Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 18 (1—2), 1967, pp. 217—229.

A GENERALIZATION OF A PROBLEM OF STEINHAUS

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(*Presented by* A. RÉNYI)

§ 1.

H. STEINHAUS [1] raised the following problem: "Does there exist a family F of measurable functions such that

a) $|f(t)|=1$ for $f(t)\in F$

b) for each sequence $\{f_n(t)\}$, where $f_n(t)\in F$ the sequence

$$\frac{1}{n}\sum_{k=1}^{n} f_k(t)$$

is divergent for almost all t?"

D. G. AUSTIN proved that if $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ is a sequence of random variables with possible values 0 and 1, then there exists an increasing sequence of integers $n_1, n_2, \ldots, n_k, \ldots$ and a random variable η such that

$$\mathbf{P}\left(\frac{\xi_{n_1}+\xi_{n_2}+\ldots+\xi_{n_k}}{k}\to\eta\right)=1$$

that is for the subsequence $\xi_{n_1}, \xi_{n_2}, \ldots, \xi_{n_k}, \ldots$ the strong law of large numbers holds.

A. RÉNYI proved that the same holds for any sequence of uniformly bounded random variables (oral communication).

P. RÉVÉSZ [2] showed that the condition

$$\mathbf{M}(\xi_n^2)\leqq K \qquad (n=1,2,\ldots)$$

for some $K>0$, guarantees the existence of a subsequence for which the strong law of large numbers is valid. (He proved the following sharper theorem:

If

$$\mathbf{M}(\xi_n^2)\leqq K \qquad (n=1,2,\ldots),$$

then there exists a subsequence $\xi_{n_1}, \xi_{n_2}, \ldots, \xi_{n_k}, \ldots$ and a random variable η such that

$$\sum_{i=1}^{\infty} c_i(\xi_{n_i}-\eta)$$

is convergent with probability 1, provided that $c_1, c_2, \ldots, c_n, \ldots$ is an arbitrary sequence of real numbers for which

$$\sum_{k=1}^{\infty} c_k^2 < +\infty.)$$

He asked: whether we can replace the condition $\mathbf{M}(\xi_n^2) \leqq K$ by the condition

$$\mathbf{M}(|\xi_n|^{1+\varepsilon}) \leqq K \qquad (n=1,2,\ldots)$$

with some ε $(0 \leqq \varepsilon < 1)$.

In this paper we prove that this conjecture holds with $\varepsilon = 0$, that is the following theorem is true:

THEOREM 1. *If $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ is a sequence of random variables for which*

$$\liminf_{n=+\infty} \mathbf{M}(|\xi_n|) < +\infty,$$

then there exists an increasing sequence $n_1, n_2, \ldots, n_k, \ldots$ of integers and an integrable random variable η, for which

$$\mathbf{P}\left(\frac{\xi_{n_1} + \xi_{n_2} + \ldots + \xi_{n_k}}{k} \to \eta\right) = 1.$$

Moreover we prove the following

THEOREM 1a. *If $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ is a sequence of random variables for which*

$$\liminf_{n=+\infty} \mathbf{M}(|\xi_n|) < +\infty,$$

then there exists a subsequence $\{\eta_n\}$ of the sequence $\{\xi_n\}$ and an integrable random variable η such that for an arbitrary subsequence $\{\tilde{\eta}_n\}$ of the sequence $\{\eta_n\}$ holds:

$$\mathbf{P}\left(\lim_{n=+\infty} \frac{\tilde{\eta}_1 + \tilde{\eta}_2 + \ldots + \tilde{\eta}_n}{n} = \eta\right) = 1.$$

The theorem is the best possible in the following sense:

THEOREM 2. *If $\{a_n\}$ is an arbitrary sequence of positive numbers for which*

$$\lim_{n=+\infty} a_n = +\infty,$$

then there exists a sequence of random variables

$$\xi_1, \xi_2, \ldots, \xi_n, \ldots \quad \textit{with} \quad \mathbf{M}(|\xi_n|) = a_n,$$

such that for any subsequence of this sequence the strong law of large numbers is not valid.

Moreover there exists a sequence $\{\eta_n\}$ of independent identically distributed random variables (with $\mathbf{M}(|\eta_n|) = 1$) such that for the sequence $\xi_n = a_n \cdot \eta_n$ (and for any of its subsequences) the strong law of large numbers is not valid.

Clearly Theorem 1 implies the undermentioned corollary (which is also a consequence of the individual ergodic theorem of Birkhoff):

COROLLARY. *If $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ are equivalent (symmetrically dependent) random variables with finite expectation, then there exists an integrable random variable η such that*

$$\mathbf{P}\left(\lim_{n=+\infty} \frac{\xi_1 + \xi_2 + \ldots + \xi_n}{n} = \eta\right) = 1.$$