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ON FINITE Δ-SYSTEMS OF ERDŐS AND RADO

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1. Introduction

The main purpose of this paper is to give some new estimations for a problem stated in [1] and to show its connection with Ramsey's theorem.

First, we have to repeat necessary definitions from [1]:

"A system Σ_1: Y_ν ($\nu \in N$) of sets Y_ν, where ν ranges over the index set N, is said to contain the system Σ_0: X_μ ($\mu \in M$) if, for every μ_0 of M, the set X_{μ_0} occurs in Σ_1 at least as often as in Σ_0, i.e., if

$$|\{\nu: \nu \in N;\ Y_\nu = X_{\mu_0}\}| \geqq |\{\mu: \mu \in M;\ X_\mu = X_{\mu_0}\}|.$$

If Σ_1 contains Σ_0 and, at the same time, Σ_0 contains Σ_1, then we do not distinguish between the systems Σ_0 and Σ_1.

The system Σ_0 is called an (a, b)-system if it consists of a (not necessarily distinct) sets of cardinal b, i.e., if $|M| = a$ and $|X_\mu| = b$ for $\mu \in M$. The system Σ_0 is called a Δ-system if it has the property that the intersections of any two of its sets (not necessarily distinct sets but sets having distinct indices μ) have the same value, i.e. if for

$$\mu_0, \mu_1, \mu_2, \mu_3 \in M; \quad \mu_0 \neq \mu_1; \quad \mu_2 \neq \mu_3$$

we always have $X_{\mu_0} X_{\mu_1} = X_{\mu_2} X_{\mu_3}$. More specifically, Σ_0 is a Δ (a)-system with kernel K if $|M| = a$ and $X_{\mu_0} X_{\mu_1} = K$ whenever $\mu_0, \mu_1 \in M$; $\mu_0 \neq \mu_1$.

. . . Expressions such as

$$(>a,\ \leqq b)\text{-system}, \quad \Delta(>a)\text{-system}$$

have their obvious meaning."

We shall concern ourselves only with a, b finite ($1 \leqq a, b < \aleph_0$).

Erdős and Rado proved that given any a, b there exists a least number $f(a, b)$ such that every $(>f,\ \leqq b)$-system contains a $\Delta(>a)$-system. In fact, they proved

$$a^{b+1} < f(a, b) \leqq b!\, a^{b+1} \left(1 - \frac{1}{2!\, a} - \frac{2}{3!\, a^2} - \dots - \frac{b-1}{b!\, a^{b-1}}\right)$$

and conjectured $f(a, b) \leqq c^b a^{b+1}$ for some absolute positive constant c.

We shall show

$$a^{b+1} \left(1 + \frac{1}{a}\right)^{\left[\frac{b}{2}\right]} \leqq f(a, b) \ \text{ for } a \text{ even}$$

$$a^{b+1} \left(1 + \frac{1}{2a} - \frac{1}{2a^2}\right)^{\left[\frac{b}{2}\right]} \leqq f(a, b) \ \text{ for } a \text{ odd}$$

(Theorem 3). For $a \geqq b^2 - b + 1$, $b \geqq 3$, we shall prove

$$f(a, b) \leqq \frac{(b+1)!}{2b} a^{b+1} \left(1 - \frac{1}{3a} - \frac{1}{3a^2}\right)$$

(Theorem 5).

We shall also show that the existence of $f(a, b)$ is a consequence of Ramsey's theorem.

Erdős and Rado denoted by $\Phi(a, b)$ the least number provided that every $(>\Phi, \leqq b)$-system Σ: $X_\mu (\mu \in M)$ which satisfies $X_\mu \neq X_\nu$ for $\mu \neq \nu$ contains a $\Delta(>a)$-system. It is easy to see that

$$f(a, b) = a\Phi(a, b).$$

Hence, we shall confine ourselves only with systems of distinct sets.

One may "polarize" the notion of $\Phi(a, b)$ in the following way:

Define $\pi(a_0, a_1, \ldots, a_{b+1}, b)$ as the least number such that every $(>\pi, \leqq b)$-system contains a $\Delta(>a_j)$-system with kernel K, $|K| = j$ for some j. Especially, put $\pi(n, a, b) = \pi(n, a, a, \ldots, a, b)$.

Evidently, $\pi(n, a, 1) = \pi(n, 1, b) = n$.

2. Results

THEOREM 1.

(1)
$$\pi(n, a, 2) \geqq na + \left[\frac{n}{\left[\frac{a+1}{2}\right]}\right] \cdot \left[\frac{a}{2}\right]$$

where $[x]$ *is the greatest integer not exceeding* x. *Especially,* $\Phi(a, 2) \geqq a^2 + a$ *for* a *even,* $\Phi(a, 2) \geqq a^2 + \frac{a-1}{2}$ *for* a *odd.*

THEOREM 2. *If* $n = 1, 2$ *or* $a = 2$ *then in* (1) *the equality holds.*

THEOREM 3. *If* $n \leqq a$ *then*

$$\pi(n, a, b) \geqq \Phi(a, 2)^{\frac{b}{2} - 1} \cdot \pi(n, a, 2) \quad \text{for } b \text{ even}$$

$$\pi(n, a, b) \geqq \Phi(a, 2)^{\frac{b-1}{2}} \cdot n \quad \text{for } b \text{ odd.}$$

If $n = ka + q$, $k \geqq 0$, $1 \leqq q \leqq a$ *then*

$$\pi(n, a, b) \geqq k \cdot \Phi(a, b) + \pi(q, a, b).$$

Especially,

(2)
$$\begin{cases} \Phi(a, b) \geqq a^b \left(1 + \dfrac{1}{a}\right)^{\left[\frac{b}{2}\right]} & \text{for } a \text{ even} \\ \Phi(a, b) \geqq a^b \left(1 + \dfrac{1}{2a} - \dfrac{1}{2a^2}\right)^{\left[\frac{b}{2}\right]} & \text{for } a \text{ odd.} \end{cases}$$