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RANDOM CENTRAL LIMIT THEOREMS FOR MARTINGALES

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1. Introduction. Suppose $\{X_n\}$ is a strictly stationary ergodic process such that

(1) $$\mathbf{E}(X_1) = 0,\ \mathbf{E}(X_n|X_1, \ldots, X_{n-1}) = 0,\ n \geqq 2$$

and

(2) $$\mathbf{E}(X_1^2) = 1.$$

Let

(3) $$\zeta_n = X_1 + X_2 + \ldots + X_n, \quad \zeta_0 = 0;$$

and

(4) $$\eta_n = n^{-\frac{1}{2}} \sum_{i=1}^{n} X_i.$$

It is known that η_n is asymptotically normal with mean 0 and variance 1 from a theorem of BILLINGSLEY [1]. Under a strong mixing condition on $\{X_n\}$, we shall now obtain a random version of this theorem similar to the theorems obtained by RÉNYI [5] and BLUM, HANSON, ROSENBLATT [2] for sums of independent random variables.

Suppose that $\{\nu_n\}$ is a sequence of positive integer valued random variables such that $\{\nu_n\}$ is independent of $\{X_n\}$ and $\nu_n \to +\infty$ in probability. Then, it follows, from the arguments in RÉNYI [5], that

(5) $$\lim_{n\to\infty} \mathbf{P}(\eta_{\nu_n} \leqq x) = \Phi(x),$$

where

(6) $$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt.$$

We shall now assume that the stationary process $\{X_n\}$ satisfies the following modified version of the strong mixing condition of ROSENBLATT [7]. Let m_a^b denote the σ-field generated by the random variables X_n, $a \leqq n \leqq b$. Then

(7) $$\sup_{A \in m_1^k,\ B \in m_{k+n}^{\infty}} |\mathbf{P}(B|A) - \mathbf{P}(B)| \leqq \alpha(n) \quad \text{where} \quad \alpha(n) \to 0 \quad \text{as} \quad n \to \infty.$$

THEOREM 1. *Under condition* (7), *if* $\{\nu_n\}$ *is a sequence of positive integer valued random variables such that* $n^{-1}\nu_n$ *converges in probability to a positive random variable* μ *having a discrete distribution, then* (5) *holds.*

THEOREM 2. *Under condition* (7), *if* $\{\nu_n\}$ *is a sequence of positive integer valued random variables such that* $n^{-1}\nu_n$ *converges in probability to a positive random variable* μ, *then* (5) *holds.*

Note that theorem 1 is a special case of theorem 2. We shall prove theorem 1 and make use of it in the proof of theorem 2. Proofs of the above theorems follow the corresponding proofs of RÉNYI [5], BLUM, HANSON and ROSENBLATT [2] for sums of independent random variables.

2. Some lemmas. In this section, we shall state and prove some lemmas which will be used later.

LEMMA 1. *If* $\{\tau_n\}$ *is a sequence of random variables with* $\mathbf{E}(\tau_k|\tau_1, \ldots, \tau_{k-1})=0$ *and* $\sigma_k^2=\text{var}\,[\tau_1+\tau_2+\ldots+\tau_k]^2<\infty$ *for* $1\leqq k\leqq n$, *then for any* $\varepsilon>0$

$$\mathbf{P}\big(\operatorname*{Max}_{1\leqq k\leqq n}|\tau_1+\tau_2+\ldots+\tau_k|>\varepsilon\big) \leqq \frac{\sigma_n^2}{\varepsilon^2}. \tag{8}$$

Proof of this lemma can be found in DOOB [4].

LEMMA 2. *Let* $W_n, X_{mn}, Y_{m,n}^{(j)}$ *and* $Z_{m,n}^{(j)}$ *be random variables for* $m, n=1, 2, \ldots$ *and* $j=1, \ldots, k$. *Suppose*

$$W_n = X_{m,n}+\sum_{j=1}^{k} Y_{m,n}^{(j)} Z_{m,n}^{(j)}$$

and

(a) $\lim\limits_{m\to\infty}\ \limsup\limits_{n} \mathbf{P}(|Y_{m,n}^{(j)}|>\varepsilon) = 0$ *for every* $\varepsilon>0$ *and* $1\leqq j\leqq k$;

(b) $\lim\limits_{M\to\infty}\ \limsup\limits_{m}\ \limsup\limits_{n} \mathbf{P}(|Z_{m,n}^{(j)}|>M)=0$ *for* $j=1, \ldots, k$;

(c) *the distributions of* $\{X_{m,n}\}$ *converge to the distribution function* F *for each fixed* m.

Then the distribution function of $\{W_n\}$ *converges to* F.

Proof of this lemma can be found in BLUM, HANSON and ROSENBLATT [2].

DEFINITION. A sequence of random variables $\{\eta_n\}$ is said to be mixing in the sense of Rényi if for any event A with $\mathbf{P}(A)>0$, and for any real number x,

$$\lim_n \mathbf{P}(A_n|A)=\lim_n \mathbf{P}(A_n)$$

where $A_n=[\eta_n\leqq x]$.

LEMMA 3. *Under the strong mixing condition on the process* $\{X_n\}$, *the sequence* $\{\eta_n\}$ *is mixing.*

PROOF. Let $A_k=(\eta_k\leqq x)$. By Theorem 2 of RÉNYI [6] it is enough to prove that for any A_k,

$$\lim_n \mathbf{P}(A_n|A_k)=\lim_n \mathbf{P}(A_n).$$

Let $j_n=[n^{1/4}]$ and consider the sequence of random variables $n^{-1/2}\zeta_{k+j_n}$. Sincy $\mathbf{E}(\zeta_{k+j_n})=0$ and $\text{Var}\,(\zeta_{k+j_n})=(k+j_n)$, $n^{-1/2}\zeta_{k+j_n}$ converges to zero in probabilite as n tends to ∞. Therefore

$$\lim_n \mathbf{P}(A_n|A_k)=\lim_n \mathbf{P}(n^{-1/2}\{\zeta_n-\zeta_{k+j_n}\}\leqq x|A_k). \tag{9}$$