Mathematica Pannonica ***New Series*** **29 /NS 3/ (2023) 2, 282–290**
DOI: 10.1556/314.2023.00029

HOW TO APPROACH STABILITY OF BI-CONTINUOUS SEMIGROUPS?

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Communicated by László Tóth

Original Research Paper
Received: Feb 17, 2023 • Accepted: Nov 15, 2023
First published online: Nov 24, 2023

ABSTRACT

This paper serves as a kick-off to address the question of how to define and investigate the stability of bi-continuous semigroups. We will see that the mixed topology is the key concept in this framework.

KEYWORDS

Bi-continuous semigroups, stability, mixed topology

MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 47D06; Secondary 35B35, 46A70

INTRODUCTION

The study of asymptotics of operator semigroups yields powerful tools for the exploration of the convergence of solutions of evolution equations. In the last decades, a lot of research has been dedicated to asymptotic behaviour of operator semigroups and especially to the study of the relations between their asymptotic and spectral properties, see for example the works of Batty [7, 8, 9], van Neerven [10, 45, 46], Eisner [6, 23, 24] or Glück [4, 19, 30].

However, this theory has been developed for strongly continuous one-parameter semigroups of linear operators, or C_0-semigroups for short. These operator semigroups yield solutions of abstract Cauchy problems associated to evolution equations. A comprehensive study of C_0-semigroups can for example be found in the monographs by Engel and Nagel [25], Goldstein [31] or Pazy [42]. Nevertheless, even if the theory of C_0-semigroups is powerful and well-developed it also has certain restrictions. For example stochastic differential equations, Ornstein–Uhlenbeck processes or Feller processes give generally speaking rise to transition semigroups which are not strongly continuous. To overcome these restrictions, one considers (weaker) locally convex topologies such that the operator semigroup becomes strongly continuous with respect to this new topology. This idea gave rise to the study of so-called bi-continuous semigroups.

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Bi-continuous semigroups, which have been introduced and investigated by Kühnemund [40, 41], are an auspicious approach to study operator semigroups which are not C_0-semigroups, i.e., they are not strongly continuous with respect to the Banach space norm. In the last years there has been a lot of development regarding the theory of bi-continuous semigroups, e.g., by Farkas [26, 27, 28], Budde [11, 13, 14, 15, 16] or Kruse [38, 39], just to mention a few.

In this paper we want to suggest how to study stability of bi-continuous semigroups. The problem that arises immediately is that one is not longer dealing with a norm but also with a locally convex topology. Of course, the theory of operator semigroups on locally convex space is well-developed, see for example [5, 17, 21, 35], however, stability of operator semigroups on locally convex spaces has only be treated, according to the best knowledge of the author, in the work [34] of Jacob and Wegner. It turns out that in the situation of locally convex spaces there are more concepts of stability in comparison to the classical C_0-semigroup case. As a matter of fact, we make use of the so-called mixed topology, cf. [47] or [18, Chapter I, Prop. 1.10], in order to deal with the norm topology and the additional locally convex topology at the same time.

The paper is organized as follows: in the first chapter, we recall some basic facts about bi-continuous semigroups and the mixed topology. Afterwards, in Section 2 we introduce stable bi-continuous semigroups and make some useful remarks regarding this stability concept. In the last section, we provide an example by means of multiplication and adjoint semigroups.

1. PRELIMINARIES

1.1. Bi-continuous semigroups

First of all, we recall the definition of bi-continuous semigroups. As mentioned above, the main idea is to equip the underlying Banach space with an additional locally convex topology. The following assumptions, as proposed by Kühnemund [41, Assump. 1], on the interaction between the norm topology and the locally convex topology will be made throughout the paper.

ASSUMPTION 1.1.
(i) τ is a locally convex Hausdorff topology and is coarser than the norm-topology on X, i.e., the identity map $(X, \|\cdot\|) \to (X, \tau)$ is continuous.
(ii) τ is sequentially complete on norm-bounded sets, i.e., every $\|\cdot\|$-bounded τ-Cauchy sequence in τ-convergent.
(iii) The dual space of (X, τ) is norming for X, i.e.,

$$\|x\| = \sup_{\substack{\varphi \in (X,\tau)' \\ \|\varphi\| \leq 1}} |\varphi(x)|.$$

In what follows, we call a triple $(X, \|\cdot\|, \tau)$ satisfying the previous assumptions a *bi-admissible space.*

Typical examples of spaces satisfying Assumption 1.1 are $\mathrm{C_b}(\mathbb{R})$, the space of bounded continuous functions, equipped with the supremum norm and the compact-open topology, the dual of a Banach space X' equipped with the dual norm and the weak*-topology, as well as the space of bounded linear operators on a Banach space $\mathscr{L}(E)$ together with the operator norm and the strong operator topology. Now, we introduce the notion of bi-continuous semigroups, cf. [41, Def. 3].

DEFINITION 1.2. Let X be a Banach space with norm $\|\cdot\|$ together with a locally convex topology τ, such that the conditions in Assumption 1.1 are satisfied. We call a family of linear bounded operators $(T(t))_{t\geq 0}$ a *bi-continuous semigroup* on X if the following holds.

(i) $(T(t))_{t\geq 0}$ satisfies the *semigroup property*, i.e., $T(t+s) = T(t)T(s)$ and $T(0) = I$ for all $s, t \geq 0$.
(ii) $(T(t))_{t\geq 0}$ is *strongly τ-continuous*, i.e., the map $\varphi_x : [0, \infty) \to (X, \tau)$ defined by $\varphi_x(t) := T(t)x$ is continuous for every $x \in X$.
(iii) $(T(t))_{t\geq 0}$ is *exponentially bounded*, i.e., there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq M\mathrm{e}^{\omega t}$ for each $t \geq 0$.

(iv) $(T(t))_{t\geq 0}$ is *locally-bi-equicontinuous*, i.e., if $(y_n)_{n\in\mathbb{N}}$ is a norm-bounded sequence in X which is τ-convergent to 0, then also $(T(s)y_n)_{n\in\mathbb{N}}$ is τ-convergent to 0 uniformly for $s \in [0, t_0]$ for each fixed $t_0 \geq 0$.

There has been a lot of research on bi-continuous semigroups in the past years. For more information regarding the theory of bi-continuous semigroups we refer the reader to the existing literature, for example to [16, 41] for generation type results, to [11, 12, 13, 14, 27, 28] for perturbation theory and [15, 29, 39] for other general theory and applications of bi-continuous semigroups.

1.2. The mixed topology

In this section, we review the so-called mixed topology. The mixed topology becomes interesting whenever one deals with a vector spaces with two different topologies. Especially, for our case of bi-continuous semigroups this is interesting as one deals both with a Banach space equipped with the norm topology and an additional locally convex topology. The idea came up when Alexiewicz started to study spaces with two norms [1, 2, 3]. Later on, Wiweger generalized this to vector spaces with locally convex topologies, cf. [47]. For unexplained topological notations in the context of locally convex topologies, we refer to [18] or [47].

DEFINITION 1.3. Let E be a vector space, $\mathcal{B}$ a bornology of countable type with basis $(B_n)_{n\in\mathbb{N}}$ and τ a locally convex topology on E. Assume that $\mathcal{B}$ and τ are compatible. For a sequence $(U_n)_{n\in\mathbb{N}}$ of absolutely convex τ-neighbourhoods of zeros we set

$$\gamma((U_n)_{n\in\mathbb{N}}) := \bigcup_{n\in\mathbb{N}} (U_1 \cap B_1 + \cdots + U_n \cap B_n).$$

The set of all such sets $\gamma((U_n)_{n\in\mathbb{N}})$ is a basis of neighbourhoods for zero for a locally convex topology on E. We will call this new locally convex topology the *mixed topology* and will denote this topology by $\gamma := \gamma(\mathcal{B}, \tau)$. If $\mathcal{B}$ is a bornology induced by a norm $\|\cdot\|$ on E we write $\gamma(\|\cdot\|, \tau)$.

The following properties are direct consequences of the construction in Definition 1.3, see for example [18, Chapter I, Prop. 1.5].

PROPOSITION 1.4. Let E be a vector space, $\mathcal{B}$ a bornology of countable type and τ a locally convex topology on E. Assume that $\mathcal{B}$ and τ are compatible. Denote by γ the corresponding mixed topology on E.

(i) γ is finer than τ.
(ii) γ and τ coincide on the sets in $\mathcal{B}$.
(iii) γ is the finest linear topology on E which coincides with τ on the sets in $\mathcal{B}$

Some authors use Proposition 1.4 as the definition of the mixed topology. The following result is important to characterize convergence with respect to the mixed topology, cf. [47, Thm. 2.3.1] or [18, Chapter I, Prop. 1.10].

PROPOSITION 1.5. Let E be a vector space, $\mathcal{B}$ a bornology of countable type with basis $(B_n)_{n\in\mathbb{N}}$ and τ a locally convex topology on E. Furthermore, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E. The following assertions are equivalent:

(a) $x_n \to x$ as $n \to \infty$ with respect to the mixed topology γ.
(b) $(x_n)_{n\in\mathbb{N}}$ is $\mathcal{B}$-bounded and $x_n \to x$ as $n \to \infty$ with respect to τ.

Locally convex topologies, can either be characterized by a neighbourhood basis of zero or by continuous seminorms, which sometimes is more convenient, see for example [43]. As it is pointed out in [18, p. 41] the mixed topology can also be characterized by seminorms. To do so, let $\mathscr{P}_\tau$ be the family of continuous seminorms generating the locally convex topology τ. Now, for each $n \in \mathbb{N}$ we choose a seminorm $p_n \in \mathscr{P}_\tau$ and define

$$p(x) := \inf\left\{\sum_{k=0}^{n} p_k(x_k) : x = x_1 + \cdots + x_n, x_k \in B_k, 1 \leq k \leq n\right\}, \quad x \in X.$$

Then the set of all such seminorms determines the mixed topology, see also [20]. This characterisation is not very useful for applications, but in certain cases a much simpler representation for the seminorms can be given, cf. [18, p. 41] and [26, Def. A.1.1]

DEFINITION 1.6. Let $(X, \|\cdot\|, \tau)$ be a bi-admissible space according to Assumption 1.1. For a sequence $(p_n)_{n\in\mathbb{N}}$ in $\mathscr{P}_\tau$ and a sequence $(a_n)_{n\in\mathbb{N}} \in \mathrm{c}_0$ with $a_n \geq 0$ for all $n \in \mathbb{N}$ define a seminorm

$$\widetilde{p}_{(p_n,a_n)_{n\in\mathbb{N}}}(x) := \sup_{n\in\mathbb{N}} a_n p_n(x), \quad x \in X.$$

By taking all those seminorms together, say

$$\widetilde{\mathscr{P}} := \left\{ \widetilde{p}_{(p_n,a_n)_{n\in\mathbb{N}}} : (p_n)_{n\in\mathbb{N}} \subseteq \mathscr{P}_\tau, (a_n)_{n\in\mathbb{N}} \in \mathrm{c}_0, a_n \geq 0 \right\},$$

this determines another locally convex topology, the so-called *submixed topology* $\widetilde{\gamma}$.

By construction, one has that $\tau \subseteq \widetilde{\gamma} \subseteq \gamma$ and $\widetilde{\gamma}$ has the same convergent sequences as γ. In some situations, the mixed topology and the submixed topology even coincide, cf. [18, Chapter I, Prop. 4.5] and [47, Thm. 3.1.1].

PROPOSITION 1.7. Let $(X, \|\cdot\|, \tau)$ be a bi-admissible space according to Assumption 1.1. Assume that either

(i) the closed unit ball $B_{\|\cdot\|} := \{x \in X : \|x\| \leq 1\}$ is τ-compact or,
(ii) for every $x \in X$, $\varepsilon > 0$ and $p \in \mathscr{P}_\tau$ there exist $y, z \in X$ such that $x = y + z$, $p(z) = 0$ and $\|y\| \leq p(x) + \varepsilon$.

Then $\gamma = \widetilde{\gamma}$.

2. STABILITY OF BI-CONTINUOUS SEMIGROUPS

To study the stability of bi-continuous semigroups, one has to take both the norm topology and the locally convex topology τ into account. Stability of C_0-semigroups on Banach spaces has been extensively studied and can for example be found in the monograph by Eisner [23]. A few years back, Jacob and Wegner investigated the asymptotics of C_0-semigroups on locally convex spaces where the situation actually changes entirely, cf. [34]. We want to make use of those results as one can consider bi-continuous semigroups as operator semigroups on spaces equipped with the mixed topology. For the sake of completeness, we recall the definition of C_0-semigroups on locally convex spaces, see for example [5, 17, 21, 35].

DEFINITION 2.1. Let X be a locally convex space equipped with a locally convex topology τ. We say that a family $(T(t))_{t\geq 0}$ of linear and continuous operators on X is a C_0*-semigroup* if the following holds.

(i) $T(t+s) = T(t)T(s)$ and $T(0) = \mathrm{I}$ for all $s, t \geq 0$.
(ii) $\tau\lim_{t\to t_0} T(t)x = T(t_0)x$ for all $t_0 \geq 0$ and all $x \in X$.

In addition, we call a C_0-semigroup *bounded* if for each bounded subset $B \subseteq X$ and each $p \in \mathscr{P}_\tau$ there exists a constant $C \geq 0$ such that

$$\sup_{x\in B} p(T(t)x) \leq C$$

for all $t \geq 0$.

We recall the following stability notions for semigroups on locally convex spaces, cf. [34, Def. 2.2].

DEFINITION 2.2. Let (X, τ) be a locally convex space and $(T(t))_{t\geq 0}$ a C_0-semigroup on X. We say that $(T(t))_{t\geq 0}$ is

(i) *strongly exponentially stable* if $\forall x \in X\ \exists \omega > 0 :\ \tau\lim_{t\to\infty} \mathrm{e}^{\omega t}T(t)x = 0$,
(ii) *strongly stable* if $\forall x \in X :\ \tau\lim_{t\to\infty} T(t)x = 0$,
(iii) *uniformly stable* if for each bounded subset of X and for each $p \in \mathscr{P}_\tau$ one has that $\lim_{t\to\infty} \sup_{x\in B} p(T(t)x) = 0$.

Following the line of the previous definition, we now introduce asymptotic behavior for bi-continuous semigroups (which actually will be operator semigroups with asymptotic behavior with respect to the mixed topology).

DEFINITION 2.3. Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup on a Banach space X with respect to the locally convex topology τ. By γ we denote the mixed topology on X. We say that $(T(t))_{t\geq 0}$ is

(i) a *strongly exponentially stable bi-continuous semigroup* if
$$\forall x \in X\ \exists \omega > 0 : \ \gamma \lim_{t\to\infty} \mathrm{e}^{\omega t} T(t)x = 0,$$

(ii) a *strongly stable bi-continuous semigroup* if
$$\forall x \in X : \ \gamma \lim_{t\to\infty} T(t)x = 0,$$

(iii) a *uniformly stable bi-continuous semigroup* if for each γ-bounded subset and every $p \in \mathscr{P}_\gamma$ one has $\lim_{t\to\infty} \sup_{x\in B} p(T(t)x) = 0$.

We observe that we can substitute the γ-boundedness in Definition 2.3(iii) by $\|\cdot\|$-boundedness by [18, Chapter I, Prop. 1.11].

REMARK 2.4. It is noteworthy, that the concept of strong continuity of a semigroup on a locally convex space (Definition 2.1) and the concept of strong continuity of a bi-continuous semigroup (Definition 1.2) differ. In fact, Definition 2.1 requires that each operator $T(t)$, $t \geq 0$, is both linear and continuous. In general, the continuity condition is not fulfilled for a bi-continuous semigroup $(T(t))_{t\geq 0}$ when considering $(T(t))_{t\geq 0}$ an an operator semigroup with respect to the mixed topology, cf. [29, Ex. 4.1]. Effectively, Definition 1.2 only requires each operator $T(t)$, $t \geq 0$, to be continuous with respect to the Banach space norm. Nonetheless, the stability concepts of Definition 2.3 still make sense.

Recall from [44, p. 273] that a locally convex space X is called C-*sequential*, if every convex sequentially open subset of X is already open. Moreover, a locally convex space is called *semi-Montel* if its bounded sets are relatively compact. These concepts are important for the following result which in fact transfers [34, Thm. 2.7] to the setting of bi-continuous semigroups. As a matter of fact, we consider C-sequential semi-Montel spaces in the result below as Montel spaces, used in [34, Thm. 2.7], are barrelled semi-Montel spaces, which is a too strong assumption as in this case $\tau = \|\cdot\|$ by [18, Chapter I, Prop. 1.15].

PROPOSITION 2.5. Let X be a Banach space equipped with an additional locally convex topology such that Assumption 1.1 is satisfied. Let γ denote the mixed topology and assume that (X, γ) is a semi-Montel space. Furthermore, let $(T(t))_{t\geq 0}$ be bi-continuous semigroup which is γ-equicontinuous on X. Then, $(T(t))_{t\geq 0}$ is a uniformly stable bi-continuous semigroup if and only if it is a strongly stable bi-continuous semigroup.

Proof. The implication "$\Longrightarrow$" is obvious. For the implication "$\Longleftarrow$" let $\mathscr{P}_\gamma$ denote the family of continuous seminorms generating the mixed topology γ. By assumption $(T(t))_{t\geq 0}$ is γ-equicontinuous, i.e., for given $q \in \mathscr{P}_\gamma$ there exists $p \in \mathscr{P}_\gamma$ and $C \geq 0$ such that $q(T(t)x) \leq Cp(x)$ for all $x \in X$ and $t \geq 0$. Let $\varepsilon > 0$ and $B \in \mathcal{B}_X$ be arbitrary but fixed. Then we notice that for $\varepsilon_0 := \frac{\varepsilon}{1+C}$ the sets $B_p(x, \varepsilon_0) := \{y \in X : p(x - y) < \varepsilon_0\}$, where $x \in \overline{B}$, forms an open cover for $\overline{B}$. Due to the assumption that X is semi-Montel, there exist $x_1, \dots, x_n \in \overline{B}$ such that $\overline{B} \subseteq \bigcup_{m=1}^{n} B_p(x_m, \varepsilon)$. For fixed $1 \leq m \leq n$ there exists $t_m \geq 0$ such that $q(T(t)x_m) < \varepsilon_0$ for $t \geq t_m$. Define $t_* := \max_{1\leq m\leq n} t_m$ and let $x \in B \subseteq \overline{B}$ be arbitrary. Then there exists $1 \leq m \leq n$ such that $x \in B_p(x_m, \varepsilon_0)$. For $t \geq t_*$ we see that
$$q(T(t)x) \leq q(T(t)(x_n - x)) + q(T(t)x_n) \leq Cp(x - x_n) + q(T(t)x_n) < C\varepsilon_0 + \varepsilon_0 = \varepsilon$$
As $x \in B$ was arbitrarily chosen, obtain that $\sup_{x\in B} q(T(t)x) < \varepsilon$ for all $t \geq t_*$. □

REMARK 2.6. One is able to formulate a weaker version of Proposition 2.5. Under the assumption that (X, γ) is both C-sequential and semi-Montel, by [39, Thm. 3.17] or [36, Thm. 7.4] the semigroup $(T(t))_{t\geq 0}$ is quasi-equicontinuous with respect to the mixed topology γ. If one assumes that the semigroup is exponential stable in the sense that there exists $M \geq 0$ and $\omega < 0$ such that $\|T(t)\| \leq M e^{\omega t}$ for all $t \geq 0$, then the γ-equicontinuity of $(T(t))_{\geq 0}$ follows automatically. However, this is in general already a strong assumption one has to make on the operator semigroup itself.

3. EXAMPLES

3.1. The multiplication semigroup

Consider the space $X := \mathrm{C_b}(\mathbb{R})$ of bounded continuous functions over $\mathbb{R}$ equipped with the supremum norm $\|\cdot\|_\infty$ and the compact-open topology τ_{co}. As mentioned above, the space $\mathrm{C_b}(\mathbb{R})$ satisfies Assumption 1.1, see also [40, Ex. 1.6]. Recall that τ_{co} is induced by the family of seminorms given by

$$p_K(f) := \sup_{x\in K} |f(x)|, \quad K \subseteq \mathbb{R}, f \in \mathrm{C_b}(\mathbb{R}),$$

where K is a compact subset of $\mathbb{R}$. It was shown by Wiweger [47, Example D)] that $\mathrm{C_b}(\mathbb{R})$ satisfies Proposition 1.7(ii) so that the mixed and the submixed topology coincide on $\mathrm{C_b}(\mathbb{R})$. This fact was also used by Goldys et al. when they studied diffusion semigroups on spaces of continuous functions or Markov processes, cf. [32, 33].

Let $q : \mathbb{R} \to \mathbb{C}$ be a continuous function satisfying $\sup_{x\in\mathbb{R}} \mathrm{Re}(q(x)) < \infty$ and consider on $\mathrm{C_b}(\mathbb{R})$ the multiplication semigroup given by

$$(T(t)f)(x) := e^{tq(x)} f(x), \quad t \geq 0, f \in \mathrm{C_b}(\mathbb{R}), x \in \mathbb{R}.$$

Then, $(T(t))_{t\geq 0}$ is a bi-continuous semigroup on $\mathrm{C_b}(\mathbb{R})$ with respect to the compact-open topology, cf. [13, Sect. 6.2]. We now will investigate the stability of this bi-continuous semigroup according to Definition 2.3. Firstly, assume that $\sup_{x\in\mathbb{R}} \mathrm{Re}(q(x)) < 0$, then

$$\widetilde{p}_{(p_{K_n},a_n)_{n\in\mathbb{N}}}(T(t)f) = \sup_{n\in\mathbb{N}} \sup_{x\in K_n} a_n e^{t\mathrm{Re}(q(x))} |f(x)| \leq \widetilde{p}_{p_{K_n},a_n}(f) \cdot \sup_{n\in\mathbb{N}} \sup_{x\in K_n} e^{t\mathrm{Re}(q(x))} \overset{t\to\infty}{\to} 0,$$

so that $(T(t))_{t\geq 0}$ is a strongly stable bi-continuous semigroup according to Definition 2.3. The previous estimate also shows that under the assumption that $\sup_{x\in\mathbb{R}} \mathrm{Re}(q(x)) < 0$ the semigroup $(T(t))_{t\geq 0}$ is equicontinuous. We also obtain for any $\|\cdot\|$-bounded (or equivalently γ-bounded) subset $B \subseteq \mathrm{C_b}(\mathbb{R})$ that

$$\sup_{f\in B} \widetilde{p}_{(p_{K_n},a_n)_{n\in\mathbb{N}}}(T(t)f) \leq \sup_{f\in B} \widetilde{p}_{p_{K_n},a_n}(f) \cdot \sup_{n\in\mathbb{N}} \sup_{x\in K_n} e^{t\mathrm{Re}(q(x))} \leq \sup_{f\in B} \sup_{n\in\mathbb{N}} a_n \|f\|_\infty \cdot \sup_{n\in\mathbb{N}} \sup_{x\in K_n} e^{t\mathrm{Re}(q(x))}$$

which also tends to 0 for $t \to \infty$, showing that $(T(t))_{t\geq 0}$ is a uniformly stable bi-continuous semigroup according to Definition 2.3. Hence, the strong stability that we observed previously is in fact just a conclusion for that. However, we want to emphasize that the other implication, i.e., Proposition 2.5 is not applicable here as $\mathrm{C_b}(\mathbb{R})$ fails to be semi-Montel due to the fact that the closed $\|\cdot\|_\infty$-unit ball of $\mathrm{C_b}(\mathbb{R})$ is not γ-compact, see also [37, Rem. 2.5(a)].

If we just assume that $\sup_{x\in\mathbb{R}} \mathrm{Re}(q(x)) < \infty$, then for $\omega > 0$ we have

$$e^{\omega t} \cdot \widetilde{p}_{(p_{K_n},a_n)_{n\in\mathbb{N}}}(T(t)f) = e^{\omega t} \cdot \sup_{n\in\mathbb{N}} \sup_{x\in K_n} a_n e^{t\mathrm{Re}(q(x))} |f(x)| \qquad \leq \widetilde{p}_{p_{K_n},a_n}(f) \cdot \sup_{n\in\mathbb{N}} \sup_{x\in K_n} e^{t(\omega+\mathrm{Re}(q(x)))},$$

showing that $(T(t))_{t\geq 0}$ is not a strongly exponentially stable bi-continuous semigroup excepted for the case when $\sup_{x\in\mathbb{R}} \mathrm{Re}(q(x)) < 0$ which implies strongly stability for this bi-continuous semigroup.

3.2. The adjoint semigroup

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. Then we can consider the adjoint semigroup $(T'(t))_{t\geq 0}$ on the dual Banach space X'. Equipped with the dual norm $\|\cdot\|_{X'}$ and the weak*-topology τ_{w^*} the dual space X' satisfies Assumption 1.1, see also [40, p. 78]. Recall that the weak*-topology is generated by the family of seminorms given by

$$p_x(\varphi) = |\varphi(x)|, \quad \varphi \in X', x \in X.$$

The dual semigroup $(T'(t))_{t\geq 0}$ on the dual space X' is generally speaking not strongly continuous [40, p. 77] as one can see by considering the left translation semigroup on $\mathrm{L}^1(\mathbb{R})$, cf. [25, Chapter I, Sect. 4(c)]. However, dual semigroups of C_0-semigroups are bi-continuous semigroups with respect to the weak*-topology on X', cf. [40, Prop. 3.18]. We will now apply our theory to dual semigroups.

REMARK 3.1. Let us comment on the mixed topology $\gamma\left(\|\cdot\|_{X'}, \tau_{\mathrm{w}^*}\right)$ which is needed for our stability concepts:

(a) From [39, Rem. 3.19(b)] we obtain that $\gamma\left(\|\cdot\|_{X'}, \tau_{\mathrm{w}^*}\right)$ is C-sequential if X is separable.

(b) Moreover, by [37, Rem. 2.5(a)] the space (X', γ) is semi-Montel as Proposition 1.7(i) is fullfilled, see also [39, Ex. 3.11(b)]. In particular, the mixed and submixed topology coincide in this case.

Let us consider the nilpotent left-translation semigroup $(T(t))_{t\geq 0}$ on the Banach space $X = \mathrm{L}^1([0,1])$. This semigroup is a strongly stable C_0-semigroup according to [23, Chapter III, Ex. 3.2]. As $\mathrm{L}^1([0,1])$ is known to be separable, we conclude by Remark 3.1 that the dual space $\mathrm{L}^\infty([0,1])$ equipped with the mixed topology γ is C-sequential and semi-Montel. The adjoint semigroup $(T'(t))_{t\geq 0}$ is the nilpotent right-translation semigroup on $\mathrm{L}^\infty([0,1])$. This semigroup is a strongly stable bi-continuous semigroup. Indeed, let $f \in \mathrm{L}^\infty([0,1])$ and $(g_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{L}^1([0,1])$, then

$$\widetilde{p}_{(p_{g_n}, a_n)_{n\in\mathbb{N}}}(T'(t)f) = \sup_{n\in\mathbb{N}} a_n \left| \int_0^1 (T'(t)f)(x) g_n(x)\,\mathrm{d}x \right| = \sup_{n\in\mathbb{N}} a_n \left| \int_0^1 f(x)(T(t)g_n)(x)\,\mathrm{d}x \right| \to 0$$

uniformly in $n \in \mathbb{N}$ as $T(t) = 0$ for all for all $t > 1$ and $(a_n)_{n\in\mathbb{N}}$ is bounded. One can also show strong stability when considering the Banach space valued spaces $\mathrm{L}^1([0,1], Y)$ as this also has been done [15] in the framework of flows on networks. However, one has to assume that Y' has the Radon–Nikodym property as in this case the dual space of $\mathrm{L}^1([0,1], Y)$ will be $\mathrm{L}^\infty([0,1], Y')$, see also [22, Chapter IV, Section 1]. However, also in this case Proposition 2.5 and Remark 2.6 fail as $(T'(t))_{t\geq 0}$ is neither γ-equicontinuous nor exponentially stable.

REMARK 3.2. The above example is somehow a toy example, especially as the semigroup is nilpotent and therefore also exponentially stable in the classical sense. It is noteworthy that if the C_0-semigroup $(T(t))_{t\geq 0}$ is weakly stable, then the adjoint semigroup $(T'(t))_{t\geq 0}$ is weak*-stable. However, the whole point of the stability discssion for bi-continuous semigroups in this paper is to take both the norm and the locally convex topology into account by means of the mixed topology.

ACKNOWLEDGEMENT

The author wants to thank the anonymous referee for his fruitful and valuable feedback which helped to improve the article. This work is based on the research supported by the National Research Foundation (Grant number: SRUG220317136). It is acknowledged that opinions, findings and conclusions or recommendations expressed in any publication generated by this supported research is that of the author(s). The National Research Foundation accepts no liability whatsoever in this regard.

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