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LARGE TRANSVERSALS TO SMALL FAMILIES OF UNIT DISKS

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Abstract. We determine conditions under which a finite family $\mathcal{F}$ of disjoint unit disks has a transversal line that intersects all but at most one member of the family. This problem is closely related to the Katchalski–Lewis conjecture for plane convex figures. We show that $T(4)$ implies $T-1$ if $|\mathcal{F}| \leqq 7$.

1. Introduction

In this paper, a convex figure is a compact convex set in the Euclidean plane E^2 with non-empty interior. In particular, we denote unit disks by A, B, C, X, Y. We use lower case letters for points of E^2 and H, L, M, N, S for lines. The *convex hull* of the sets A, B is denoted by $[A,B]$.

We say that a family $\mathcal{F}$ of non-overlapping translates of a convex figure has a *transversal* and the property T if there is a line that intersects all members of $\mathcal{F}$. If there is a line that meets all but at most k members of $\mathcal{F}$ then $\mathcal{F}$ has property $T-k$. Furthermore, when all k-membered subfamilies of $\mathcal{F}$ have property T, then we say that $\mathcal{F}$ is of property $T(k)$. Finally, an m-transversal of $\mathcal{F}$ is a line meeting m members of $\mathcal{F}$.

Transversal theory grew out of Helly's theorem which plays a prominent role in geometry. It says that if every $n+1$ members of a family of compact convex subsets of R^n have a common point then all of them have a common point. It is quite natural to replace the intersection point by a line and ask under what conditions the theorem remains true. In this paper we shall be concerned with the case of the Euclidean plane.

In 1989, Tverberg [14] proved that $T(5) \Rightarrow T$ for a disjoint family of translates of a compact convex set, a conjecture of Grünbaum in [8]. In gen-

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eral, we know that neither $T(3)$ nor $T(4)$ is enough to guarantee the same [9]. It was believed for a long time that $T(4)$ implied T for a disjoint family of unit disks; cf. Grünbaum [8]. However, as shown by the counterexample of Aronov, Goodman, Pollack, and Wenger [1], that is not the case. For $T(3)$, Katchalski and Lewis [13] proved that there exists a universal constant k_3 such that $T(3)$ implies $T - k_3$ for any finite family of disjoint translates of an arbitrary convex figure. They estimated k_3 to be about 192π and conjectured that $k_3 = 2$. It was shown, using a construction with unit disks, by Bezdek [2] that $k_3 \geqq 2$. The upper estimate was improved by Tverberg [15] and later by Holmsen [10]. The currently best known upper bound for k_3 is 22, established by Holmsen in [11]. Holmsen [10] also constructed examples which show that $k_3 \geqq 4$, and so the Katchalski–Lewis conjecture must be modified accordingly. In the same article, it was shown that $k_3 = 4$ for finite families of unit squares whose sides are parallel to the coordinate axes.

Unit disks played an important role in the history of this subject. The first major result in this direction was Danzer's [5] proof that $T(5) \Rightarrow T$. His proof was later simplified by Aronov, Goodman, Pollack, and Wenger [1]. All known examples seem to point to $T(3) \Rightarrow T - 2$ and $T(4) \Rightarrow T - 1$ for unit circles. The Katchalski–Lewis constant k_3 can be better estimated for unit circles than for general convex figures, and this supports the conjecture above.

Kaiser [12] proved that $T(3) \Rightarrow T - 12$ for a finite disjoint family of unit disks in the plane. This was recently improved to $T - 2$ by Heppes in [3] and [4].

2. The property $(T - 1)$

Henceforth, $\mathcal{F}$ denotes a family of at least six mutually disjoint unit disks in the plane with the property $T(4)$. For two disjoint disks, the *tangential separators* are the two common tangents that separate them.

THEOREM. *If* $|F| \leqq 7$ *then* F *has a* $\big(|\mathcal{F}| - 1\big)$*-transversal.*

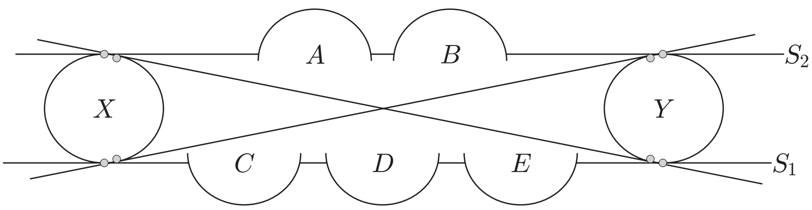

Fig. 1