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MAPPINGS PRESERVING SOME GEOMETRICAL FIGURES

S.-M. JUNG* (Chochiwon)

Abstract. We introduce new characterizations of linear isometries. More precisely, we prove that if a one-to-one mapping $f : \mathbf{R}^n \to \mathbf{R}^n$ $(n > 1)$ maps the periphery of every regular triangle (quadrilateral or hexagon) of side length $a > 0$ onto the periphery of a figure of the same type with side length $b > 0$, then there exists a linear isometry $I : \mathbf{R}^n \to \mathbf{R}^n$ up to translation such that $f(x) = (b/a)I(x)$.

1. Introduction

Let X and Y be normed spaces. A mapping $f : X \to Y$ is called an isometry if f satisfies the equality

$$\big\| f(x) - f(y) \big\| = \|x - y\|$$

for all $x, y \in X$. A distance $r > 0$ is said to be preserved (conservative) by a mapping $f : X \to Y$ if $\big\| f(x) - f(y) \big\| = r$ for all $x, y \in X$ with $\|x - y\| = r$.

If f is an isometry, then every distance $r > 0$ is conservative by f, and conversely. We can now raise a question whether each mapping that preserves certain distances is an isometry. Indeed, A. D. Aleksandrov [1] had raised a question whether a mapping $f : X \to X$ preserving a distance $r > 0$ is an isometry, which is now known to us as the Aleksandrov problem. Without loss of generality, we may assume $r = 1$ when X is a normed space (see [14]). F. S. Beckman and D. A. Quarles [2] solved the Aleksandrov problem for finite-dimensional real Euclidean spaces $X = \mathbf{R}^n$ $(1 < n < \infty)$ (see also [4, 8, 10, 15]):

Theorem 1. *If a mapping $f : \mathbf{R}^n \to \mathbf{R}^n$ $(1 < n < \infty)$ preserves a distance $r > 0$, then f is a linear isometry up to translation.*

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It seems to be interesting to investigate whether the 'distance $r > 0$' in the above theorem can be replaced by some properties characterized by 'geometrical figures' without loss of its validity.

In Theorem 5 of this paper, we will prove that if a one-to-one mapping $f : \mathbf{R}^n \to \mathbf{R}^n$ $(n > 1)$ maps the periphery of every regular triangle (quadrilateral or hexagon) of side length $a > 0$ onto the periphery of a figure of the same type with side length $b > 0$, then there exists a linear isometry $I : \mathbf{R}^n \to \mathbf{R}^n$ up to translation such that $f(x) = (b/a)I(x)$.

Throughout this paper, by the triangles, quadrilaterals and hexagons, we will denote the peripheries of the geometrical figures. A side of a triangle (a quadrilateral or a hexagon) without its endpoints (vertices) is called an open side.

2. A property determined by regular triangles

For a given integer $n \geqq 2$ and a constant $r > 0$, we use T_r^n to denote the set of all regular triangles in $\mathbf{R}^n$ whose side length is r.

We notice that if two distinct regular triangles $P_1, P_2 \in T_r^n$ intersect each other in infinitely many points, which are spread out on two open sides of P_1, then P_2 is either coincident with P_1 or a shift of P_1 along one side of P_1 including infinitely many intersection points. For the latter case, P_2 has to intersect one of the relevant open sides of P_1 in exactly one point.

Theorem 2. *Given integers $m, n \geqq 2$ and constants $a, b > 0$, let $f : \mathbf{R}^m \to \mathbf{R}^n$ be a one-to-one mapping under which the image of each regular triangle in T_a^m belongs to T_b^n. Then, the equality $\left| f(x) - f(y) \right| = b$ holds true for all $x, y \in \mathbf{R}^m$ with $|x - y| = a$.*

Proof. Let $x, y \in \mathbf{R}^m$ be separated by a distance a and choose another point $z \in \mathbf{R}^m$ such that the three points comprise the vertices of a triangle $P_1 \in T_a^m$. Furthermore, choose a point $w \in \mathbf{R}^m \setminus \{x\}$ coplanar with x, y, z such that y, z, w are the vertices of a triangle $P_2 \in T_a^m$.

Let x', y', z' be the vertices of the image of the triangle P_1 under f. Assume that the image of the side $\overline{yz}$ is spread out on the open sides $\overline{x'y'}$ and $\overline{y'z'}$. If each of the open sides includes more than one image point of $\overline{yz}$, then the image of the triangle P_2 would coincide with the image of P_1, which is contrary to the injectivity of f.

Without loss of generality, assume that $\overline{x'y'}$ contains infinitely many image points of $\overline{yz}$ and the open side $\overline{y'z'}$ contains only one image point of $\overline{yz}$, say u', and let u be the unique point of $\overline{yz}$ satisfying $u' = f(u)$, where u is allowed to be y or z.

Choose another regular triangle $P_3 \in T_a^m$, which is distinct from P_1 and P_2, with the following properties: