Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 18 (3—4), 1967, pp. 321—328.

OPERATIONS WITH STRUCTURES

By

L. LOVÁSZ (Budapest)

(*Presented by* A. RÉNYI)

Introduction

We deal in this paper with (relational) structures $\langle H, R_1, \ldots, R_l\rangle$ with finitely many finitary relations over a set H. H is not necessarily non-empty. We shall consider only the case when $l=1$, i.e. structures of form $\langle H, R\rangle$ but our results extend without difficulty for the general case.

Our main concern will be in the direct product of finite structures (i.e. $\langle H, R\rangle$ with finite domain H). In [1] the question was discussed under what conditions it is true that any two direct factorizations of a structure have a common refinement. It was mentioned that if the structures A, B have this "refinement-property" then e.g. $A^2\cong B^2$ implies $A\cong B$. We shall prove a general theorem from which it follows that for finite A, B the last implication always holds. On the other hand, it is easy to see that not all finite structures have the refinement-property (or the unique prime factorization-property).

The same is true with an arbitrary natural number n instead of 2. Further, if A, B, C are finite structures, and the relation of C is not irreflexiv (i.e. there is an element c in C such that $R(c, \ldots, c)$ holds, where R is the relation of C), then $AC\cong BC$ implies $A\cong B$. Our general result states that under certain conditions, a "polynomial" formed from structures assumes every value only once (up to isomorphism).

In § 1 we define the necessary notions, among them the (cardinal) sum and the (direct) product of two structures and a new operation on structures which will be called exponentiation. This operation has a remarkable resemblance to ordinary exponentiation in the domain of the natural numbers, when we bring it into contact with the sum and the product operation on structures. The relevant identities will be proved in § 2. In § § 1—2 we do not suppose that the structures are finite.

In § 3 we prove our main theorem from which the result mentioned above (concerning "polynomials" of structures) will follow easily. We mention that the operation of exponentiation is not indispensable in our arguments in § 3, i.e. the necessary notions derived from it could be introduced more directly. This will be pointed out on the due place. However, the "exponentiation" seems to us to be very natural in the present context and to be interesting also for its own sake.

§ 1. If N is a set we denote its cardinality by $|N|$. Let φ, ψ be mappings. By $\mathrm{Dom}\,\varphi$, $\mathrm{Rng}\,\varphi$ we denote the definition domain and the range of φ, respectively. The result of application of φ on $a\in\mathrm{Dom}\,\varphi$ is denoted by $a\varphi$. The product $\varphi\psi$ is defined if and only if $\mathrm{Rng}\,\varphi\subseteq\mathrm{Dom}\,\psi$. In this case $\mathrm{Dom}\,(\varphi\psi)=\mathrm{Dom}\,\varphi$ and $\varphi\psi$ is determined by the equation $a(\varphi\psi)=(a\varphi)\psi$ $(a\in\mathrm{Dom}\,\varphi=\mathrm{Dom}\,\varphi\psi)$. If φ is one-to-one then φ^{-1} is defined by $(a\varphi^{-1})\varphi=a$ $(a\in\mathrm{Rng}\,\varphi)$. We have in this case $\mathrm{Dom}\,\varphi^{-1}=\mathrm{Rng}\,\varphi$, $\mathrm{Rng}\,\varphi^{-1}=\mathrm{Dom}\,\varphi$. If $M\subseteq\mathrm{Dom}\,\varphi$ then $M\varphi=\{x\varphi\colon x\in M\}$.

Let k be a natural number, $k \geqq 1$. By a *k-dimensional structure* we mean a pair $\langle S, R\rangle$, where S is a set and $R \subseteqq S^k$. S is the *domain* and R is the *relation* of $A = \langle S, R\rangle$. S and R are also denoted by $S(A)$ and $R(A)$ in dependence of A. The elements of A are the elements of $S(A)$. Obviously, for $k=2$, the 2-dimensional structures are the directed graphs without parallel edges. On the other hand, if $\mathcal{S}$ is an algebraic structure with domain S and finitary operations $m_1, \ldots, m_l$ then we can correspond to $\mathcal{S}$ the $k+l$ dimensional structure $\langle S, R\rangle$ where k is the maximum of the numbers k_i of places of m_i $(i=1, \ldots, l)$ and R is defined as the set of the $k+l$ tuples $\langle x_1, \ldots, x_k, m_1(x_1, \ldots, x_{k_1}), \ldots, m_l(x_1, \ldots, x_{k_l})\rangle$. What is important for us is that this correspondence is a one-to-one and is preserved under isomorphism and direct product. Hence our results in § 4 extend also to finite algebraic structures.

In what follows we consider structures of a fixed dimension k. By A, B, C, D, E, F, G (possibly with indices) we always mean structures.

If $S(A)=\emptyset$, $R(A)=\emptyset$ then we denote A by O. $A_p^{(k)}$ is the structure with the domain consisting of the natural numbers $1, 2, \ldots, p$ and with the identity relation; i.e. $(x_1, \ldots, x_k) \in R(A_p^{(k)})$ if and only if $x_1 = \ldots = x_k$.

We denote the set of elements x of the structure such that $(x, \ldots, x) \in R(A)$ by $Q(A)$.

Let M, N be sets, $\varphi_1, \ldots, \varphi_k$ mappings of M into N (i.e. Dom $\varphi_i = M$, Rng $\varphi_i \subseteqq N$). $[\varphi_1, \ldots, \varphi_k]$ denotes that mapping of M^k into N^k for which the image of $(x_1, \ldots, x_k) \in M^k$ is $(x_1\varphi_1, \ldots, x_k\varphi_k) \in N^k$.

If A is a structure and for the mapping φ we have Dom $\varphi = S(A)$ then $B = A\varphi$ denotes the structure defined as follows. $S(A\varphi) = S(A)\varphi = \text{Rng}\,\varphi$, $R(A\varphi) = \{(x_1, \ldots, x_k)[\varphi, \ldots, \varphi]: (x_1, \ldots, x_k) \in R(A)\}$. In this case we call φ a *homomorphism of A onto B*. If, in addition, φ is one-to-one, then φ is *isomorphism of A onto B*, and we write $A \cong B$. $H(A, B)$ will denote the set of all homomorphisms of A onto B.

Let $e = (x_1, \ldots, x_k)$, $f = (y_1, \ldots, y_k)$. Then $e \cdot f$ denotes $((x_1, y_1), \ldots, (x_k, y_k))$. Similarly, if $\langle x_1, x_2, \ldots\rangle$, $\langle y_1, y_2, \ldots\rangle$ are vectors of the same (finite or infinite) length then $\langle x_1, x_2, \ldots, \ldots\rangle \cdot \langle y_1, y_2, \ldots\rangle = \langle (x_1, y_1), (x_2, y_2), \ldots\rangle$.

Let A and B be structures. If $A' \cong A$ and $B' \cong B$, furthermore A' and B' have no element in common then the structure C defined by $S(C) = S(A') \cup S(B')$ and $R(C) = R(A') \cup R(B')$ is called a *(cardinal) sum* of A and B. Obviously, all cardinal sums of A and B are isomorphic. Therefore we may denote an arbitrary one of them by $A+B$ and this indeterminacy will not cause any difficulty. Certainly, in case $S(A) \cap S(B) = \emptyset$ we define $A+B$ by putting $A'=A$, $B'=B$ in the above construction. AB is the *direct product* of A and B, i.e. $S(AB) = S(A) \cdot S(B)$ (where for any sets M, N $M \cdot N$ means the cartesian product of M and N) and if $e \in S(A)^k$, $f \in S(B)^k$ then $e \cdot f \in R(A \cdot B)$ if and only if $e \in R(A)$ and $f \in R(B)$.

To define the exponentiation, we mean by A^B the structure for which $S(A^B) = S(A)^{S(B)}$ (i.e. the set of all mappings of $S(B)$ into $S(A)$) and if $\varphi_1, \ldots, \varphi_k \in S(A^B)$ then $(\varphi_1, \ldots, \varphi_k) \in R(A^B)$ is equivalent to $R(B)[\varphi_1, \ldots, \varphi_k] \subseteqq R(A)$.

We remark that the following are identically true:

$$A+O \cong O+A \cong A, \quad A_p^{(k)} \cdot A \cong A \cdot A_p^{(k)} \cong A \cdot p = \underbrace{A}_{1} + \ldots + \underbrace{A}_{p},$$

$$A_1^{(k)} \cdot A \cong A \cdot A_1^{(k)} \cong A,$$

$$O \cdot A \cong A \cdot O \cong O, \quad A^{A_p^{(k)}} \cong A^p,$$