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COHOMOGENEITY ONE G-PSEUDOMANIFOLDS

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Abstract

In this note we prove structure theorems for cohomogeneity one G-pseudomanifolds (i.e. with a one dimensional orbit space), which generalize the well-known results for smooth G-manifolds, see [1, 2]. The concept of a G-pseudomanifold was introduced by the author in [3], where the basic theory is developed.

§1. Introduction

To begin we first recall some definitions given in [3] which will be used in this work.

Given a topological space X its *open cone*, denoted by cX, is defined as follows: $cX = X \times [0,1)/(x,0) \sim (x',0)$. Let $[x,r]$ denote the corresponding equivalence class and $*$ the distinguished point $[x,0]$. For $X = \emptyset$ we let $cX = \{*\}$.

Now let G be a compact Lie group. By a *G-space* we mean a completely regular space X, together with a continuous action $G \times X \to X$.

Let $G_x = \{g \in G : g \cdot x = x\}$ be the isotropy subgroup of G at $x \in X$. Also let X/G denote the corresponding orbit space with the quotient topology induced by the canonical projection $\pi : X \to X/G$.

There is a canonical action on cX, which is the following: $g \cdot [x,r] = [g \cdot x, r]$, where $g \in G$, $x \in X$, $r \in [0,1)$. Notice that the distinguished point $*$ of cX is G-fixed.

If X is a G-space, we now define the concept of a conical slice.

Definition 1.1. Given an orbit P in X, then a slice S_x in X at $x \in P$ is called a *conical slice* of P if it satisfies the following condition:

There is a compact H-space L (possibly empty), without fixed points, called a *link* of P, where $H = G_x$, together with an H-equivariant homeomorphism $\phi : S_x \to \Re^{i_0} \times cL$, for an integer $i_0 \geqq 0$, where H acts trivially on $\Re^{i_0}$.

For the definition of slices, and their existence in a completely regular G-space, see [1], pp. 82–86.

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Now we define the concept of a G-pseudomanifold.

DEFINITION 1.2. A (-1)-*dimensional* G-*pseudomanifold* is the empty set. An n-*dimensional* G-*pseudomanifold* $(n \geqq 0)$ is a G-space X, with a connected orbit space X/G, satisfying the following condition:

(C) Each orbit P in X has a conical slice $S_x \cong^{\phi} \Re^{i_0} \times cL$, such that L is an $n-i-1$-dimensional H-pseudomanifold, where $i = i_0 + \dim G/H \neq n-1$.

Note that this definition is more restrictive than the one given in [3].

Here are some examples.

(1) Let G be a compact Lie group acting locally linearly on a completely regular manifold M, [1, p. 171]. Assume that the orbit type filtration of M has no strata of codimension one, and M/G is connected. Claim M is a G-pseudomanifold.

The proof is by induction on the dimension of M. For $\dim M = 0$ it is trivial. Assume that the claim is valid for manifolds with dimension strictly smaller than $\dim M$.

Given an orbit P in M, choose a point $x \in P$ with $G_x = H$. Consider a linear slice $S_x \cong E$ at x in M, where H acts orthogonally on a Euclidean space E. Let $V = (E^H)^{\perp}$ denote the orthogonal complement of E^H with respect to an H-invariant inner product on E. Put L to be the unit sphere of V with respect to the associated H-invariant metric.

Since $L \cong S^l$, where $l+1 = \dim V$, we can put a smooth structure on L, which is independent of the choice of an orthonormal basis for V. Clearly, $H \times L \to L$ is locally linear [1, p. 308], since H acts smoothly on V, and L is a submanifold.

Hence there is an H-equivariant homeomorphism $S_x \cong E^H \oplus (E^H)^{\perp} \cong \Re^{i_0} \times cL$, where $i_0 = \dim E^H$. By the inductive hypothesis, we have that L is an H-pseudomanifold. Therefore M is a G-pseudomanifold.

(2) Let G be a compact Lie group which acts smoothly on a paracompact manifold M, and H a closed normal subgroup of G. Then $\Phi : G \times M \to M$ induces a canonical action $\widetilde{\Phi} : G/H \times M/H \to M/H$. We shall examine when M/H is a G/H-pseudomanifold.

Choose a Riemannian metric on M invariant under G. Then we have $N_x(H \cdot x) = T_x(H \cdot x)^{\perp} \oplus N_x(G \cdot x)$ for $x \in M$, where N_x denotes the orthogonal complement in $T_x(M)$, and $\perp$ the orthogonal complement in $T_x(G \cdot x)$. Now for some $r > 0$, there is a Riemannian normal coordinate system at x of radius r. Let U_x be the union of geodesic segments of length less than r, starting from x in a direction orthogonal to $H \cdot x$. Then U_x is a linear K-slice at x in M, for $K = G_x \cap H$. Clearly, U_x contains a linear G_x-slice S_x.

Now if $\pi : M \to M/H$ is the canonical projection then $\pi(U_x) \cong U_x/K$. Hence $\pi(S_x) \cong S_x/K$, which we consider embedded in M/H. Clearly, the