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ZERO-DIVISOR GRAPH OF $C(X)$

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Abstract. In this article the zero-divisor graph $\Gamma\big(C(X)\big)$ of the ring $C(X)$ is studied. We associate the ring properties of $C(X)$, the graph properties of $\Gamma\big(C(X)\big)$ and the topological properties of X. Cycles in $\Gamma\big(C(X)\big)$ are investigated and an algebraic and a topological characterization is given for the graph $\Gamma\big(C(X)\big)$ to be triangulated or hypertriangulated. We have shown that the clique number of $\Gamma\big(C(X)\big)$, the cellularity of X and the Goldie dimension of $C(X)$ coincide. It turns out that the dominating number of $\Gamma\big(C(X)\big)$ is between the density and the weight of X. Finally we have shown that $\Gamma\big(C(X)\big)$ is not triangulated and the set of centers of $\Gamma\big(C(X)\big)$ is a dominating set if and only if the set of isolated points of X is dense in X if and only if the socle of $C(X)$ is an essential ideal.

1. Introduction

Let $C(X)$ be the ring of all real valued continuous functions on a completely regular Hausdorff space X. As in [1] and [13], by the zero-divisor graph $\Gamma\big(C(X)\big)$ of $C(X)$ we mean the graph with vertices nonzero zero-divisors of $C(X)$ such that there is an edge between vertices f, g if and only if $f \neq g$ and $fg = 0$. Our main purpose in this article is to study the relations between ring properties of $C(X)$, graph properties of $\Gamma\big(C(X)\big)$ and topological properties of the space X. In this section we determine the distance between vertices, radius, diameter and the girth of $\Gamma\big(C(X)\big)$ by properties of a completely regular Hausdorff space X. In Section 2 we will investigate cycles of $\Gamma\big(C(X)\big)$. It turns out that the cycles in $\Gamma\big(C(X)\big)$ have only length 3 or 4. Graphical characterizations of regularity and almost regularity of the ring $C(X)$ and topological characterizations for the graph $\Gamma\big(C(X)\big)$ to be

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triangulated, hypertriangulated and complemented are given in this section. Section 3 is devoted to the dominating number and the clique number of $\Gamma(C(X))$. Finally a topological characterization and an algebraic characterization for the set of centers of $\Gamma(C(X))$ to be a dominating set are given in this section.

Recall that for two vertices f and g of $\Gamma(C(X))$, $d(f,g)$ is the length of the shortest path from f to g. The diameter of $\Gamma(C(X))$ is denoted by $\operatorname{diam}\Gamma(C(X))$ and is defined by $\operatorname{diam}\Gamma(C(X)) = \sup\{d(f,g) : f,g \in \Gamma(C(X))\}$ and the girth $\operatorname{gr}\Gamma(C(X))$ of $\Gamma(C(X))$ is defined as the length of the shortest cycle in $\Gamma(C(X))$. For every $f \in C(X)$, the zeroset $Z(f)$ is the set of zeros of f, i.e., $Z(f) = \{x \in X : f(x) = 0\}$ and the reader is referred to [6], [7], [8] and [9] for undefined terms and notation.

The following lemma topologically characterizes the concept of distance in $\Gamma(C(X))$. First we need the following sublemma.

SUBLEMMA 1.1. *For every $f,g \in \Gamma(C(X))$, there exists a vertex $h \in \Gamma(C(X))$ adjacent to both f and g if and only if $\operatorname{int} Z(f) \cap \operatorname{int} Z(g) \neq \emptyset$.*

PROOF. Suppose $x \in \operatorname{int} Z(f) \cap \operatorname{int} Z(g)$ and define $h \in C(X)$ such that $h(x) = 1$ and $h(X \setminus \operatorname{int} Z(f) \cap \operatorname{int} Z(g)) = \{0\}$. Clearly $h \in \Gamma(C(X))$ and $hf = hg = 0$, i.e., h is adjacent to both f and g. Conversely if there exists $h \in \Gamma(C(X))$ adjacent to f and g, then $fh = gh = 0$ implies that $X \setminus Z(h) \subseteqq Z(f) \cap Z(g)$ and therefore $\operatorname{int} Z(f) \cap \operatorname{int} Z(g) \neq \emptyset$.

LEMMA 1.2. *Suppose that $f,g \in \Gamma(C(X))$. Then*

(i) $d(f,g) = 1$ *if and only if* $Z(f) \cup Z(g) = X$.

(ii) $d(f,g) = 2$ *if and only if* $Z(f) \cup Z(g) \neq X$ *and* $\operatorname{int} Z(f) \cap \operatorname{int} Z(g) \neq \emptyset$.

(iii) $d(f,g) = 3$ *if and only if* $Z(f) \cup Z(g) \neq X$ *and* $\operatorname{int} Z(f) \cap \operatorname{int} Z(g) = \emptyset$.

PROOF. (i) is evident. To prove (ii), first suppose that $d(f,g) = 2$, then by part (i), clearly $Z(f) \cup Z(g) \neq X$ and there exists $h \in \Gamma(C(X))$ such that h is adjacent to both f and g. Hence by Sublemma 1.1, $\operatorname{int} Z(f) \cap \operatorname{int} Z(g) \neq \emptyset$. Conversely, let $Z(f) \cup Z(g) \neq X$ and $\operatorname{int} Z(f) \cap \operatorname{int} Z(g) \neq \emptyset$. Then $d(f,g) > 1$ and by Sublemma 1.1, there is a vertex adjacent to both f and g, i.e., $d(f,g) = 2$. To establish (iii), let $d(f,g) = 3$. Clearly $Z(f) \cup Z(g) \neq X$, by part (i). By Sublemma 1.1, it is also evident that $\operatorname{int} Z(f) \cap \operatorname{int} Z(g) = \emptyset$. Conversely, suppose that $Z(f) \cup Z(g) \neq X$ and $\operatorname{int} Z(f) \cap \operatorname{int} Z(g) = \emptyset$. By parts (i) and (ii), $d(f,g) > 2$. Now if the vertex h is adjacent to f and the vertex k is adjacent to g, then $fh = gk = 0$. Hence $(X \setminus Z(h)) \cap (X \setminus Z(k))$