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CONSTRUCTION OF NEW OPERATORS BY COMPOSITION OF INTEGRAL-TYPE OPERATORS AND DISCRETE OPERATORS

Ulrich ABEL[1] and Vijay GUPTA[2,*]

[1] Department MND, Technische Hochschule Mittelhessen, Wilhelm-Leuschner-Straße 13, 61169 Friedberg, Germany
[2] Department of Mathematics, Netaji Subhas University of Technology, Sector 3 Dwarka, New Delhi 110078, India

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ABSTRACT

In this paper, we propose some new positive linear approximation operators, which are obtained from a composition of certain integral type operators with certain discrete operators. It turns out that the new operators can be expressed in discrete form. We provide representations for their coefficients. Furthermore, we study their approximation properties and determine their moment generating functions, which may be useful in finding several other convergence results in different settings.

KEYWORDS

Approximation by positive operators, moment generating function, rate of convergence, Voronovskaya-type formula

MATHEMATICS SUBJECT CLASSIFICATION (2020)

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1. INTRODUCTION

In the theory of approximation, linear operators play an important role. Several operators are proposed in the last seven decades. Actually in the late sixties an integral modification of the Bernstein polynomial was introduced by Durrmeyer. Eighteen years later, the Durrmeyer-type modification of Szász–Mirakyan and Baskakov operators were proposed in the year 1985 independently by different researchers. Since then, many other Durrmeyer-type hybrids and standard forms have been proposed and their characteristics discussed.

The new operators are of great interest for researchers in constructive approximation theory. Some operators are constructed through generating functions, some exponential-type operators are constructed through differential equations. In this paper we provide a new and different approach to construct new operators through composition methods.

Recently Gupta [10] obtained two new operators by combining exponential operator associated to x^3 with Szász–Mirakyan and Laguerre type operators.

* Corresponding author. E-mail: vijaygupta2001@hotmail.com

The recent article [5] continues a series of papers on composition and decomposition of positive linear operators. In particular it contains an observation [5, Eq. (25)] that the Baskakov operators are compositions of the Post-Widder and Szász–Mirakyan operators. This was previously mentioned in [8, page 2066].

We observe here that if we take the composition of some integral-type operator with a discrete operator, we get a purely discrete operator. More precisely, we take the composition $U_{m,n} = V_m \circ W_n$ of two operators

$$(V_m f)(x) = \int_I k_m(x,t)\, f(t)dt,$$

where I is a certain real interval and $k_m(x,t)$ is a kernel function, and

$$(W_n f)(x) = \sum_k w_{n,k}(x) f\left(x_{n,k}\right).$$

For a suited class of functions f, we obtain a discrete operator

$$\left(U_{m,n} f\right)(x) = \sum_k u_{k,m,n}(x) f\left(x_{n,k}\right).$$

In several concrete cases it is possible to give a closed expression for the coefficients

$$u_{k,m,n}(x) = \int_I k_m(x,t)\, w_{n,k}(t)\, dt.$$

Throughout the paper we tacitly assume that the functions f have the property that interchanging the order of the limit processes integration and summation is justified. We provide some new operators and establish their moment generating functions. Also we give some basic convergence results for the new operators including a Voronovskaja-type formula.

2. PRELIMINARIES

We start with some notation. For positive integers n, let

$$z^{\underline{n}} = z(z-1)\cdots(z-n+1), \quad z^{\underline{0}} = 1,$$
$$z^{\overline{n}} = z(z+1)\cdots(z+n-1), \quad z^{\overline{0}} = 1$$

denote the falling and the rising factorial, respectively. Frequently, we use the Pochhammer symbol $(z)_n \equiv z^{\overline{n}} = \Gamma(z+n)/\Gamma(z)$, for the rising factorial, which is common in the theory of hypergeometric functions. Throughout the paper we consider, for each real A, the function $\exp_A$ defined by $\exp_A(x) = \exp(Ax)$. Furthermore, we denote by e_r $(r = 0,1,2,\ldots)$ the monomials $e_r(x) = x^r$. For real numbers x, we define the function $\psi_x = e_1 - xe_0$.

At the beginning we list the approximation operators which are treated in this paper.

The Bernstein polynomial associated to a function $f: [0,1] \to \mathbb{R}$ is defined by

$$(B_n f)(x) = \sum_{\nu=0}^{n} p_{n,\nu}(x) f\left(\frac{\nu}{n}\right),$$

where $p_{n,\nu}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}$ are the Bernstein basis polynomials.

The Bernstein–Durrmeyer operator is defined by

$$(\overline{B}_n f)(x) = (n+1)\sum_{\nu=0}^{n} p_{n,\nu}(x) \int_0^1 p_{n,\nu}(t)\, f(t)dt,$$

provided that all integrals exist.

The Szász–Mirakyan operators S_n associate to each function $f: [0,+\infty) \to \mathbb{R}$ of at most exponential growth the function

$$(S_n f)(x) = \sum_{k=0}^{\infty} s_k(nx)\, f\left(\frac{k}{n}\right),$$

where $s_k(x) = e^{-x}x^k/k!$ $(k = 0,1,2,\ldots)$.

Their Durrmeyer variant is defined by

$$(\overline{S}_n f)(x) = n \sum_{j=0}^{\infty} s_j\,(nx) \int_0^{\infty} s_j(nt) f(t) dt,$$

provided that all integrals exist. Note that $|f(t)| = O(e^{\gamma t})$ as $t \to +\infty$ implies that $\left(\overline{S}_n f\right)(x)$ is well-defined, for $n > \gamma$.

The Post–Widder operator is given by

$$(P_n f)(x) = \frac{n^n}{x^n} \frac{1}{\Gamma(n)} \int_0^{\infty} e^{-nt/x} t^{n-1} f(t)\, dt.$$

The operator due to Jain and Pethe [15] in the slightly modified form given by Abel and Ivan [1] is defined by

$$(L_{n,c} f)(x) = \left(\frac{c}{1+c}\right)^{ncx} \sum_{k=0}^{\infty} \frac{(ncx)_k}{k!\,(1+c)^k} f\left(\frac{k}{n}\right).$$

The Müller gamma operator is given by

$$(G_n f)(x) = \frac{x^{n+1}}{n!} \int_0^{\infty} t^n e^{-xt} f\left(\frac{n}{t}\right) dt.$$

The moment generating function (m.g.f.) of a linear operator U on a space of functions, is defined by

$$\left(U \exp_A\right)(x),$$

provided that $U \exp_A$ exists. It can be utilized to obtain the moments Ue_r of the operator U via

$$(Ue_r)(x) = \left[\left(\frac{d}{dA}\right)^r \left(U \exp_A\right)(x)\right]_{A=0}.$$

The m.g.f. of several approximation operators are gathered in [12]. Direct computation confirms the m.g.f. of the above-mentioned operators:

- Szász–Mirakyan operator [12, Eq. (8)]:

$$\left(S_n \exp_A\right)(x) = \exp\left(nx\left(e^{A/n} - 1\right)\right) \tag{2.1}$$

- Szász–Mirakyan–Durrmeyer operator:

$$(\overline{S}_n \exp_A)(x) = \frac{n}{n-A} \exp\left(\frac{nAx}{n-A}\right) \qquad (n > A) \tag{2.2}$$

- Post–Widder operator:

$$\left(P_n \exp_A\right)(x) = \left(\frac{n}{n-Ax}\right)^n \qquad (n > Ax) \tag{2.3}$$

- The operator due to Jain and Pethe:

$$\left(L_{n,c} \exp_A\right)(x) = \left(\frac{c}{1+c-e^{A/n}}\right)^{ncx}, \tag{2.4}$$

  for $n > A/\log(1+c)$; (see [13, (1.9.4)]).
- Müller gamma operator: For real numbers $A < 0$, it holds

$$(G_n \exp_A)(x) = \frac{2}{n!} (-Anx)^{(n+1)/2} K_{n+1}\left(2\sqrt{-Anx}\right),$$

  where the modified Bessel function of the second kind is defined by (6.1); see Lemma 6.2.

In the following sections we determine the m.g.f. for the new composition operators presented in this paper.

For studying the approximation of the new composition operators, we apply the following general approximation theorem [4, Theorem 2.1]. Let $I \subseteq \mathbb{R}$ be an interval and let $C_b(I)$ be the space of all real-valued continuous and bounded functions on I.

PROPOSITION 2.1. Let $W_n : C_b(I) \to C_b(I)$ and $W : C_b(I) \to C_b(I)$ be positive linear operators. Furthermore, let $Z^x, Z_1^x, Z_2^x, \dots$ be I-valued random variables with probability distributions depending on a parameter $x \in I$. Suppose that for each $f \in C_b(I)$ the functions $x \mapsto Ef(Z^x)$ and $x \mapsto Ef\left(Z_n^x\right)$, $x \in I$, $n \geq 1$, are continuous on I. Suppose that for each $s \in \mathbb{R}$ and $x \in I$,

$$\lim_{n\to\infty} \left(W_n \exp_{is}\right)(x) = \left(W \exp_{is}\right)(x)$$

and, for each $x \in I$, the function $s \mapsto \left(W \exp_{is}\right)(x)$ is continuous on $\mathbb{R}$. Then,

$$\lim_{n\to\infty} (W_n f)(x) = (Wf)(x)$$

for all $f \in C_b(I)$ and $x \in I$.

Here $W_n \exp_{is}$ is defined via $\exp_{is}(t) = \cos(st) + i \sin(st)$.

3. COMPOSITION OF BERNSTEIN–DURRMEYER OPERATOR AND BERNSTEIN OPERATOR

The composition of Bernstein–Durrmeyer operator $\overline{B}_m$ and Bernstein operator B_n yields

$$\left(B_{m,n} f\right)(x) := \left(\overline{B}_m \circ B_n f\right)(x) = \sum_{\nu=0}^{m} p_{m,\nu}(x)\,(m+1) \int_0^1 p_{m,\nu}(t) \sum_{k=0}^{n} p_{n,k}(t)\, f\left(\frac{k}{n}\right) dt.$$

THEOREM 3.1. The composition $B_{m,n} f := \overline{B}_m \circ B_n$ has the representation

$$\left(B_{m,n} f\right)(x) = \sum_{k=0}^{\min\{m,n\}} b_{m,n,k}(x) f\left(\frac{k}{n}\right),$$

where

$$b_{m,n,k}(x) = \binom{m+n+1}{n}^{-1} \sum_{i=0}^{k} \sum_{j=0}^{n-k} \binom{m}{i+j} \binom{k}{i} \binom{n-k}{j} p_{i+j,i}(x).$$

Proof. By definition, we obtain

$$\begin{aligned}
\left(B_{m,n} f\right)(x) &= (m+1) \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \sum_{\nu=0}^{m} p_{m,\nu}(x) \int_0^1 p_{m,\nu}(t)\, p_{n,k}(t)\, dt \\
&= (m+1) \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \sum_{\nu=0}^{m} p_{m,\nu}(x) \binom{m}{\nu} \binom{n}{k} B(\nu+k+1, m-\nu+n-k+1) \\
&= \frac{(m+1)!}{(m+n+1)!} \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) \sum_{\nu=0}^{m} \binom{m}{\nu} x^{\nu} (1-x)^{m-\nu} (\nu+k)^{\underline{k}} (m-\nu+n-k)^{\underline{n-k}} \\
&= \frac{(m+1)!}{(m+n+1)!} \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) \sum_{\nu=0}^{m} \binom{m}{\nu} \left[\left(\frac{\partial}{\partial y}\right)^{k} y^{\nu+k} \left(\frac{\partial}{\partial z}\right)^{n-k} z^{m-\nu+n-k} \right]_{\substack{y=x \\ z=1-x}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
b_{m,n,k}(x) &= \frac{(m+1)!}{(m+n+1)!} \binom{n}{k} \left[\left(\frac{\partial}{\partial y}\right)^{k} \left(\frac{\partial}{\partial z}\right)^{n-k} y^k z^{n-k} (y+z)^m \right]_{\substack{y=x \\ z=1-x}} \\
&= \frac{(m+1)!}{(m+n+1)!} \binom{n}{k} \sum_{i=0}^{k} \sum_{j=0}^{n-k} \binom{k}{i} \binom{n-k}{j} \frac{k!}{i!} x^i \frac{(n-k)!}{j!} x^i (1-x)^j m^{\underline{i+j}} \\
&= \frac{(m+1)!n!}{(m+n+1)!} \sum_{i=0}^{k} \sum_{j=0}^{n-k} \binom{k}{i} \binom{n-k}{j} \frac{x^i}{i!} \frac{(1-x)^j}{j!} m^{\underline{i+j}} \\
&= \binom{m+n+1}{n}^{-1} \sum_{i=0}^{k} \sum_{j=0}^{n-k} \binom{k}{i} \binom{n-k}{j} \binom{m}{i+j} p_{i+j,i}(x).
\end{aligned}$$

This completes the proof of the assertion. □

PROPOSITION 3.2. For each real A, the moment generating function of the operator $B_{m,n}$ is given by

$$B_{m,n} \exp_A = B_m \, {}_2F_1 \left(-n, m \cdot +1; m+2; 1-e^{A/n}\right),$$

where ${}_2F_1$ denotes the hypergeometric function.

Proof. By direct computation, one verifies that

$$\left(B_n \exp_A\right)(x) = \left(1 - x + xe^{A/n}\right)^n.$$

Using the integral representation

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt \qquad (\operatorname{Re} c > \operatorname{Re} b > 0)$$

of the hypergeometric function ${}_2F_1$, provided that z is not a real number greater than or equal to 1, we obtain

$$\int_0^1 t^\nu (1-t)^{m-\nu}(1-zx)^n\,dt = \frac{1}{m+1}\binom{m}{\nu}^{-1} {}_2F_1(-n, \nu+1; m+2; z),$$

which is a polynomial of degree n in the variable z. We have

$$\begin{aligned}\left(\overline{B}_m (1-ze_1)^n\right)(x) &= (m+1)\sum_{\nu=0}^m p_{m,\nu}(x)\int_0^1 \binom{m}{\nu} t^\nu (1-t)^{m-\nu}(1-zt)^n\,dt \\ &= \sum_{\nu=0}^m p_{m,\nu}(x)\, {}_2F_1(-n,\nu+1;m+2;z).\end{aligned}$$

Putting $z = 1 - e^{A/n}$ gives the desired formula. □

Recalling that $\psi_x(t) = t - x$, the first few central moments of $B_{m,n}$ are given by

$$\begin{aligned}\left(B_{m,n}\psi_x^0\right)(x) &= 1, \\ \left(B_{m,n}\psi_x^1\right)(x) &= \frac{1-2x}{m+2}, \\ \left(B_{m,n}\psi_x^2\right)(x) &= \frac{m+2n+1}{(m+2)(m+3)n} + \frac{2(m-3)n + m(m-1)}{(m+2)(m+3)n} x(1-x).\end{aligned}$$

Since,

$$\begin{aligned}\left(B_{n,n}\psi_x^1\right)(x) &= (1-2x)n^{-1} + O_x\left(n^{-2}\right), \\ \left(B_{n,n}\psi_x^2\right)(x) &= 3x(1-x)n^{-1} + O_x\left(n^{-2}\right), \\ \left(B_{n,n}\psi_x^3\right)(x) &= 22x(1-x)(1-2x)n^{-2} + O_x\left(n^{-3}\right), \\ \left(B_{n,n}\psi_x^4\right)(x) &= 27(x(1-x))^2 n^{-2} + O_x\left(n^{-3}\right)\end{aligned}$$

as $n \to \infty$, we have the Voronovskaja-type formula

$$\lim_{n\to\infty} n\left(\left(B_{n,n}f\right)(x) - f(x)\right) = (1-2x)f'(x) + \frac{3}{2}x(1-x)f''(x),$$

provided that the function f is bounded on $[0,1]$ and has a second derivative $f''(x)$.

4. COMPOSITION OF SZÁSZ–MIRAKYAN–DURRMEYER OPERATOR AND JAIN–PETHE OPERATOR

For $c > 0$, the operator due to Jain and Pethe [15] in the slightly modified form given by Abel and Ivan [1] is defined by

$$(L_{n,c}f)(x) = \left(\frac{c}{1+c}\right)^{ncx} \sum_{k=0}^{\infty} \frac{(ncx)_k}{k!\,(1+c)^k} f\left(\frac{k}{n}\right).$$

Note that the operators $L_{n,c}$ are well-defined, for all sufficiently large n, since the infinite sum is convergent if $n > A/\log(1+c)$, provided that $|f(t)| \le Ke^{At}$ $(t \ge 0)$. The special case $c = 1$ is the Lupaş operator [16] defined by

$$(L_n f)(x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{k!2^k} f\left(\frac{k}{n}\right).$$

Taking the composition of $\overline{S}_m$ and $L_{n,c}$, we obtain the approximation operator $Q_{m,n,c}$ as follows:

$$\begin{aligned}\left(Q_{m,n,c}f\right)(x) &:= \left(\overline{S}_m \circ L_{n,c}f\right)(x)\\ &= m\sum_{\nu=0}^{\infty} s_\nu(mx) \int_0^\infty s_\nu(mt) \sum_{k=0}^{\infty}\left(\frac{c}{1+c}\right)^{nct} \frac{(nct)_k}{k!\,(1+c)^k} f\left(\frac{k}{n}\right) dt.\end{aligned}$$

THEOREM 4.1. For $m > nc\log\frac{c}{1+c}$, a concise form of $Q_{m,n,c}$ is given by

$$\left(Q_{m,n,c}f\right)(x) = \sum_{k=0}^{\infty} q_{m,n,c,k}(x) f\left(\frac{k}{n}\right),$$

with

$$q_{m,n,c,k}(x) = \frac{m}{k!\,(1+c)^k} \exp\left(\frac{mnc\log\frac{c}{1+c}}{m - nc\log\frac{c}{1+c}}x\right) \sum_{i=0}^{k} \begin{bmatrix}k\\ i\end{bmatrix} \frac{i!\,(nc)^i}{\left(m - nc\log\frac{c}{1+c}\right)^{i+1}} L_i\left(-\frac{m^2x}{m - nc\log\frac{c}{1+c}}\right).$$

where L_i denotes the Laguerre polynomial of degree i and $\begin{bmatrix}k\\ i\end{bmatrix}$ are the signless Stirling numbers of the first kind.

A concise representation of the Laguerre polynomial is the Rodrigues' formula

$$L_n(x) = \frac{1}{n!}e^x\left(\frac{d}{dx}\right)^n\left(x^n e^{-x}\right).$$

The explicit form is given by

$$L_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{k!}. \tag{4.1}$$

In the proof we take advantage of the relation

$$L_n(x) = M(-n, 1, x), \tag{4.2}$$

[2, Eq. (13.6.9)], where

$$M(a,b,x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!} \tag{4.3}$$

is the confluent hypergeometric function (also denoted by ${}_1F_1$) and $(a)_k = \Gamma(a+k)/\Gamma(a)$, $k = 0, 1, 2, \ldots$, is the Pochhammer symbol.

REMARK 4.2. In the special case $m = n$, the composition of Szász–Durrmeyer operators and Jain-Pethe operators is given by

$$\begin{aligned}&\left(Q_{n,n,c}f\right)(x)\\ &= \exp\left(\frac{nc\log\frac{c}{1+c}}{1 - c\log\frac{c}{1+c}}x\right) \sum_{k=0}^{\infty} \frac{1}{k!\,(1+c)^k} f\left(\frac{k}{n}\right) \sum_{i=0}^{k} \begin{bmatrix}k\\ i\end{bmatrix} \frac{i!c^i}{\left(1 - c\log\frac{c}{1+c}\right)^{i+1}} L_i\left(-\frac{nx}{1 - c\log\frac{c}{1+c}}\right).\end{aligned}$$

and

$$\left(Q_{n,n,1}f\right)(x)=\exp\left(\frac{-nx\log 2}{\log(2e)}\right)\sum_{k=0}^{\infty}\frac{1}{k!2^k}f\left(\frac{k}{n}\right)\sum_{i=0}^{k}\begin{bmatrix}k\\i\end{bmatrix}\frac{i!}{(\log(2e))^{i+1}}L_i\left(-\frac{nx}{\log(2e)}\right).$$

Proof of Theorem 4.1. Using $(u)_k=\sum_{i=0}^{k}\begin{bmatrix}k\\i\end{bmatrix}u^i$ we can write

$$\begin{aligned}(Q_{m,n,c}f)(x)&=m\sum_{\nu=0}^{\infty}s_\nu(mx)\int_0^\infty s_\nu(mt)\sum_{k=0}^{\infty}\left(\frac{c}{1+c}\right)^{nct}\frac{(nct)_k}{k!\,(1+c)^k}f\left(\frac{k}{n}\right)dt\\&=m\sum_{k=0}^{\infty}\frac{1}{k!\,(1+c)^k}f\left(\frac{k}{n}\right)\sum_{i=0}^{k}\begin{bmatrix}k\\i\end{bmatrix}n^ic^i\sum_{\nu=0}^{\infty}\frac{m^\nu}{\nu!}s_\nu(mx)\int_0^\infty e^{-t\left(m-nc\log\frac{c}{1+c}\right)}t^{\nu+i}dt\\&=\sum_{k=0}^{\infty}q_{m,n,c,k}(x)f\left(\frac{k}{n}\right),\end{aligned}$$

with

$$\begin{aligned}q_{m,n,c,k}(x)&=\frac{m}{k!\,(1+c)^k}\sum_{i=0}^{k}\begin{bmatrix}k\\i\end{bmatrix}\frac{n^ic^i}{\left(m-nc\log\frac{c}{1+c}\right)^{i+1}}e^{-mx}\sum_{\nu=0}^{\infty}\frac{\left(m^2x\right)^\nu}{\nu!\nu!}\frac{(i+\nu)!}{\left(m-nc\log\frac{c}{1+c}\right)^\nu}\\&=\frac{me^{-mx}}{k!\,(1+c)^k}\sum_{i=0}^{k}\begin{bmatrix}k\\i\end{bmatrix}\frac{(nc)^i\,i!}{\left(m-nc\log\frac{c}{1+c}\right)^{i+1}}M\left(1+i,1,\frac{m^2x}{m-nc\log\frac{c}{1+c}}\right),\end{aligned}$$

where M denotes the confluent hypergeometric function. Using Kummer's transformation [2, Eq. (13.1.27)]

$$M(a,b,z)=e^zM(b-a,b,-z)$$

and equation (4.2) we get

$$\begin{aligned}M\left(1+i,1,\frac{m^2x}{m-nc\log\frac{c}{1+c}}\right)&=\exp\left(\frac{m^2x}{m-nc\log\frac{c}{1+c}}\right)M\left(-i,1,-\frac{m^2x}{m-nc\log\frac{c}{1+c}}\right)\\&=\exp\left(\frac{m^2x}{m-nc\log\frac{c}{1+c}}\right)L_i\left(-\frac{m^2x}{m-nc\log\frac{c}{1+c}}\right).\end{aligned}$$

Furthermore,

$$\begin{aligned}e^{-mx}\exp\left(\frac{m^2x}{m-nc\log\frac{c}{1+c}}\right)&=\exp\left(\frac{-mx\left(m-nc\log\frac{c}{1+c}\right)+m^2x}{m-nc\log\frac{c}{1+c}}\right)\\&=\exp\left(\frac{mnc\log\frac{c}{1+c}}{m-nc\log\frac{c}{1+c}}x\right).\end{aligned}$$

Hence,

$$q_{m,n,c,k}(x)=\frac{m}{k!\,(1+c)^k}\exp\left(\frac{mnc\log\frac{c}{1+c}}{m-nc\log\frac{c}{1+c}}x\right)\sum_{i=0}^{k}\frac{\begin{bmatrix}k\\i\end{bmatrix}i!\,(nc)^i}{\left(m-nc\log\frac{c}{1+c}\right)^{i+1}}L_i\left(-\frac{m^2x}{m-nc\log\frac{c}{1+c}}\right),$$

which completes the proof. □

PROPOSITION 4.3. For each real A, the moment generating function of the operator $Q_{m,n,c}$ is given by

$$\left(Q_{m,n,c}\exp_A\right)(x)=\frac{1}{1-\frac{n}{m}c\log\left(\frac{c}{1+c-e^{A/n}}\right)}\exp\left(\frac{nc\log\left(\frac{c}{1+c-e^{A/n}}\right)}{1-\frac{n}{m}c\log\left(\frac{c}{1+c-e^{A/n}}\right)}x\right).$$

provided that the conditions $n>A/\log(1+c)$ and $nc\log\left(\frac{c}{1+c-e^{A/n}}\right)<m$ are satisfied.

Proof. By (2.4), we have, for $n > A/\log(1+c)$,

$$\left(L_{n,c}\exp_A\right)(x) = \left(\frac{c}{1+c-e^{A/n}}\right)^{ncx} = \exp\left(ncx\log\left(\frac{c}{1+c-e^{A/n}}\right)\right).$$

It follows that

$$\begin{aligned}\left(Q_{m,n,c}\exp_A\right)(x) &= \left(\overline{S}_m \circ L_{n,c}\exp_A\right)(x) = \left(\overline{S}_m \exp_{nc\log\left(\frac{c}{1+c-e^{A/n}}\right)}\right)(x)\\ &= \frac{m}{m - nc\log\left(\frac{c}{1+c-e^{A/n}}\right)}\exp\left(\frac{mnc\log\left(\frac{c}{1+c-e^{A/n}}\right)x}{m-nc\log\left(\frac{c}{1+c-e^{A/n}}\right)}\right),\end{aligned}$$

where the last equality is a consequence of (2.2). □

REMARK 4.4. In the particular case $m = n$, we have

$$(Q_{n,n,c}\exp_A)(x) = \left(1 - c\log\left(\frac{c}{1+c-e^{A/n}}\right)\right)^{-1}\exp\left(\frac{ncx\left(\log c - \log(1+c-e^{A/n})\right)}{1-c\left(\log c - \log(1+c-e^{A/n})\right)}\right).$$

THEOREM 4.5. If r is a natural number then for $f \in C_b[0,\infty)$ (the class of bounded continuous functions on the positive real axis), we have

$$\begin{aligned}\lim_{n\to\infty}\left(Q_{m,n,c}f\right)(x) &= \left(\overline{S}_m f\right)(x),\\ \lim_{m\to\infty}\left(Q_{m,n,c}f\right)(x) &= (L_{n,c}f)(x),\\ \lim_{n\to\infty}\left(Q_{n,n,c}f\right)(x) &= f(x),\\ \lim_{n\to\infty}\left(Q_{r,r,c}f\left(\frac{\cdot}{n}\right)\right)(nx) &= f(x).\end{aligned}$$

Proof. By Proposition 4.3, we have

$$\lim_{n\to\infty}(Q_{m,n,c}\exp_{is})(x) = \frac{m}{m-is}\exp\left(\frac{misx}{m-is}\right) = \left(\overline{S}_m\exp_{is}\right)(x),$$

and

$$\lim_{m\to\infty}(Q_{m,n,c}\exp_{is})(x) = \left(\frac{c}{1+c-e^{is/n}}\right)^{ncx} = \left(L_{n,c}\exp_{is}\right)(x),$$

where $L_{n,c}$ is the Jain-Pethe operator. By Remark 4.4, we have

$$\lim_{n\to\infty}\left(Q_{n,n,c}\exp_{is}\right)(x) = e^{isx}$$

and

$$\lim_{n\to\infty}\left(Q_{r,r,c}\exp_{is/n}\right)(nx) = e^{isx}.$$

The desired conclusion follows by Proposition 2.1 (also see [3, Th. 1]). □

5. COMPOSITION OF BASKAKOV–SZÁSZ OPERATOR AND SZÁSZ–MIRAKYAN OPERATOR

The hybrid Baskakov–Szász operator $\overline{V}_n$ (see [14]) is defined by

$$\left(\overline{V}_n f\right)(x) = n\sum_{\nu=0}^{\infty} v_{n,\nu}(x)\int_0^{\infty} s_\nu(nt)f(t)dt,$$

where $v_{n,\nu}(x) = \frac{(n)_\nu}{\nu!}\frac{x^\nu}{(1+x)^{n+\nu}}$ is the Baskakov basis function and $s_\nu(x) = e^{-x}x^\nu/\nu!$ $(\nu = 0,1,2,\ldots)$ is the Szász basis function.

The composition of $\overline{V}_m$ and the Szász–Mirakyan operator S_n, provides us a new approximation operator $C_{m,n}$ as follows:

$$\left(C_{m,n}f\right)(x) := \left(\overline{V}_m \circ S_n f\right)(x) = m\sum_{\nu=0}^{\infty} v_{m,\nu}(x)\int_0^{\infty} s_\nu(mt)\sum_{k=0}^{\infty} s_k(nt)f\left(\frac{k}{n}\right)dt.$$

THEOREM 5.1. A concise form of $C_{m,n}$ is given by

$$\left(C_{m,n}f\right)(x)=\sum_{k=0}^{\infty}c_{k,m,n}(x)f\left(\frac{k}{n}\right),$$

where

$$c_{k,m,n}(x)=\frac{mn^k}{(m+n)^{k+1}}(1+x)^{-m}{}_2F_1\left(k+1,m;1;\frac{mx}{(m+n)(1+x)}\right).$$

In the special case $m=n$ we get

$$c_{k,n,n}(x)=\frac{1}{2^{k+1}(1+x)^n}{}_2F_1\left(k+1,n;1;\frac{x}{2(1+x)}\right).$$

Proof of Theorem 5.1. We have

$$\begin{aligned}(C_{m,n}f)(x)&=m\sum_{\nu=0}^{\infty}v_{m,\nu}(x)\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)\int_0^{\infty}\frac{e^{-(m+n)t}(mt)^{\nu}(nt)^k}{k!\nu!}dt\\&=m\sum_{\nu=0}^{\infty}v_{m,\nu}(x)m^{\nu}\sum_{k=0}^{\infty}\frac{n^k(k+\nu)!}{k!\nu!\,(m+n)^{k+\nu+1}}f\left(\frac{k}{n}\right)\\&=\frac{m}{(1+x)^m}\sum_{k=0}^{\infty}\frac{n^k}{(m+n)^{k+1}}f\left(\frac{k}{n}\right)\sum_{\nu=0}^{\infty}\binom{k+\nu}{\nu}\frac{(m)_\nu}{\nu!}\left(\frac{mx}{(m+n)(1+x)}\right)^{\nu},\end{aligned}$$

such that

$$c_{k,m,n}(x)=\frac{m}{(1+x)^m}\frac{n^k}{(m+n)^{k+1}}\sum_{\nu=0}^{\infty}\binom{k+\nu}{\nu}\frac{(m)_\nu}{\nu!}\left(\frac{mx}{(m+n)(1+x)}\right)^{\nu}.$$

Since

$$\sum_{\nu=0}^{\infty}\binom{k+\nu}{\nu}\frac{(m)_\nu}{\nu!}z^{\nu}={}_2F_1(k+1,m;1;z)$$

now the result follows. □

PROPOSITION 5.2. For each real A, the moment generating function of the operator $C_{m,n}$ is given by

$$\left(C_{m,n}\exp_A\right)(x)=\frac{m(m-n\left(e^{A/n}-1\right))^{n-1}}{\left(m-n\left(e^{A/n}-1\right)(1+x)\right)^m},$$

provided that $m>n\left(e^{A/n}-1\right)(1+x)$.

Proof. By direct computation, one verifies that

$$\left(\overline{V}_m\exp_A\right)(x)=\frac{m\,(m-A)^{n-1}}{(m-A\,(1+x))^m}.$$

Hence, by equation (2.1), we get

$$\begin{aligned}\left(C_{m,n}\exp_A\right)(x)&=\left(\overline{V}_m\circ S_n\exp_A\right)(x)=\left(\overline{V}_m\exp_{n\left(e^{A/n}-1\right)}\right)(x)\\&=\frac{m\left(m-n\left(e^{A/n}-1\right)\right)^{n-1}}{\left(m-n\left(e^{A/n}-1\right)(1+x)\right)^m},\end{aligned}$$

which proves our claim. □

REMARK 5.3. In the special case $m=n$, we obtain

$$\left(C_{n,n}\exp_A\right)(x)=\frac{\left(2-e^{A/n}\right)^{n-1}}{\left(2+x-e^{A/n}(1+x)\right)^n},$$

provided that $n>A/\log\left((2+x)/(1+x)\right)$.

THEOREM 5.4. If r is a natural number then, for $f \in C_b[0,\infty)$ (the class of bounded continuous functions on the positive real axis), we have

$$\lim_{n\to\infty}\left(C_{n,n}f\right)(x) = f(x),$$
$$\lim_{n\to\infty}\left(C_{r,r}f\left(\frac{\cdot}{n}\right)\right)(nx) = (P_r f)(x),$$
$$\lim_{n\to\infty}\left(C_{rn,rn}f\left(n\,\cdot\right)\right)\left(\frac{x}{n}\right) = \left(G_r^{[0]}f\right)(x).$$

In the preceding theorem P_r is Post–Widder operator as defined in Section 2. The operator $G_n^{[\alpha]}$, defined for $x \in [0,\infty)$, $\alpha > -1$ and $n \in \mathbb{N}$ by

$$\left(G_n^{[\alpha]}f\right)(x) = e^{-nx/2}\sum_{k=0}^{\infty}\frac{1}{2^{k+\alpha+1}}L_k^{(\alpha)}\left(-\frac{nx}{2}\right)f\left(\frac{k}{n}\right)$$

was discussed in [18, Eq. (3.4)]. Here $L_n^{(\alpha)}$ denote the generalized Laguerre polynomials, or associated Laguerre polynomials, explicitly given by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n}(-1)^k\frac{(k+\alpha+1)_{n-k}}{(n-k)!k!}x^k.$$

Proof of Theorem 5.4. By Remark 5.3, we have

$$\lim_{n\to\infty}(C_{n,n}\exp_{is})(x) = e^{isx},$$

and

$$\lim_{n\to\infty}\left(C_{r,r}\exp_{is/n}\right)(nx) = \left(1-\frac{isx}{r}\right)^{-r} = (P_r e^{ist})(x).$$

By [11, Lemma 1], we conclude that

$$\lim_{n\to\infty}\left(C_{rn,rn}\exp_{isn}\right)\left(\frac{x}{n}\right) = \frac{1}{2-e^{is/r}}\exp\left(\frac{e^{is/r}-1}{2-e^{is/r}}x\right) = \left(G_r^{[0]}\exp_{is}\right)(x).$$

The desired conclusion follows by Proposition 2.1 and also from [3, Th. 1]. □

The first few central moments of $C_{m,n}$ are given by

$$\left(C_{m,n}\psi_x^0\right)(x) = 1,$$
$$\left(C_{m,n}\psi_x^1\right)(x) = \frac{1}{m},$$
$$\left(C_{m,n}\psi_x^2\right)(x) = \frac{m+2n}{m^2 n} + \frac{x\,(x+2)}{m} + \frac{x}{n}.$$

Since,

$$\left(C_{n,n}\psi_x^3\right)(x) = \frac{2x\left(x^2+6x+11\right)}{n^2} + \frac{13}{n^3},$$
$$\left(C_{n,n}\psi_x^4\right)(x) = \frac{3x^2\,(x+3)^2}{n^2} + \frac{6x^4+44x^3+133x^2+181x}{n^3} + \frac{75}{n^4},$$

we have the Voronovskaja-type formula

$$\lim_{n\to\infty} n\left(\left(C_{n,n}f\right)(x) - f(x)\right) = f'(x) + \frac{1}{2}x\,(x+3)\,f''(x),$$

provided that $f \in C_b\,[0,+\infty)$ has a second derivative $f''(x)$.

6. COMPOSITION OF MÜLLER GAMMA OPERATOR AND SZÁSZ–MIRAKYAN OPERATOR

The Müller gamma operators (see Sec. 2) can alternatively be rewritten in the form

$$(G_n f)(x) = \frac{(nx)^{n+1}}{n!} \int_0^\infty t^{-n-2} e^{-nx/t} f(t)\, dt.$$

Composition of Müller gamma and Szász–Mirakyan operator yields

$$\begin{aligned}\left(G_{m,n} f\right)(x) := \left(G_m \circ S_n f\right)(x) &= \frac{(mx)^{m+1}}{m!} \int_0^\infty t^{-m-2} e^{-mx/t} \sum_{k=0}^\infty e^{-nt} n^k \frac{t^k}{k!} f\left(\frac{k}{n}\right) dt \\ &= \frac{(mx)^{m+1}}{m!} \sum_{k=0}^\infty \frac{n^k}{k!} f\left(\frac{k}{n}\right) \int_0^\infty t^{k-m-2} e^{-(nt+mx/t)} dt.\end{aligned}$$

THEOREM 6.1. A concise form of $G_{m,n}$ is given by

$$\left(G_{m,n} f\right)(x) = \sum_{k=0}^\infty g_{k,m,n}(x) f\left(\frac{k}{n}\right),$$

where

$$g_{k,m,n}(x) = \frac{2\,(mnx)^{(m+1+k)/2}}{m!k!} K_{m+1-k}\left(2\sqrt{mnx}\right)$$

and K_ν denotes the modified Bessel function of the second kind.

Proof. Applying the integral representation

$$K_\nu(az) = \frac{z^\nu}{2} \int_0^\infty t^{-\nu-1} \exp\left(-\frac{a}{2}\left(t + \frac{z^2}{t}\right)\right) dt, \tag{6.1}$$

of the modified Bessel function of the second kind [17, page 39] (cf. [7, 8.432, formula 7, p. 917]), with $\nu = m+1-k$, $a = 2n$ and $z = \sqrt{mx/n}$, we get

$$g_{k,m,n}(x) = 2\frac{(mx)^{m+1} n^k}{m!k!} \left(\frac{mx}{n}\right)^{(k-m-1)/2} K_{m+1-k}\left(2n\sqrt{mx/n}\right),$$

which proves the desired formula. □

In the special case $m = n$ we get

$$g_{k,n,n}(x) = \frac{2\,(n\sqrt{x})^{n+k+1}}{n!k!} K_{n+1-k}\left(2n\sqrt{x}\right).$$

LEMMA 6.2. For real numbers $A < 0$, it holds

$$(G_n \exp_A)(x) = \frac{2}{n!} (-Anx)^{(n+1)/2} K_{n+1}\left(2\sqrt{-Anx}\right).$$

Proof. By the definition, we have

$$(G_n \exp_A)(x) = \frac{x^{n+1}}{n!} \int_0^\infty t^n \exp\left(-xt - \frac{-An}{t}\right) dt.$$

Using the integral representation (6.1) with $\nu = -n-1$, $a = 2x$ and $z = \sqrt{-An/x}$, we get the desired formula. □

PROPOSITION 6.3. For real numbers $A < 0$, the operator $G_{m,n}$ satisfies

$$(G_{m,n} \exp_A)(x) = \frac{2}{m!} \left(n\left(1 - e^{A/n}\right) mx\right)^{(m+1)/2} K_{m+1}\left(2\sqrt{n\left(1 - e^{A/n}\right) mx}\right).$$

In the special case $m = n$, we have

$$\left(G_{n,n} \exp_A\right)(x) = \frac{2n^n}{\Gamma(n)} \left(\left(1 - e^{A/n}\right) x\right)^{(n+1)/2} K_{n+1}\left(2n\sqrt{\left(1 - e^{A/n}\right) x}\right)$$

Proof of Prop. 6.3. By equation (2.1), we have

$$G_{m,n}\exp_A = G_m \exp_{n\left(e^{A/n}-1\right)}$$

and noting that $e^{A/n}-1<0$, for $A<0$, application of Lemma 6.2 completes the proof. □

The first few central moments of $G_{m,n}$ are given by

$$\begin{aligned}
\left(G_{m,n}\psi_x^0\right)(x) &= 1,\\
\left(G_{m,n}\psi_x^1\right)(x) &= 0,\\
\left(G_{m,n}\psi_x^2\right)(x) &= \frac{x^2}{m-1}+\frac{x}{n}\\
\left(G_{m,n}\psi_x^3\right)(x) &= \frac{4x^3}{(m-1)(m-2)}+\frac{3x^2}{(m-1)n}+\frac{x}{n^2},\\
\left(G_{m,n}\psi_x^4\right)(x) &= \frac{3(m+6)x^4}{(m-1)(m-2)(m-3)}+\frac{6(m+2)x^3}{(m-1)(m-2)n}+\frac{(3m+4)x^2}{(m-1)n^2}+\frac{x}{n^3}.
\end{aligned}$$

Since, for $s=3$ and $s=4$, $\left(G_{n,n}\psi_x^s\right)(x)=O_x\left(n^{-2}\right)$ as $n\to\infty$, we have the Voronovskaja-type formula

$$\left(G_{n,n}f\right)(x)=f(x)+\frac{x(1+x)}{2n}f''(x)+o_x(1/n)\qquad(n\to\infty),$$

provided that $f\in C_b[0,+\infty)$ has a second derivative $f''(x)$.

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