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A GENERALIZATION OF KŐNIG'S THEOREM

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Introduction. A well known theorem of D. Kőnig states that the maximum number of (pairwise) independent edges of a bipartite graph G equals to the minimum number of vertices covering all edges of G. This theorem has a lot of generalizations, equivalents and applications in different branches of mathematics. We are going to give a generalization related to the original problem concerning bipartite graphs. As an application of this we prove a conjecture of P. ERDŐS concerning chromatic number of finite set-systems.

§ 1. Let A, B be two disjoint finite sets. We consider bipartite graphs the edges of which join vertices of A to vertices of B. If we say bipartite graph we always think of a graph of this type. If G is a bipartite graph and $X \subseteqq A$ then let X^G denote the set of those vertices which are joined to a vertex of X by an edge of G.

ORE gave the following equivalent of Kőnig's theorem: *The maximum number of independent edges of a bipartite graph G is*

$$|A| - \max_{X \subseteqq A} \{|X| - |X^G|\}.$$

In other words, a bipartite graph critical concerning the property that for any $X \subseteqq A$

$$|X^G| \geqq |X| - \delta,$$

consists of independent edges.

In connexion with this theorem the following question arises. We order an integer $f(X)$ to every $X \subseteqq A$. Let K denote the class of those bipartite graphs G for which every $X \subseteqq A$ satisfies

$$|X^G| \geqq f(X). \tag{1}$$

An element G of K is said to be *critical* if no proper subgraph of it belongs to K. Which graphs are critical?

I do not know the answer in the general case, only supposing

$$f(X \cup Y) + f(X \cap Y) \geqq f(X) + f(Y) \quad \text{if} \quad X \cap Y \neq \varnothing. \tag{2}$$

$$f(X \cup Y) \leqq f(X) + f(Y) \quad \text{if} \quad X \cap Y = \varnothing. \tag{3}$$

An example for a function satisfying (2) and (3) is

$$f(X) = \begin{cases} a|X| + b, & \text{if} \quad X \neq \varnothing \quad (a, b \geqq 0), \\ 0, & \text{if} \quad X = \varnothing. \end{cases}$$

THEOREM. *A $G \in K$ is critical if and only if every $x \in A$ has in G valency $f(\{x\})$.*

Let G be an arbitrary graph of K and let α_G denote the system of those subsets of A for which

$$|X^G| = f(X).$$

We may suppose, that $\emptyset \in \alpha_G$, i.e. $f(\emptyset)=0$, since otherwise $K=\emptyset$. Our theorem can be formulated in the form that G is critical if and only if $\{x\} \in \alpha_G$ for every $x \in A$.

We need the following

LEMMA. *If $X, Y \in \alpha_G$ then $X \cap Y \in \alpha_G$.*

PROOF OF THE LEMMA. If $X \cap Y = \emptyset$, then it is trivial. Otherwise we may use (2):

$$|(X \cup Y)^G| + |(X \cap Y)^G| \leqq |X^G \cup Y^G| + |X^G \cap Y^G| =$$

$$= |X^G| + |Y^G| = f(X) + f(Y) \leqq f(X \cup Y) + f(X \cap Y).$$

Since $G \in K$, here the equality must hold. This proves $X \cap Y \in \alpha_G$.

PROOF OF THE THEOREM. The "if" part is trivial. Suppose that G is a critical element of K. Let $x \in A$ and let $y_1, \ldots, y_\varphi$ be the vertices of B joined to x by the edges $E_1, \ldots, E_\varphi$, respectively. If we omit E_i $(1 \leqq i \leqq \varphi)$ then the remaining graph G_i does not belong to K and hence there is an $X_i \subseteqq A$ such that

$$|X_i^{G_i}| < f(X_i),$$

i.e.

$$X_i \in \alpha_G, \quad x \in X_i, \quad y_i \notin X_i^{G_i}. \tag{4}$$

By our lemma $Y = X_0 \cap \ldots \cap X_\varphi \in \alpha_G$. Put $Y_0 = Y - \{x\}$, then by (3) and (4)

$$|Y^G| = \varphi + |Y_0^G| \geqq f(\{x\}) + f(Y_0) \geqq f(Y).$$

Since here the equality must hold because of $Y \in \alpha_G$, we obtain $\varphi = f(\{x\})$, Q.E.D.

§ 2. Let $\mathcal{H}$ be a finite set-system. We put $P(\mathcal{H}) = \bigcup_{E \in \mathcal{H}} E$. The elements of $P(\mathcal{H})$ and $\mathcal{H}$ are called the *vertices* and *simplices* of $\mathcal{H}$, respectively. If the simplices of $\mathcal{H}$ have the same cardinality β then we say that $\mathcal{H}$ is a *uniform β-system*.

The notion of a finite set-system is a generalization of the notion of finite undirected graphs without isolated vertices; since we can identify a graph with the system of its edges. The concepts and theorems of graph theory have more possible generalizations for set-systems. In [3] and [4] some questions of this type are detailed.

An *α-colouring* of $\mathcal{H}$ means a function defined on $P(\mathcal{H})$ the range of which is a set of natural numbers $\leqq \alpha$. Such a colouring is *correct* if every simplex of $\mathcal{H}$ contains vertices to which different natural numbers are ordered. It is *strictly correct* if the integers ordered to the elements of any simplex of $\mathcal{H}$ cover the interval $[1, \alpha]$.