Acta Math. Hungar.
101 (4) (2003), 323–344.

ON SUMMABILITY OF WEIGHTED LAGRANGE INTERPOLATION. I (GENERAL WEIGHTS)

L. SZILI and P. VÉRTESI (Budapest)*

Abstract. The paper is devoted to the study of summability of weighted Lagrange interpolation on the roots of orthogonal polynomials with respect to a weight function w. Starting from the Lagrange interpolation polynomials we shall construct a wide class of discrete processes (using summations) which are uniformly convergent in a suitable Banach space $\left(C_\varrho, \|\cdot\|_\varrho\right)$ of continuous functions (ϱ denotes a weight). We shall give such conditions with respect to w, ϱ, $\left(C_\varrho, \|\cdot\|_\varrho\right)$ and to summation methods for which the uniform convergence holds. Error estimates for the approximation will also be considered.

1. Introduction and formulation of the problem

The aim of this paper is to construct a wide class of discrete processes (using algebraic polynomials) which are uniformly convergent in suitable spaces of continuous functions.

In the trigonometric case (when the fundamental point system is equidistant) sequences of this type can be constructed in a unified way (see [12] and [13]). The algebraic case is more complicated.

One of the most natural discrete approximating tools is the Lagrange interpolation. However, as it was proved by G. Faber in 1914, there is *no* point system for which the corresponding sequence of Lagrange interpolation polynomials $L_n f$ $(n \in \mathbf{N})$ would converge uniformly for every continuous function. On the other hand by replacing $L_n f$ $(n \in \mathbf{N})$ with a suitable summation, one can get convergence.

Let $I := (a, b) \subset \mathbf{R}$ be a bounded or an unbounded interval on the real line and let $w : I \to [0, +\infty)$ be a weight function for which there exists a

*Research supported by the Hungarian National Scientific Research Foundation (OTKA) under Grant Ns. T032719, T32872, T37299 and T22943.

Key words and phrases: weighted interpolation, weighted approximation, summation methods, uniform convergence.

2000 Mathematics Subject Classification: 40C05, 40D05, 40G05, 41A05, 41A10.

unique system of orthonormal polynomials $p_n(w)$ (i.e. $0 < \int_I w < +\infty$ and $\int_I p_n(w)p_m(w)w = \delta_{m,n}$ for all $m, n \in \mathbf{N}_0 := \{0, 1, 2, \dots\}$).

For an infinite real matrix

$$\Theta = (\theta_{l,n})_{l=0,\dots,n-1}^{n\in\mathbf{N}} \qquad \big(n \in \mathbf{N} := \{1, 2, \dots\}\big)$$

we shall consider a suitable summation of Lagrange interpolation on the root system $U_n(w)$ $(n \in \mathbf{N})$ of $p_n(w)$'s. These polynomials will be denoted by $L_n^{\Theta}\big(f, U_n(w), \cdot\big)$.

We also take another weight function $\varrho : I \to [0, +\infty)$ and a Banach space $\big(C_\varrho(I), \|\cdot\|_\varrho\big)$, where $C_\varrho(I)$ is a linear subspace of real valued continuous functions defined on I (shortly $C(I)$) and

$$\|f\|_\varrho := \sup_{x\in I} \big|(f\varrho)(x)\big| \qquad \big(f \in C_\varrho(I)\big).$$

We shall investigate the following problem: *How can the matrix* Θ, *weights* w, ϱ *and the space* $\big(C_\varrho(I), \|\cdot\|_\varrho\big)$ *be chosen satisfying*

$$\lim_{n\to+\infty} \Big\|\Big(f - L_n^{\Theta}\big(f, U_n(w), \cdot\big)\Big)\Big\|_\varrho = 0$$

for all $f \in C_\varrho(I)$*?*

The goal of this paper is to give *general* conditions with respect to $w, \varrho, \big(C_\varrho(I), \|\cdot\|_\varrho\big)$ and Θ which guarantee the uniform convergence of $L_n^{\Theta} f$ $(n \in \mathbf{N})$ for every $f \in C_\varrho(I)$.

In subsequent papers we will show that our general conditions hold for *many special weights* (for example Jacobi-, exponential- and Laguerre ones).

2. Θ-summation of Lagrange interpolation

Suppose that the weight function satisfies the following conditions:

$$\text{(W)} \qquad \begin{cases} w : I \to \mathbf{R} \text{ is nonnegative, measurable in Lebesgue's sense,} \\ \displaystyle\int_I w > 0 \text{ and } \int_I x^n w(x)\,dx < +\infty \text{ for all } n \in \mathbf{N}_0. \end{cases}$$

Then for all $n \in \mathbf{N}_0$ there exists a unique polynomial

$$p_n(w, x) = \gamma_n(w)x^n + \cdots, \qquad \gamma_n(w) > 0$$