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STRONG APPROXIMATION BY FOURIER–LAPLACE SERIES ON THE UNIT SPHERE $\mathbf{S}^{n-1}$

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Abstract. We study the strong approximation properties of the Cesàro means of order δ of the Fourier–Laplace expansion of functions integrable on the unit sphere $\mathbf{S}^{n-1}$, where $\delta \geqq \lambda := (n-2)/2$, the latter being the critical index for Cesàro summability of Fourier–Laplace series on $\mathbf{S}^{n-1}$. The main purpose of this paper is to extend known results from the unit circle $\mathbf{S}^1$ to the general sphere $\mathbf{S}^{n-1}$ with $n \geqq 3$. We prove six theorems. To prove Theorems 1–3, our machinery is based on the equiconvergent operator $E_N^\delta(f)$ of the Cesàro means $\sigma_N^\delta(f)$ on $\mathbf{S}^{n-1}$ introduced by Wang Kunyang for $\delta > -1$. We prove in Theorem 6 that $E_N^\delta(f)$ is also equiconvergent with $\sigma_N^\delta(f)$ for $\delta > 0$ in the case of strong approximation. To prove Theorems 4 and 5, we rely on known equivalence relations between K-functionals and moduli of continuity.

1. Introduction

Let $n \geqq 2$ be an integer, and denote by

$$\mathbf{S}^{n-1} := \left\{ x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n : \sum_{k=1}^{n} x_k^2 = 1 \right\}$$

the unit sphere with center at the origin in the n-dimensional Euclidean space $\mathbf{R}^n$. We equip $\mathbf{S}^{n-1}$ with the surface Lebesgue measure $d\sigma(x)$ and

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write $f \in L^p(\mathbf{S}^{n-1})$ for some $1 \leqq p \leqq \infty$ if

$$\|f\|_p := \begin{cases} \left\{ \displaystyle\int_{\mathbf{S}^{n-1}} \left| f(x) \right|^p d\sigma(x) \right\}^{1/p} & \text{for } 1 \leqq p < \infty, \\ \operatorname{ess\,sup} \left\{ \left| f(x) \right| : x \in \mathbf{S}^{n-1} \right\} & \text{for } p = \infty. \end{cases}$$

The so-called translation operator S_θ on $\mathbf{S}^{n-1}$ was introduced by Rudin [9] as follows: given $0 < \theta < \pi$, $f \in L^1(\mathbf{S}^{n-1})$, and $x \in \mathbf{S}^{n-1}$, let

$$S_\theta(f)(x) := \frac{1}{|\mathbf{S}^{n-2}| \sin^{n-2}\theta} \int_{\{y \in \mathbf{S}^{n-1} : xy = \cos\theta\}} f(y)\, d\ell_{x,\theta}(y),$$

where $|\mathbf{S}^{n-2}|$ means the surface area of $\mathbf{S}^{n-2}$, xy is the inner product of $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n) \in \mathbf{S}^{n-1}$ defined by

$$xy := \sum_{k=1}^{n} x_k y_k,$$

and $d\ell_{x,\theta}(y)$ is the Lebesgue measure on the parallel $\{y \in \mathbf{S}^{n-1} : xy = \cos\theta\}$ (which is a discrete measure in case $n = 2$). The S_θ is a natural extension of the notion of the ordinary translation from the unit circle $\mathbf{S}^1$ to $\mathbf{S}^{n-1}$ when $n \geqq 3$. It is well known that S_θ is a bounded operator on $L^p(\mathbf{S}^{n-1})$. More exactly, for all $0 < \theta < \pi$ and $1 \leqq p \leqq \infty$ we have

$$\|S_\theta\|_{(p,p)} := \sup \left\{ \left\| S_\theta(f) \right\|_p \big/ \|f\|_p : 0 \neq f \in L^p(\mathbf{S}^{n-1}) \right\} = 1.$$

Let r be a positive real number. We recall that the difference operator Δ_θ^r of order r with increment θ is defined in terms of S_θ as follows:

$$\Delta_\theta^r := (I - S_\theta)^{r/2} = \sum_{k=0}^{\infty} \frac{\psi^{(k)}(0)}{k!} (S_\theta)^k,$$

where I is the identity operator, $(S_\theta)^0 := I$, and

$$\psi(t) := (1-t)^{r/2} = \sum_{k=0}^{\infty} \binom{r/2}{k} (-1)^k t^k, \quad |t| < 1.$$