq x+h} \lambda^2_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(n)\lambda^2_{\mathrm{sym}^i g\times \mathrm{sym}^j g}(n) r_Q(n) \\
&= \sum_{x<n\leq x+h} (\lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(n)\lambda_{\mathrm{sym}^i g\times \mathrm{sym}^j g}(n))(\lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(n)\lambda_{\mathrm{sym}^i g\times \mathrm{sym}^j g}(n) r_Q(n)) \\
&\ll_{f,g,D,\varepsilon} x^{\varepsilon} \sum_{x<n\leq x+h} \lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(n)\lambda_{\mathrm{sym}^i g\times \mathrm{sym}^j g}(n) r_Q(n) \\
&\ll_{f,g,D,\varepsilon} x^{\varepsilon}((x+h)^{1-\frac{1}{(i+1)^2(j+1)^2}+\varepsilon} + x^{1-\frac{1}{(i+1)^2(j+1)^2}+\varepsilon}) \\
&\ll_{f,g,D,\varepsilon} x^{1-\frac{1}{(i+1)^2(j+1)^2}+\varepsilon}. \qquad (4.1)
\end{aligned}$$

By the assumption on h and (3.2), we have

$$\sum_{x<n\leq x+h} \lambda^2_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(n)\lambda^2_{\mathrm{sym}^i g\times \mathrm{sym}^j g}(n) r_Q(n) = hP'_{(i+1)^2-1}(\log x) + O\left(x^{1-\frac{42}{42(i+1)^4(j+1)^4-16(i+1)^2-3}+\varepsilon}\right) \gg x^{\theta}, \qquad (4.2)$$

where $P'_{(i+1)^2-1}(\log x)$ is some polynomial of degree $(i+1)^2-1$. Then from (4.1) and (4.2) we get the contradiction. Hence from above we know that the sequence $\{\lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(n)\lambda_{\mathrm{sym}^i g\times \mathrm{sym}^j g}(n)\}_{\substack{n=Q(X)\\ X\in\mathbb{Z}^2}}$ has at least one sign change in the interval $(x, x+x^{\theta}]$ with $1-\frac{42}{42(i+1)^4(j+1)^4-16(i+1)^2-3} < \theta < 1$. Therefore, the sequence $\{\lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(n)\lambda_{\mathrm{sym}^i g\times \mathrm{sym}^j g}(n)\}_{\substack{n=Q(X)\\ X\in\mathbb{Z}^2}}$ has at least $\gg x^{1-\theta}$ sign changes in the interval $(x, 2x]$ for sufficiently large x.

ACKNOWLEDGEMENTS

The author would like to extend his sincere gratitude to Professors Guangshi Lü, Bin Chen, Bingrong Huang and Research fellow Zhiwei Wang for their constant encouragement and valuable suggestions. The author is extremely grateful to the anonymous referees for their meticulous checking, for thoroughly reporting countless typos and inaccuracies as well as for their valuable comments. These corrections and additions have made the manuscript clearer and more readable.

FUNDING

This work was financially supported in part by The National Key Research and Development Program of China (Grant No. 2021YFA1000700), Natural Science Basic Research Program of Shaanxi (Program Nos. 2023-JC-QN-0024, 2023-JC-YB-077), Foundation of Shaanxi Educational Committee (2023-JC-YB-013) and Shaanxi Fundamental Science Research Project for Mathematics and Physics (Grant No. 22JSQ010).

DECLARATION OF COMPETING INTEREST

The author declares that there is no conflict of interests regarding the publication of this paper.

DATA AVAILABILITY

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

REFERENCES

[1] Banerjee, S. and Pandey, M. K. Signs of Fourier coefficients of cusp form at sum of two squares. *Proc. Indian Acad. Sci. Math. Sci. 130*, 1 (2020), Paper No. 2, 9 pp.

[2] Bourgain, J. Decoupling, exponential sums and the Riemann zeta function. *J. Amer. Math. Soc. 30* (2017), 205–224.

[3] Clozel, L. and Thorne, J. A. Level-raising and symmetric power functoriality. I. *Compos. Math. 150* (2014), 729–748.

[4] Clozel, L. and Thorne, J. A. Level-raising and symmetric power functoriality. II. *Ann. of Math. 181* (2015), 303–359.

[5] Clozel, L. and Thorne, J. A. Level-raising and symmetric power functoriality. III. *Duke Math. J. 166* (2017), 325–402.

[6] Deligne, P. La Conjecture de Weil. I. *Inst. Hautes Études Sci. Pull. Math. 43* (1974), 273–307.

[7] Dieulefait, L. Automorphy of $Symm^5(GL(2))$ and base change. *J. Math. Pures Appl. (9) 104*, (4) (2015), 619–656.

[8] Gelbart, S. and Jacquet, H. A relation between automorphic representations of $GL(2)$ and $GL(3)$. *Ann. Sci. École Norm. Sup. 11*, 4 (1978), 471–542.

[9] Gun, S., Kohnen, W., and Rath, P. Simultaneous sign change of Fourier-coefficients of two cusp forms. *Arch. Math. 105* (2015), 413–424.

[10] Hua, G. D. The simultaneous sign changes of Hecke eigenvalues over an integral binary quadratic form. *Acta Math. Hungar. 167*, 2 (2022), 476–491.

[11] A. Ivić, Exponential pairs and the zeta function of Riemann. *Stud. Sci. Math. Hungar. 15* (1980), 157–181.

[12] Iwaniec, H. *Topics in Classical Automorphic Forms.* Grad. Stud. Math., Vol. 17, Amer. Math. Soc., (Providence, 1997).

[13] Iwaniec, H. and Kowalski, E. *Analytic Number Theory.* Amer. Math. Soc. Colloquium Publ., Vol. 53, Amer. Math. Soc., (Providence, 2004).

[14] Jacquet, H. and Shalika, J. A. On the Euler products and the classification of automorphic representations I. *Amer. J. Math. 103*, 3 (1981), 499–558.

[15] Jacquet, H. and Shalika, J. A. On the Euler products and the classification of automorphic forms II. *Amer. J. Math. 103* (1981), 777–815.

[16] Kim, H. and Shahidi, F. Functorial products for $GL_2 \times GL_3$ and functorial symmetric cube for GL_2, with an appendix by C. J. Bushnell and G. Heniart. *Ann. of Math. 155* (2002), 837–893.

[17] Kim, H. and Shahidi, F. Cuspidality of symmetric power with applications. *Duke Math. J. 112* (2002), 177–107.

[18] Kim, H. Functoriality for the exterior square of GL_4 and symmetric fourth of GL_2, Appendix 1 by D. Ramakrishan, Appendix 2 by H. Kim and P. Sarnak. *J. Amer. Math. Soc. 16* (2003), 139–183.

[19] Knopp, M., Kohnen, W., and Pribitkin, W. On the signs of Fourier coefficients of cusp forms. *Ramanujan J. 7* (2003), 269–277.

[20] Kohnen, W. and Sengupta, J. Signs of Fourier coefficients of two cusp forms of different weights. *Proc. Amer. Math. Soc. 137* (2009), 3563–3567.

[21] Kohnen, W. and Martin, Y. Sign changes of Fourier coefficients of cusp forms supported on prime power indices. *Int. J. Number Theory 10*, 8 (2014), 1921–1927.

[22] Kumari, M. and Murty, M. R. Simultaneous non-vanishing and sign changes of Fourier coefficients of modular forms. *Int. J. Number Theory 14*, 8 (2018), 2291–2301.

[23] Liu, J. Y. and Ye, Y. B. *Perron's formula and the prime number theorem for automorphic L-functions.* Pure Appl. Math. Q. 3, No. 2, 2007, pp. 481–497.

[24] Lau, Y.-K. and Lü, G. S. Sums of Fourier coefficients of cusp forms. *Q. J. Math. 62*, 3 (2011), 687–716.

[25] Lao, H. X. and Luo, S. Sign changes and non-vanishing of Fourier coefficients of holomorphic cusp forms. *Rocky Mountain J. Math. 51*, 5 (2021), 1701–1714.

[26] Liu, H. F. On the asymptotic distribution of Fourier coefficients of cusp forms. *Bull. Braz. Math. Soc. (N.S.) 54*, 21 (2023), 17 pp.
[27] Murty, M. R. Oscillations of the Fourier coefficients of modular forms. *Math. Ann. 262* (1983), 431–446.
[28] Meher, J., Shankhadhar, K. D., and Viswanadham, G. K. A short note on sign changes. *Proc. Indian Acad. Sci. (Math. Sci.) 123*, 3 (2013), 315–320.
[29] Newton, J. and Thorne, J. A. Symmetric power functoriality for holomorphic modular forms. *Publ. Math. Inst. Hautes Études Sci. 134* (2021), 1–116.
[30] Newton, J. and Thorne, J. A. Symmetric power functoriality for holomorphic modular forms. II. *Publ. Math. Inst. Hautes Études Sci. 134* (2021), 117–152.
[31] Perelli, A. General L-functions. *Ann. Mat. Pura Appl. 130* (1982), 287–306.
[32] Ramachandra, K. and Sankaranarayanan, A. Notes on the Riemann zeta-function. *J. Indian Math. Soc. (N.S.) 57*, 1-4 (1991), 67–77.
[33] Rudnick, Z. and Sarnak, P. Zeros of principal L-functions and random matrix theory. *Duke Math. J. 81* (1996), 269–322.
[34] Siegel, C. L. Berechnung von Zetafunctionen an ganzzanhligen Stellen. *Nachr. Akad. Wiss. Göttingen Math. Phys. K1. II 2* (1969), 87–102.
[35] Shahidi, F. On certain L-functions. *Amer. J. Math. 103* (1981), 297–355.
[36] Shahidi, F. Third symmetric power L-functions for $GL(2)$. *Compos. Math. 70* (1989), 245–273.
[37] Vaishya, L. Signs of Fourier coefficients of cusp forms at integers represented by an integral binary quadratic form. *Proc. Indian Acad. Sci. Math. Sci. 131*, 2 (2021), Paper No. 41, 14 pp.
[38] Venkatasubbareddy, K. and Sankaranarayanan, A. On the tetra, penta, hexa, hepta and octa product L-functions. *Eur. J. Math. 9*, 1 (2023), Paper No. 17, 24 pp.