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ON THE POINT-GROUP AND LINE-GROUP OF A GRAPH[1]

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(Presented by G. HAJÓS)

One of the most important concepts associated with a graph G is its group of automorphisms $\Gamma(G)$, a permutation group which acts on the points of G. This group induces another permutation group $\Gamma_1(G)$ which acts on the lines of G. Now any permutation group can, of course, be considered simply as an abstract group. Thus it is natural to ask the questions in which the terminology follows CARMICHAEL [1]: when are $\Gamma(G)$ and $\Gamma_1(G)$ *isomorphic* as abstract groups and when are they *identical* as permutation groups? These questions are answered in Theorems 1 and 2.

When A and B are permutation groups acting on the sets X and Y respectively, we will write $A \cong B$ to mean that A and B are isomorphic groups. However $A \equiv B$ indicates not only isomorphism but that A and B are identical permutation groups. More specifically $A \cong B$ if there is a $1-1$ map $h: A \leftrightarrow B$ such that for all α_1, α_2 in A, $h(\alpha_1\alpha_2) = h(\alpha_1)h(\alpha_2)$. To define $A \equiv B$ precisely, we also require another $1-1$ map $f: X \leftrightarrow Y$ such that for all x in X and α in A

$$f(\alpha x) = h(\alpha) f(x).$$

A *graph* G consists of a finite set $V(G)$ of *points* $v_1, \ldots, v_p$, together with a prescribed set $X(G)$ of unordered pairs of distinct points of $V(G)$. Each such pair of points u and v is a *line* $x = uv$ of the graph G. In this case we say that the points u and v are *adjacent* and that the line x is *incident* with each of the points u and v. The *complete graph* K_p of p points has every two distinct points adjacent. The graph $K_p - x$ is obtained from K_p by removing any line x. The graph $K_m \cdot K_n$ is obtained by identifying any point of K_m with any point of K_n.

An *automorphism* of G is a permutation of the set $V(G)$ of points of G which preserves adjacency. Let $\Gamma(G)$ be the group of all automorphisms of G. Usually $\Gamma(G)$ is called the group of G, but here it will be called the *point-group* of G. Of course $\Gamma(G)$ induces a group $\Gamma_1(G)$ of permutations acting on the set $X(G)$ of lines of G. We will call $\Gamma_1(G)$ the *line-group* of G.

We now use $K_4 - x$ to illustrate the difference between the point-group and the line-group. In Figure 1 we have a copy of $K_4 - x$ with points labelled v_1, v_2, v_3, v_4 and lines x_1, x_2, x_3, x_4, x_5.

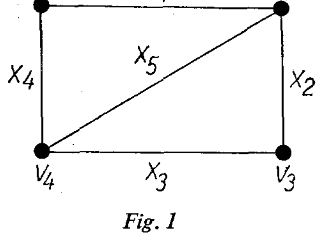

Fig. 1

[1] Invited address at the Bolyai Mathematical Society in Budapest on May 19th, 1967.

The point-group $\Gamma(K_4-x)$ consists of the four permutations

$$(v_1)(v_2)(v_3)(v_4)$$
$$(v_1)(v_3)(v_2v_4)$$
$$(v_2)(v_4)(v_1v_3)$$
$$(v_1v_3)(v_2v_4).$$

The identity permutation of the point-group induces the identity permutation on the lines, while $(v_1)(v_3)(v_2v_4)$ induces a permutation on the lines which fixes x_5, interchanges x_1 with x_4 and x_2 with x_3. Thus the line-group $\Gamma_1(K_4-x)$ consists of the following permutations, induced respectively by the above members of the point-group:

$$(x_1)(x_2)(x_3)(x_4)(x_5)$$
$$(x_1x_4)(x_2x_3)(x_5)$$
$$(x_3x_4)(x_1x_2)(x_5)$$
$$(x_1x_3)(x_2x_4)(x_5).$$

Of course the line-group and the point-group of K_4-x are isomorphic. But they are certainly not identical permutation groups since $\Gamma_1(K_4-x)$ has degree 5 and $\Gamma(K_4-x)$ has degree 4. Note that the line x_5 is fixed by every member of the line-group. Even the permutation group obtained from $\Gamma_1(K_4-x)$ by restricting its object set to x_1, x_2, x_3, x_4 only is not identical with $\Gamma(K_4-x)$, since these two isomorphic permutation groups of the same degree have different cycle structure. Furthermore, even when two permutation groups have the same degree and the same cycle structure, they still need not be identical.

SABIDUSSI [4] has demonstrated the sufficiency of the following conditions using group theoretic methods. This proof uses very elementary graphical concepts.

THEOREM 1. *The line-group and the point-group of a graph G are isomorphic if and only if G has at most one isolated point and K_2 is not a component of G.*

PROOF. Let α' be the permutation in $\Gamma_1(G)$ which is induced by the permutation α in $\Gamma(G)$. By the definition of multiplication in $\Gamma_1(G)$, we have $\alpha'\beta'=(\alpha\beta)'$ for all α, β in $\Gamma(G)$. Thus the mapping $\alpha\to\alpha'$ is a group homomorphism from $\Gamma(G)$ onto $\Gamma_1(G)$. Hence $\Gamma(G)\cong\Gamma_1(G)$ if and only if the kernel of this mapping is trivial.

To prove the necessity, assume $\Gamma(G)\cong\Gamma_1(G)$. Then $\alpha\neq 1$ (the identity permutation) implies $\alpha'\neq 1$. If G has distinct isolated points v_1 and v_2, we can define $\alpha\in\Gamma(G)$ by $\alpha(v_1)=v_2$, $\alpha(v_2)=v_1$ and $\alpha(v)=v$ for all $v\neq v_1, v_2$. Then $\alpha\neq 1$ but $\alpha'=1$. If K_2 is a component of G, take the line of K_2 to be $x=v_1v_2$ and define $\alpha\in\Gamma(G)$ exactly as above to obtain $\alpha\neq 1$ but $\alpha'=1$.

To prove the sufficiency, assume that G has at most one isolated point and and that K_2 is not a component of G. If $\Gamma(G)$ is trivial, then obviously $\Gamma_1(G)$ fixes every line and hence $\Gamma_1(G)$ is trivial. Therefore suppose there exists $\alpha\in\Gamma(G)$ with $\alpha(u)=v\neq u$. Then the degree of u is equal to the degree of v. Since u and v are not isolated, this degree is not zero.