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AKADÉMIAI KIADÓ

A DENSITY THEOREM FOR DEDEKIND ZETA FUNCTIONS

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ABSTRACT

We apply a recent general zero density theorem of us (valid for a large class of complex functions) to improve earlier density theorems of Heath-Brown and Paul–Sankaranarayanan for Dedekind zeta functions attached to a number field K of degree n with $n > 2$.

KEYWORDS

Dedekind zeta functions, zeros of Dedekind zeta functions, zero-density theorems for Dedekind zeta functions

MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 11R42; Secondary 11M41

1. INTRODUCTION

Let K be an algebraic number field of degree $n := [K : Q]$, and

$$\zeta_K(s) := \sum_{\mathfrak{a} \neq 0} \frac{1}{(N\mathfrak{a})^s} \tag{1.1}$$

its Dedekind zeta function where $\mathfrak{a}$ runs through all integral ideas of K and $N\mathfrak{a}$ denotes the norm of $\mathfrak{a}$. (In case of $n = 1$, i.e. $K = Q$ we will write $\zeta_Q(s) = \zeta(s)$ for Riemann's zeta function.) The series is absolutely convergent for $\sigma > 1$ where $s = \sigma + it$ and has an analytic continuation to the whole complex plane and it is regular except for a simple pole at $s = 1$. The case $n = 1$ is that of the Riemann zeta function and the density of zeros, that is the function ($\varrho = \beta + i\gamma$)

$$N_K(\sigma, T) = \sum_{\substack{\zeta_K(\varrho)=0 \\ \beta \geq \sigma, |\gamma| \leq T}} 1 \tag{1.2}$$

plays a similar role in the distribution of prime ideals of K as the analogous $N(\sigma, T)$ in case of Riemann's $\zeta(s)$ in the distribution of rational primes.

It is well known [6] that

$$N_K(0, T) \sim \frac{n}{\pi} T \log T. \tag{1.3}$$

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According to [4] it seems that Sokolovsky [10] was the first to prove zero density theorems for Dedekind zeta functions. We will express the results in the form

$$N_K(1-\eta,T) \ll T^{B_K(\eta)\eta+\varepsilon}, \tag{1.4}$$

where the constants implied by the $\ll$ symbol are allowed to depend on ε and the field K.

The relation (1.3) shows that any result of type (1.4) with $B_K(\eta) \geq 1/\eta$ follows from (1.3) so we are only interested in the case

$$B_K(\eta)\eta < 1. \tag{1.5}$$

We note that with some extra effort one could replace T^ε by $(\log T)^c$ but we consider the simpler case of T^ε. (The earlier results were proved with $(\log T)^c$ but we quote them in the weaker form (1.4).)

Sokolovsky [10] proved with the above notation

$$B_K(\eta) \leq n+2-C \Big/ (n^2\log(n+2)) \tag{1.6}$$

which was improved by Heath-Brown [4] for general $n \geq 3$ to

$$B_K(\eta) \leq n \tag{1.7}$$

and for $n=2$ to

$$B_K(\eta) \leq \frac{2}{1-\eta} \quad \text{for } \eta \in [\varepsilon, 1/4] \tag{1.8}$$

and

$$B_K(\eta) = \frac{4}{1+2\eta} \quad \text{for } \eta \in [1/4, 1/2-\varepsilon]. \tag{1.9}$$

We note that for the quadratic case $n=2$ Heath-Brown [4] proved even further results which show, for example, that the density hypothesis for $\zeta_K(s)$, i.e.

$$B_K(\eta) \leq 2 \tag{1.10}$$

is also true in the range $\eta \leq 13/124 = 0.104838\ldots$. We do not detail these results since we will consider only the case $n \geq 3$.

Recently Paul and Sankaranarayanan [7] improved Heath-Brown's result (1.7) for $n \geq 7$ to

$$B_K(\eta) \leq \frac{2n}{3}+2 < n. \tag{1.11}$$

We remark that as mentioned earlier we can suppose (1.5), i.e. $B_K(\eta)\eta < 1$, so we can formulate the result (1.11) in the equivalent form

$$B_K(\eta) \leq \frac{2(n+3)}{3} \quad \text{for } \eta < \frac{3}{2(n+3)}. \tag{1.12}$$

We will prove

THEOREM 1.1. With the notation (1.2)–(1.4) we have

$$B_K(\eta) \leq \frac{2n/3}{1-2\eta} \quad \text{for } n \geq 4. \tag{1.13}$$

Further we have

$$B_K(\eta) \leq \frac{2}{1-2\eta} \quad \text{if } n=3 \quad \text{and } \eta \leq 4/29 = 0.137931\ldots \tag{1.14}$$

and

$$B_K(\eta) \leq \frac{26/21}{1-4\eta} \quad \text{for } n=3,\ \ 4/29 \leq \eta < 1/4. \tag{1.15}$$

REMARK. We have $(26/21)/(1-4\eta) > 2/(1-2\eta)$ for $\eta > 4/29$.

We state further the conditional

THEOREM 1.2. If $\zeta(1/2+it) \ll |t|^{1/8}$ then

$$B_K(\eta) \le \frac{2}{1-2\eta} \quad \text{for } n=3,\ \eta \le 1/6 \tag{1.16}$$

and

$$B_K(\eta) \le \frac{1}{1-4\eta} \quad \text{for } n=3,\ \eta \in [1/6, 1/4). \tag{1.17}$$

REMARK 1.3. It is interesting to note that the condition of Theorem 1.2 refers only for the Riemann zeta function, not for the Dedekind zeta function of K.

REMARK 1.4. The result (1.14) is nearly as strong as the density hypothesis if $n=3$ and $\eta \to 0$ since it tells

$$B_K(\eta) \le \frac{2}{1-2\eta} = 2 + O(\eta). \tag{1.18}$$

In comparison, we note that the first result of similar strength (even slightly weaker) for the Riemann zeta function,

$$B_K(\eta) = B_{\mathbb{Q}}(\eta) \le 2 + O(\eta^{0.14}) \tag{1.19}$$

was shown in 1954 by Turán [12] using his famous power sum method [13]. In contrast to this we do not need the power sum method but use a simple ingenious idea of Halász [2] in the theorem [9] from which we deduce our present result. This idea of Halász led in their joint work with Turán [3] to the first proof of the density hypothesis for $\zeta(s)$ for $\eta < c_0$ with a small positive value c_0.

REMARK 1.5. In the next section we will show that Theorem 1.1 improves the earlier best result of Heath-Brown [4] for $n=3$ and $\eta < 37/252$, further for $n \ge 4$ and $\eta < 1/6$. It is sharper for all $n \ge 7$ than the result (1.11) of Paul and Sankaranarayanan [7] for all values of η.

2. COMPARISON WITH EARLIER RESULTS

In the case $n \ge 7$ we have to compare our theorem with the estimate (1.11) coming from [7]. Due to the condition $2\eta < 3/(n+3)$ (cf. (1.12)) we have clearly

$$\frac{\frac{2n}{3}}{1-2\eta} < \frac{\frac{2n}{3}}{1-\frac{3}{n+3}} = \frac{\frac{2n}{3}}{\frac{n}{n+3}} = \frac{2(n+3)}{3}. \tag{2.1}$$

Hence, our result is stronger than the Theorem of Paul and Sankaranarayanan for every $n \ge 7$.

In case of $n \ge 4$ we have

$$\frac{2/3}{1-2\eta} < 1 \quad \text{for} \quad \eta < 1/6. \tag{2.2}$$

Consequently, our result improves (1.7) of Heath-Brown if $\eta < 1/6$ and $n \ge 4$.

In case of $n=3$ and $\eta \le 4/29$ $(< 1/6)$ the inequality (2.2) shows that our result (1.13) is better. Finally for the remaining case of $\eta \in (4/29, 37/252)$ we have

$$4\eta < 37/63 \quad \Leftrightarrow \quad 1-4\eta > 26/63 \tag{2.3}$$

which implies

$$\frac{26/21}{1-4\eta} < 3 = n \tag{2.4}$$

and shows that our result improves (1.7) for $\eta < 37/252 = 0.146825\ldots$ in case of $n=3$.

It follows again from (2.2) that our conditional Theorem 1.2 would improve (1.7) for the range $\eta < 1/6$ for $n=3$. We note that we do not consider the quadratic case for which Heath-Brown's work [4] contains important results.

3. PROOFS OF THEOREMS 1.1 AND 1.2. PREPARATION

Our proof will be based on our knowledge of the growth rate of the *Dedekind zeta function and Riemann zeta function* on the critical line and on a general zero density theorem of the present author which is based on the pioneering ideas of Halász, Montgomery and Turán. The most crucial ideas are

(i) Heath-Brown's estimate [5]

$$\zeta_K(1/2+it) \ll_{K,\varepsilon} |t|^{n/6+\varepsilon}; \tag{3.1}$$

(ii) the classical estimate

$$\zeta(1/2+it) \ll_{\varepsilon} |t|^{1/6+\varepsilon} \tag{3.2}$$

reached simultaneously by Hardy–Littlewood and van der Corput (cf. Titchmarsh [11], Theorems 5.5 and 5.12);

(iii) for $n=3$ the deep estimate of Bourgain [1] for the Riemann zeta function:

$$\zeta(1/2+it) \ll_{\varepsilon} |t|^{13/84+\varepsilon}; \tag{3.3}$$

(iv) a simple but ingenious idea of Halász [2] incorporated already in the author's general density theorem [9].

In order to formulate the mentioned general density theorem we introduce some notations. We use Lindelöf's μ-function

$$\mu_\zeta(\sigma_0) = \inf\{\mu; |\zeta(\sigma+it)| \ll (1+|t|)^\mu \ \text{ for } \ \sigma \ge \sigma_0, |s-1| \ge c_0, 1 < |t| \le T\} \tag{3.4}$$

and its analogue for the Dedekind zeta function

$$\mu_K(\sigma_0) = \inf\{\mu; |\zeta_K(\sigma+it)| \ll (1+|t|)^\mu \ \text{ for } \ \sigma \ge \sigma_0, |s-1| \ge c_0, 1 < |t| \le T\}, \tag{3.5}$$

and the functions derived from it, with arbitrary functions satisfying $\lambda_\zeta(2\eta) \le 1$ and

$$\lambda_\zeta(\eta) \ge \inf_{0<b;(b+1)\eta\le 1/2} \frac{\mu_\zeta(1-(b+1)\eta)}{b\eta}, \tag{3.6}$$

$$\lambda_K(\eta) \ge \inf_{0<a;(a+1)\eta\le 1/2} \frac{\mu_K(1-(a+1)\eta)}{a\eta}. \tag{3.7}$$

We note that as it turns out from the proof we can substitute $\lambda_\zeta(\eta)$ and $\lambda_K(\eta)$ by any functions larger than defined by the true infimum in (3.6) and (3.7). We actually do not know the exact values of them, at any rate, only upper bounds for them. We further introduce for $\lambda_K(\eta)\lambda_\zeta(2\eta) \ne 0$

$$d_K(\eta) := \lambda_K(\eta)/\lambda_\zeta(2\eta). \tag{3.8}$$

For $n \ge 4$ it is sufficient to use the classical result $\mu_\zeta(1/2) \le 1/6$ (see (3.2)) and from it with the choice $(b+1)\eta = 1/2$ and by the above remark we can use the values

$$\lambda_\zeta(\eta) = \frac{1/6}{1/2-\eta} = \frac{1/3}{1-2\eta} \Longrightarrow \lambda_\zeta(2\eta) = \frac{1/3}{1-4\eta}, \tag{3.9}$$

while for $n=3$ we will use the deep result (3.3) of Bourgain [1] which we write in the form

$$\mu_\zeta(1/2) \le R_0/6 \quad \text{with} \quad R_0 = 13/14 \tag{3.10}$$

in order to see the effect of the improvement of Bourgain more clearly. In the conditional case (Theorem 1.2) we can take $R_0 = 3/4$ to obtain $\mu_\zeta(1/2) \le 1/8$. The inequality (3.10) yields

$$\lambda_\zeta(\eta) = \frac{R_0/3}{1-2\eta} \Rightarrow \lambda_\zeta(2\eta) = \frac{R_0/3}{1-4\eta}. \tag{3.11}$$

Similarly to this the result (3.1) of Heath-Brown gives

$$\mu_K(1/2) \le n/6 \Rightarrow \lambda_K(\eta) = \frac{n/3}{1-2\eta}, \tag{3.12}$$

consequently (where we can take $R_0 = 1$ for $n \geq 4$)

$$d_K(\eta) = \frac{n(1-4\eta)}{R_0(1-2\eta)}. \tag{3.13}$$

In our general theorem ([8] or [9]) we consider a general Dirichlet series $f(s)$ in place of $\zeta_K(s)$ with the conditions

(i) $f(s) = \sum_{\nu=1}^{\infty} \frac{f_\nu}{\nu^s}$ ($f_1 \neq 0$) and $\frac{1}{f(s)} = \sum_{\nu=1}^{\infty} \frac{g_\nu}{\nu^s}$ analytic for $\sigma > 1$;

(ii) $f(s)$ can be extended analytic for $\sigma \geq 1 - \alpha_f$, $\alpha_f > 0$ up to a simple pole at $s = 1$, and

(iii) $f_\nu \ll \nu^\Delta$, $g_\nu \ll \nu^\Delta$ for every $\Delta > 0$.

We define $\mu_f(\eta)$, $\lambda_f(\eta)$ and $d_f(\eta)$ analogously to $\mu_K(\eta)$, $\lambda_K(\eta)$ and $d_K(\eta)$, defined in (3.6)–(3.8) above.

Under the above conditions our recent theorem [8] proves

$$N_f(1-\eta, T) \ll_{f,\varepsilon,\eta} T^{B_f(\eta)\eta+\varepsilon} \quad \text{for} \quad \eta < 1/4 \tag{3.14}$$

with (using $\lceil x \rceil = \min\{n \in Z^+, n \geq x\}$)

$$B_f(\eta) \leq 2\lambda_\zeta(2\eta) \max \left\{ d_f(\eta) \left\lceil d_f^{-1}(\eta) \right\rceil, 1 + \left\lceil d_f^{-1}(\eta) \right\rceil^{-1} \right\} \tag{3.15}$$

if $\lambda_f(\eta)\lambda_\zeta(2\eta) \neq 0$, further

$$B_f(\eta) \leq 2\lambda_f(\eta) \quad \text{if} \quad \lambda_\zeta(2\eta) = 0 \tag{3.16}$$

and

$$B_f(\eta) \leq 2\lambda_\zeta(2\eta) \quad \text{if} \quad \lambda_f(\eta) = 0. \tag{3.17}$$

In our case $f(s) = \zeta_K(s)$ we can take $\alpha_K = 1/2$ and (iii) is true since denoting the divisor function by $\tau(n)$ we have by the explanation following (7) of [4]

$$|g_\nu| \leq f_\nu \leq (\tau(\nu))^n \ll \nu^\Delta \quad \text{for every} \quad \Delta > 0 \tag{3.18}$$

following from classical knowledge ([6]) about the number field K.

REMARK 3.1. As noted in Turán's book [13], p. 367 the growth rate of *both* $\zeta(s)$ and $f(s)$ play a crucial role in the density estimate for the zeros of $f(s) = \zeta_K(s)$ (in our present case).

4. PROOFS OF THEOREMS 1.1 AND 1.2. CONTINUATION

As mentioned earlier we will suppose throughout that $n \geq 3$. Let us suppose $\lambda_\zeta(2\eta) > 0$. We will investigate first the value of $d_K(\eta)$. In order to beat (1.7) we need by the first term in (3.15) (for $\lambda_K(\eta)$ see (3.12))

$$n > 2\lambda_\zeta(2\eta) \cdot \frac{\lambda_K(\eta)}{\lambda_\zeta(2\eta)} \left\lceil \frac{\lambda_\zeta(2\eta)}{\lambda_K(2\eta)} \right\rceil \geq 2\lambda_K(\eta) = \frac{2n/3}{1-2\eta} \tag{4.1}$$

the condition

$$\eta < 1/6, \tag{4.2}$$

which we assume in the following. If $\eta < 1/6$ then $(1-4\eta)/(1-2\eta) > 1/2$, hence we have from (3.13)

$$d_K(\eta) = \frac{n(1-4\eta)}{R_0(1-2\eta)} > \frac{n}{2R_0} \geq 2 \quad \text{for} \quad n \geq 4 \tag{4.3}$$

and

$$d_K(\eta) > 1 \quad \text{for} \quad n = 3. \tag{4.4}$$

Consequently we have always $d_K(\eta) > 1$, so $\left\lceil d_K^{-1}(\eta) \right\rceil = 1$ and so

$$\max \left(d_K(\eta) \left\lceil d_K^{-1}(\eta) \right\rceil, 1 + \left\lceil d_K^{-1}(\eta) \right\rceil^{-1} \right) = \max(d_K(\eta), 2). \tag{4.5}$$

Hence, we have

$$B_K(\eta) \leq 2\lambda_K(\eta) = \frac{2n/3}{1-2\eta} \quad \text{for} \quad n \geq 4 \tag{4.6}$$

and, by (3.11)–(3.12),

$$B_K(\eta) \le \max\left(\frac{2n/3}{1-2\eta}, \frac{4R_0/3}{1-4\eta}\right) \quad \text{for} \;\; n = 3. \tag{4.7}$$

This means that for $n = 3$ we have

$$B_K(\eta) \le \frac{2}{1-2\eta} \quad \text{if} \quad 1 - 4\eta \ge 2R_0(1-2\eta)/3. \tag{4.8}$$

We note that $2/(1-2\eta) < 3$ iff $\eta < 1/6$. With the unconditional value $R_0 = 13/14$ this means that (4.8) holds for $\eta \le 4/29$. For $\eta \ge 4/29$ we have

$$B_K(\eta) \le \frac{26/21}{1-4\eta}. \tag{4.9}$$

This means that our estimate (4.7) improves (1.7) of Heath-Brown in case of $n = 3$ iff

$$\frac{26/21}{1-4\eta} < 3 \quad \Leftrightarrow \quad \eta < \frac{37}{252} = 0.146825\ldots \left(< \frac{1}{6}\right). \tag{4.10}$$

If we consider the conditional Theorem 1.2 when we can take $\mu(1/2) \le 1/8$, i.e. $R_0 \le 3/4$ then in case of $n = 3$ similarly to (4.8) now

$$B_K(\eta) \le \frac{2}{1-2\eta} \quad \text{if} \quad 1 - 4\eta \ge (1-2\eta)/2 \;\; \Leftrightarrow \;\; \eta \le 1/6. \tag{4.11}$$

Hence conditionally, our estimate (4.7) improves (1.7) of Heath-Brown for the whole interval $\eta < 1/6$ for $n = 3$ too as unconditionally in all cases $n \ge 4$, since by (4.6) we have clearly for $\eta < 1/6$, $n \ge 4$

$$B_K(\eta) \le \frac{2n/3}{1-2\eta} < \frac{2n/3}{2/3} = n. \tag{4.12}$$

Hence, we do not even need the deep result (3.3) of Bourgain for $n \ge 4$, similarly to the case $n = 3$, $\eta \le 1/8$, as we can easily see from (4.8). More exactly, it is sufficient to use the classical $\mu_\zeta(1/2) \le 1/6$ of Hardy–Littlewood or van der Corput.

Finally, if the Lindelöf Hypothesis holds for the Riemann zeta function, i.e. $\mu_\zeta(1/2) = 0$, then $\lambda_\zeta(2\eta) = 0$ for $\eta \le 1/4$ and then by (3.16) we have

$$B_K(\eta) \le 2\lambda_K(\eta) = \frac{2n/3}{1-2\eta} < n \tag{4.13}$$

for all $n \ge 3$ and $\eta < 1/6$, similarly to Theorem 1.2. □

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