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ON LONGEST CYCLES IN GRID GRAPHS

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0. Introduction

In 1966 T. Gallai [1] asked the following question: Do all longest paths of a connected graph contain a common vertex? The answer was given by H. Walther [6], who presented a graph without such a vertex. The analogous question about longest cycles is obviously settled by the Petersen graph. T. Zamfirescu [8] generalized these two questions to the problem: Do there exist k-connected graphs such that for any j of its vertices there is a longest path (cycle) omitting them? And what is the minimal number of vertices of such graphs? A recent survey of results in this direction can be found in H.-J. Voss' book [5]. One can ask whether there exist interesting families of graphs each member of which possesses at least one vertex contained in each of its longest cycles. H. Walther [6] showed that the planar graphs are no such family; then T. Zamfirescu and B. Grünbaum [2] presented 2-connected, respectively 3-connected, planar counterexamples. On the other hand W. T. Tutte [4] obviously provided such a family with his theorem asserting that every 4-connected planar graph is hamiltonian. The graph of Fig. 1 shows that the bipartite graphs do not form such a family either.

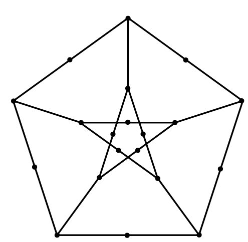

Fig. 1

In this paper we are concerned with lattice graphs. First we present a lattice graph with empty intersection of its longest cycles. Then we prove about the family of all grid graphs, a natural subclass of the family of all

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lattice graphs, that they do constitute a family of the desired kind (without being all hamiltonian). To do this, we need three lemmas on the shape of longest cycles in grid graphs. Then we specify four vertices which must lie on any longest cycle.

The *lattice* Γ is the infinite graph which is embedded in $\mathbf{R}^2$ with $\mathbf{Z}^2$ as vertex set and with an edge between any two vertices at Euclidean distance 1. A *lattice graph* is here a finite 2-connected subgraph of Γ. A lattice graph G as a subset of $\mathbf{R}^2$ is denoted by $\operatorname{set} G$. Any lattice graph G contains exactly one cycle K, called the *outer cycle* of G, such that $\operatorname{set} G$ and the unbounded component of $\mathbf{R}^2 \setminus \operatorname{set} K$ are disjoint. A lattice graph G is called a *grid graph* if every edge and every vertex of Γ in the closure of the bounded component of $\mathbf{R}^2 \setminus \operatorname{set} K$ belongs to G. This notion is due to T. Zamfirescu, who also suggested the present investigation.

As usual, $V(G)$ is the vertex set, $E(G)$ the edge set of the graph G. $[u, v]$ means the edge between the vertices u and v. For a graph G with a subgraph H, the graph $G - H$ means the graph obtained from G by deleting every vertex of H and any edge of G incident to such a vertex.

1. A lattice graph the longest cycles of which have empty intersection

C. Thomassen found the graph T of Fig. 2 (presented in [9], p. 216), such that for each vertex there exists a longest cycle of T avoiding it. By appropriately modifying T we have constructed lattice graphs with the same property. Fig. 3 shows such a graph found by G. Wegner, which is reproduced here with his kind permission.

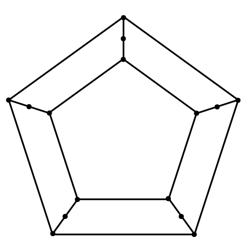

Fig. 2

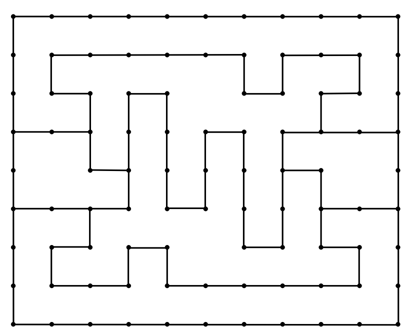

Fig. 3