

## ON FINITE DIRICHLET SERIES

By

L. LOVÁSZ (Budapest)

*To Professor P. TURÁN on his 60<sup>th</sup> birthday*

A *finite Dirichlet series* (abbreviated by FDS in the sequel) is a function of the form

$$f(s) = \sum_{i=1}^n a_i i^s \quad (a_i \text{ integer})$$

(we shall not use analytical properties of these; it is enough to consider them for integral  $s$ ). Such a function is an analogue of a polynomial and several properties of polynomials are valid for such functions too. We are going to list such properties here — from a number-theoretical point of view.

Obviously, sum, difference and product of FDSs are FDSs too. Thus, the FDSs form a ring. As MCKENZIE mentioned<sup>1</sup>, unique prime factorization holds in this ring. To show this, let  $p_i$  denote the  $i$ <sup>th</sup> prime number. Every FDS can be written uniquely in the following form:

$$f(s) = \sum_{1 \leq i_1 < \dots < i_v} a_{i_1, \dots, i_v} p_{i_1}^s p_{i_2}^s \dots p_{i_v}^s.$$

From this we can deduce that the ring of FDSs is isomorphic with the ring of polynomials with infinitely many variables (over the ring of integers). It is well known that prime factorization is unique in this latter ring.

An FDS is called *primitive*, if its coefficients are coprime. The following analogue of GAUSS' lemma can be proved just in the same way as for polynomials:

*The product of primitive FDSs is primitive.*

The main purpose of this paper is to investigate some connections between the properties of FDSs and the number-theoretical properties of their values for different values of  $s$ . If  $f(s)$  is an FDS and  $s_0$  is a non-negative integer then  $f(s_0)$  is an integer.

**THEOREM 1.** *Assume that  $f(s) = \sum_{i=1}^n a_i i^s$ ,  $g(s) = \sum_{i=1}^m b_i i^s$  ( $a_n, b_m \neq 0$ ) are FDSs such that*

$$f(s) | g(s)$$

*for every  $s \geq 0$ . Then  $n | m$  and*

$$h(s) = \left( \frac{m}{n} - 1 \right)! \frac{g(s)}{f(s)}$$

*is an FDS.*

---

<sup>1</sup> Oral communication

PROOF: Put

$$u(s) = - \sum_{i=1}^{n-1} \frac{a_i}{a_n} \left( \frac{i}{n} \right)^s$$

and let  $p$  be a sufficiently large but fixed integer. Then

$$\frac{g(s)}{f(s)} = \frac{g(s)}{a_n \cdot n^s} \cdot \frac{1}{1-u(s)} = \left\{ \sum_{j=0}^p (u(s))^j \frac{g(s)}{a_n \cdot n^s} \right\} + \frac{(u(s))^{p+1} g(s)}{a_n \cdot n^s (1-u(s))} = \left\{ \sum_{i=1}^N \beta_i \gamma_i^s \right\} + R(s)$$

where  $N$  is some positive integer,  $\beta_i$  and  $\gamma_i$  are rationals ( $1 \leq i \leq N$ ),  $\gamma_i > 0$  and for  $R(s)$  we have

$$|R(s)| \leq \left( \sum_{i=1}^{n-1} \left| \frac{a_i}{a_n} \right| \right)^{p+1} \left( \sum_{i=1}^m |b_i| \right) \frac{1}{|a_n|} \cdot \frac{1}{|1-u(s)|} \cdot \left[ \left( \frac{n-1}{n} \right)^{p+1} \frac{m}{n} \right]^s.$$

This tends to 0 if  $s \rightarrow \infty$  and  $p$  is large enough.

Now let  $\sum_{j=0}^N c_j x^j$  be a polynomial with integral coefficients and roots  $\gamma_1, \dots, \gamma_N$ .

Then

$$\sum_{j=0}^N c_j \sum_{i=1}^N \beta_i \gamma_i^{s+j} = \sum_{i=1}^N \beta_i \gamma_i^s \sum_{j=0}^N c_j \gamma_i^j = 0.$$

and hence

$$\sum_{j=0}^N c_j \frac{g(s+j)}{f(s+j)} = \sum_{j=0}^N c_j R(s+j).$$

The right-hand side tends to 0, while the left-hand side is always an integer. This implies that both vanish for  $s \geq s_0$ .

We use now the following well-known theorem: if  $\eta(s)$  is a function defined on the set of integers  $s \geq s_0$  and it satisfies the linear difference-equation

$$\sum_{j=0}^M d_j \eta(s+j) = 0$$

where the roots  $\varrho_1, \dots, \varrho_M$  of the polynomial  $\sum_{j=0}^M d_j x^j$  are different then

$$\eta(s) = \sum_{i=1}^M \alpha_i \varrho_i^s$$

with some coefficients  $\alpha_i$ . The application of this theorem yields

$$(1) \quad g(s) = f(s) \sum_{i=1}^N \alpha_i \gamma_i^s$$

with some coefficients  $\alpha_i$ ,  $s \geq s_0$ . But it is easy to see that if (1) holds for every  $s \geq s_0$ , then it holds formally, i.e. it holds for every  $s \geq 0$ .