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AKADÉMIAI KIADÓ

A NOTE ON THE ZEROS OF THE DEDEKIND ZETA FUNCTIONS

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ABSTRACT

We prove zero density theorems for Dedekind zeta functions in the vicinity of the line $\operatorname{Re} s = 1$, improving an earlier result of W. Staś.

KEYWORDS

Dedekind zeta functions, zeros of Dedekind zeta functions, zero density theorems for Dedekind zeta functions

MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 11R42; Secondary 11M41

1. INTRODUCTION

In a recent work [10] we showed density theorems for Dedekind zeta functions

$$\zeta_K(s) := \sum_{\mathfrak{a} \neq 0} (N\mathfrak{a})^{-s} \tag{1.1}$$

of algebraic number fields K of degree $n := [K : Q]$ over the rational field, where $\mathfrak{a}$ runs through all integral ideas of K and $N\mathfrak{a}$ denotes the norm of $\mathfrak{a}$. (In case of $n = 1$, $\zeta_K(s) = \zeta(s)$, the Riemann zeta function.)

The number of non-trivial zeros of $\zeta_K(s)$ – denoted by $\varrho = \beta + i\gamma$ with $0 \leq \beta \leq 1$, $|\gamma| \leq T$ – satisfy the asymptotic relation

$$N_K(T) \sim \frac{n}{\pi} T \log T. \tag{1.2}$$

The goal of zero density theorems is to show that for any σ with $\frac{1}{2} < \sigma < 1$, the quantity

$$N_K(\sigma, T) := \sum_{\substack{\zeta_K(\varrho)=0 \\ \beta \geq \sigma, |\gamma| \leq T}} 1 \tag{1.3}$$

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is much smaller than the total number $N_K(T)$ of zeros with $|\gamma| \le T$. At present it is only possible to show such results in case of

$$\eta = 1 - \sigma = O\left(\frac{1}{n}\right). \tag{1.4}$$

So we will formulate earlier and present results in the form

$$N_K(1-\eta, T) \ll_{\eta,\varepsilon,K} T^{B_K(\eta)\eta+\varepsilon} \qquad (\varepsilon > 0 \quad \text{arbitrary}), \tag{1.5}$$

where we look for results satisfying

$$B_K(\eta)\eta < 1. \tag{1.6}$$

It turns out that it suffices to restrict us for cases when $B_K(\eta)$ depends only on η and the degree $n = [K : Q]$ of the field extension but the constant $\ll_{\varepsilon,\eta,K}$ in (1.5) may depend beyond η and ε on other parameters of the field K as well.

2. EARLIER RESULTS IN CASE OF $\eta \gg 1/n$

In case of $\eta \gg 1/n$ earlier results (using the notation (1.5)–(1.6)) were

$$B_K(\eta) \le n + 2 - c/\big(n^2 \log(n+2)\big) \tag{2.1}$$

by Sokolovsky [12],

$$B_K(\eta) \le n \tag{2.2}$$

by Heath-Brown [4], and a few years ago

$$B_K(\eta) \le \frac{2n}{3} + 2 = \frac{2(n+3)}{3} \tag{2.3}$$

by B. Paul and A. Sankaranarayanan [7]. This improved the estimate (2.2) of Heath-Brown for $n \ge 7$.

Finally, in a recent work [10] the present author succeeded to improve these estimates for all $n \ge 7$ for all η, further for $\eta < 1/6$ for $4 \le n \le 6$ and for $\eta < 37/252$ for $n = 3$. The corresponding estimates were

$$B_K(\eta) \le \frac{2n}{3(1-2\eta)} \qquad \text{for } n \ge 4, \tag{2.4}$$

$$B_K(\eta) \le \frac{2}{1-2\eta} \qquad \text{if } n = 3 \text{ and } \eta \le \frac{4}{29} = 0.137931. \tag{2.5}$$

$$B_K(\eta) \le \frac{26/21}{1-4\eta} \qquad \text{if } n = 3 \text{ and } \frac{4}{29} \le \eta < 1/4. \tag{2.6}$$

REMARK 2.1. The result (2.4)–(2.6) of [10] is a relatively small improvement compared with (2.3) for large values of n, but for small values of n, it is more significant. In case of cubic fields we get almost the density hypothesis, $B_K(\eta) \le 2 + O(\eta)$, while (2.2) yields $B_K(\eta) \le 3$, (2.3) gives $B_K(\eta) \le 4$.

The proof of the estimates (2.4)–(2.6) was based on a new result of the author [8], [11] which proved a general zero density theorem for a large class of Dirichlet series (see Section 4 for the details). This general theorem was based on the pioneering work of Halász and Turán [3], in particular, to a simple but ingenious idea of Halász [2].

3. THE CASE $\eta = o(1/n)$

Although the earlier mentioned general theorem of the author [8], [11] works for all n and all $\eta < 1/4$, in the actual application for Dedekind zeta functions we need to suppose (1.6), that is

$$\frac{2n\eta}{3(1-2\eta)} < 1 \Leftrightarrow \eta < \frac{3}{2(n+3)} \qquad \text{for } n \ge 4, \tag{3.1}$$

in order to obtain non-trivial results. On the other hand, if $\eta = o(1/n)$ then W. Staś [14] proved a stronger result than any of the estimates (2.1)–(2.6). His result was completely explicitly depending on the order n and discriminant Δ of the field K. It meant an improvement as a function of η compared with the best estimate (2.2) of Heath-Brown if $\eta = o(1/n)$ although the final exponent of T was better only if $\eta \le cn^{-7/3}$. His theorem proved

$$N_K(1-\eta, T) < \exp\exp\left(c_1 n^{5600}|\Delta|^{1600}\right) T^{\frac{n^2}{10}(n\eta)^{3/2}\log^2\frac{1}{n\eta}} \tag{3.2}$$

with a constant c_1 for

$$\eta \le \frac{1}{3\exp(10^9)n}, \qquad T \ge e. \tag{3.3}$$

This result was a direct generalization of the mentioned density theorem of Halász and Turán [3] which was the first work which proved the celebrated Density Hypothesis (DH) for the Riemann zeta function, the estimate

$$N(1-\eta, T) \ll_{\varepsilon,\eta} T^{2\eta+\varepsilon} \tag{3.4}$$

or with our notation (1.5) even

$$B(\eta) < 2 \tag{3.5}$$

for a non-trivial halfplane $\eta > c_2$ of the critical strip, with a small positive c_2. The value of c_2 was improved later in many steps among others by Montgomery, Huxley, Jutila and finally Bourgain [1] to the present record $\eta < 7/32$ ($\sigma > 25/32$).

Their proof [3] used

(i) the Korobov–Vinogradov estimate $|\zeta(1-\eta, T)| \ll T^{c\eta^{3/2}}(\log T)^{2/3}$;
(ii) the second main theorem of Turán's power sum theory [15];
(iii) an idea of G. Halász [2].

The idea of Halász remained an important part of all later works showing density theorems, at least in case of $\eta < 1/4$.

Staś succeeded to generalize the same method for Dedekind zeta functions. In a first work [13] he generalized (i) to substitute it by the general estimate (i'):

$$\left|\zeta_K(1-\eta+iT)\right| \le e^{c_3 n^8|\Delta|^2} T^{600n^2(n\eta)^{3/2}} \log^{2/3} T. \tag{3.6}$$

Then, in his second work [14], he proved (3.2) using (i'), (ii) and (iii).

The goal of the present work is, to show that our recent general density theorem improves the estimate (3.2) of Staś by eliminating the logarithmic factor from the exponent of T, which tends to infinity if $\eta = o(1/n)$ and satisfies by (3.3)

$$\log^2\frac{1}{n\eta} \ge (10^9 + \log 3)^2 > 10^{18}. \tag{3.7}$$

4. THE DENSITY THEOREM AND ITS PROOF

Let us consider according to the introduction a field K with $n = [K : Q]$ and suppose that we look for an upper estimate for the number of zeros $\varrho = \beta + i\gamma$ of $\zeta_K(s)$ in a rectangle $R_{\eta,T} = \{\beta \ge 1-\eta,\ |\gamma| \le T\}$ for an $\eta \in (0, 1/4)$ in the form

$$N_K(1-\eta, T) = \sum_{\varrho\in R_{\eta,T}} 1 \ll_{\varepsilon,\eta,K} T^{B_K(\eta)\eta+\varepsilon} \tag{4.1}$$

for every $\varepsilon > 0$.

THEOREM 4.1. If $\eta < 1/4$ and $n \ge 2$ then (1.5) holds with $B_K(\eta) = 1800\sqrt{3}\,n^{7/2}\sqrt{\eta}$.

The proof will follow from the estimate (3.6) of Staś and from our general density theorem ([8], [11]) using beside classical standard knowledge of the Dedekind zeta function the mentioned idea of Halász [2]. We will not need the power sum theorem of Turán.

REMARK 4.2. Theorem 4.1 is at the first sight for small values of n weaker than the result of Staś quoted in (3.2). However due to the condition (3.3) in the range of its validity we have (3.7), so Theorem 4.1 is stronger than (3.3) for all values of n. It improves further all previous results of [12], [4], [7], [10] for $\eta < c_0(n)$.

REMARK 4.3. The theorem is trivial if $B_K(\eta)\eta \geq 1$, so we can suppose WLOG that

$$\eta^{3/2} < \frac{1}{1800\sqrt{3}n^{7/2}} \Leftrightarrow \eta < \left(1800\sqrt{3}\right)^{-2/3} n^{-7/3} = 0.00468\ldots n^{-7/3}. \tag{4.2}$$

REMARK 4.4. In case of the Riemann zeta function (i.e. $n = 1$) we proved [9]

$$B_K(\eta) \leq \left(3\sqrt{2} + o(1)\right)\sqrt{\eta} \qquad \text{for } \eta \to 0. \tag{4.3}$$

In the case of a general Dirichlet series $f(s)$ an interesting phenomenon, already observed in the original work of Halász and Turán [3], appeared: the estimate for the density of zeros of $f(s)$ depended on the growth of *both* $f(s)$ and $\zeta(s)$ (the Riemann zeta function) on vertical lines. This is not appearing now in the assertion of Theorem 4.1 since the known estimates for the growth of $\zeta_K(s)$ (cf. (3.6)) are weaker than the one for $\zeta(s)$ (at least in the non-abelian cases for $n \geq 3$).

REMARK 4.5. As it appears in (4.1) we allow the constant implied by the $\ll$ symbol to depend on ε, η and the field K.

Proof of Theorem 4.1. The proof will follow easily from our general theorem in [11] using additionally beyond classical knowledge of the Dedekind zeta function ($\zeta(s)$ denotes the Riemann zeta function):

(i) the estimate (3.6) for $\zeta_K(s)$,

(ii) the idea of Halász.

We proved in [11] the following

THEOREM A. Suppose that $f(s) = \sum_{m=1}^{\infty} f_m m^{-s}$ is a Dirichlet series with $f_1 \neq 0$ and the following properties:

$$f(s) = \sum_{m=1}^{\infty} \frac{f_m}{m^s} \qquad \text{and} \qquad M(s) = \frac{1}{f(s)} = \sum_{m=1}^{\infty} \frac{g_m}{m^s} \text{ are analytic for } \sigma > 1, \tag{4.4}$$

$$f(s) \text{ can be extended as an analytic function for } \sigma \geq \alpha_f, \tag{4.5}$$

$\alpha_f < 1$, up to a simple pole at $s = 1$,

$$f_m \ll m^{\Delta},\ g_m \ll m^{\Delta} \text{ for every } \Delta > 0, \tag{4.6}$$

$$\mu_f(\sigma_0) := \inf\left\{\mu; |f(\sigma + it)| \ll |t|^{\mu} \text{ for } \sigma \geq \sigma_0,\ 1 \leq |t|\right\} < \infty \text{ for } \sigma_0 \geq \alpha_f. \tag{4.7}$$

Let us denote with $\lambda_f(\eta)$ any function with

$$\lambda_f(\eta) \geq \min_{0 \leq a; (a+1)\eta \leq 1-\alpha_f} \frac{\mu_f(1-(a+1)\eta)}{a\eta} \tag{4.8}$$

and let $\mu_\zeta(\sigma_0), \lambda_\zeta(\eta)$ denote the same functions for $\zeta(s)$ in place of $f(s)$.

With the above conditions and notation we have

$$N_f(1-\eta, T) := \sum_{\substack{f(\beta+i\gamma)=0 \\ \beta \geq 1-\eta,\, |\gamma| \leq T}} 1 \ll_{\varepsilon,\eta,f} T^{B_f(\eta)\eta+\varepsilon} \tag{4.9}$$

for any $\eta < \min(1-\alpha_f, 1/4)$ with

$$B_f(\eta) \leq \max\left(2\lambda_f(\eta), 4\lambda_\zeta(2\eta)\right) \qquad \text{if } \lambda_f(\eta) > \lambda_\zeta(2\eta). \tag{4.10}$$

In case of $f(s) = \zeta_K(s)$ the conditions (4.4)–(4.6) are well known (see e.g. [5], Satz 154, or [6], Proposition 7.1). The arithmetical function f_m is totally multiplicative, so $g_m = \mu(m)f_m$ with the usual Möbius function $\mu(m)$ and $f_m \leq \tau_n(m)$ with the generalized divisor function of order n (see e.g. Corollary 3 of paragraph 1 of Chapter VII of [6]), so (4.4)–(4.6) hold for $f(s) = \zeta_K(s)$. Further, we obtain from the estimate (3.6) of Staś

$$|\zeta_K(1-3\eta+it)| \ll |t|^{1800\sqrt{3}n^2(n\eta)^{3/2}} \log^{2/3}|t| \tag{4.11}$$

and consequently

$$\mu_K(1-3\eta) \leq 1800\sqrt{3}n^2(n\eta)^{3/2}. \tag{4.12}$$

Now with $a = 2$ we obtain for any $n \geq 2$

$$\lambda_K(\eta) = 900\sqrt{3}n^{7/2}\eta^{1/2} > \frac{\mu_\zeta(1-6\eta)}{4\eta} =: \lambda_\zeta(2\eta), \tag{4.13}$$

using (3.6) for $n = 1$. Hence, (4.10) implies for all $n \geq 2$

$$B_K(\eta) \leq 1800\sqrt{3}n^{7/2}\eta^{1/2}. \tag{4.14}$$

□

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