RECENT RESULTS IN TOPOLOGICAL GRAPH THEORY*

By

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A *graph* G is usually defined as a finite collection V of points together with a collection X of *lines*, each of which joins two distinct points and no two of which join the same pair of points. This combinatorial definition asserts nothing about drawing graphs on surfaces such as the plane, sphere, torus, projective plane etc. The purpose of this lecture is to explore some of these topological aspects of graph theory and to describe a few unsolved problems concerning them.

In order to fix the terminology of this lecture, we begin by drawing all the graphs with four points:

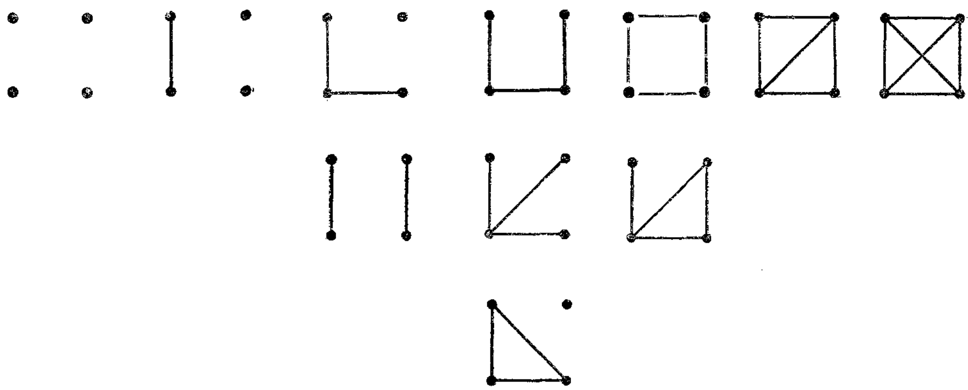

Fig. 1

Reading from left to right and top to bottom, the first of these graphs is called *totally disconnected:* it has four points and no lines. The last is the *complete graph* K_4 with four points; every pair of its points are adjacent. There are eleven different (non-isomorphic) graphs with four points, six of which are connected. The first two of these graphs having three lines are *trees.* The first of the two graphs with four lines is a *cycle.* We note that in none of these graphs does there occur any loops or parallel lines as shown in Figure 2.

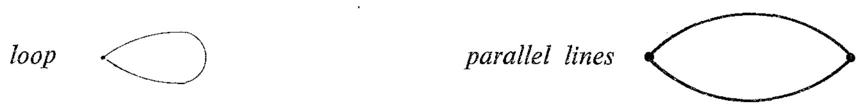

Fig. 2

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A *planar graph* is one which can be drawn in the plane in such a way that no two of its edges intersect, a *plane graph* is already so drawn. In Figure 3 the complete graph K_4 is redrawn as a plane graph differently than in Figure 1, so that it is obviously planar.

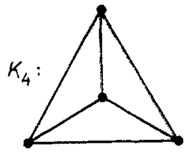

Fig. 3

The *complete bicolored graph* $K_{m,n}$ consists of m points of one color, say light, and n points of another color, say dark, in which two points are adjacent if and only if they have different colors. In Figure 4, both the complete graph K_5 and the complete bicolored graph $K_{3,3}$ are shown. It is easy to verify that neither of these graphs is planar.

Fig. 4

Two graphs are *isomorphic* if there is a $1-1$ correspondence between their sets of points which preserves adjacency. The *degree* of a point is the number of lines with which it is incident. Two graphs are *homeomorphic* if it is possible to insert new points of degree 2 into their lines in such a way that the two resulting graphs are isomorphic. A graph homeomorphic with K_4 is shown in Figure 5.

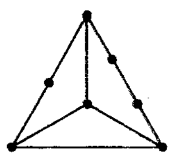

Fig. 5

With the help of these definitions we may state the first theorem of topological graph theory, due to KURATOWSKI [13].

THEOREM 0. *A graph G is planar if and only if it has no subgraph homeomorphic with* K_5 *or* $K_{3,3}$.