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ON THE STRUCTURE OF FACTORIZABLE GRAPHS

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Introduction

A graph (finite, undirected, without loops and multiple edges) will be called *factorizable*, if it contains a 1-factor (or, briefly, a factor). A graph is *critical*, if the removal of any vertex results in a factorizable graph. TUTTE [11] gave the following necessary and sufficient condition for the existence of a factor in a given graph:

A graph is factorizable if and only if removing any subset X of its vertices, at most $|X|$ of the components of the remaining graph have an odd number of vertices.

Let $\delta(G)$ denote the *deficiency* of the graph G, i.e. the minimum number of vertices the removal[1] of which yields a factorizable graph. A critical graph G has $\delta(G)=1$, conversely this is, of course, not true. A formula for $\delta(G)$, as a natural generalization of TUTTE's theorem, was given by BERGE [2]. A good survey on maximal matchings of graphs is given by the following theorem, found independently by GALLAI [4] and EDMONDS [3]. To state it we have to introduce some notations. Let G be a graph, then $V(G)$ and $E(G)$ will denote the set of its vertices and edges, resp. For $S\subseteqq V(G)$ let $G[S]$ be the subgraph spanned by S. Let D_G denote the set of those points of G which are not covered by every matching; let A_G be the set of those vertices of $V(G)-D_G$ which are joined to some vertex of D_G in G; finally, put $C_G=V(G)-A_G-D_G$. Then the theorem of GALLAI and EDMONDS asserts:

(a) *the components of $G[C_G]$ are factorizable, the components of $G[D_G]$ are critical;*

(b) *the number of components of $G[D_G]$ is $\delta(G)+|A_G|$;*

(c) *any maximal matching of G consists of a factor of $G[C_G]$ and, for any component T of $G[D_G]$, of a matching covering all but one vertex of T and of at most one edge joining T to A_G.*

An edge of G will be called *allowed* if it is contained in some maximal matching of G and *forbidden* otherwise. From the theorem of GALLAI and EDMONDS one easily deduces that the edges having an endpoint in D_G are allowed while the edges of $G[A_G]$ and the edges between A_G and C_G are forbidden. The type of the edges of $G[C_G]$ is not described by this theorem. By (c), one can pick out a component of $G[C_G]$ if one wants to describe its allowed and forbidden edges and consider it as

[1] Removal of certain vertices means their removal together with the edges incident with them. If X is a set of certain vertices of G then the graph remaining after the removal of X is denoted by $G-X$. In the case $X=\{x\}$ is one-element we write simply $G-x$. Similarly, if Q is a set of edges of G then $G-Q$ denotes the graph arising by omitting the edges of Q.

a factorizable graph. Therefore, to get a survey on the forbidden and allowed edges of a graph we may be confined to the case the graph is factorizable.

The structure of a factorizable graph seems to be an interesting problem also from the point of view that for factorizable graphs the structure given in the GALLAI–EDMONDS theorem becomes trivial: $C_G = V(G)$, $A_G = D_G = \emptyset$.

KOTZIG [8] defined a structure of factorizable graphs from which the position of allowed and forbidden edges can be seen. HETYEI [7] investigated factorizable graphs from a different point of view. Unfortunately, these papers were written in Slovak and Hungarian and their results do not seem to be well-known.

Partly in connexion with the famous conjecture of VAN DER WAERDEN (cf. [6]), the number of factors of a factorizable graph was investigated by many authors. Thus, M. HALL [5] proved that a factorizable bipartite graph with valencies $\geqq k$ "below" contains at least $k!$ different factors. BEINEKE and PLUMMER [1] proved that a 2-connected factorizable graph has two different factors. ZAKS [12] showed that a k-connected factorizable graph ($k \geqq 2$) contains at least $k!!$ factors. For k odd this result is already sharp, as shown by the complete $(k+1)$-gon. ZAKS used the existence of a *totally covered vertex*, i.e. of a vertex which is incident with allowed edges only. In connexion with this, GRÜNBAUM conjectured that in a k-connected graph there exist at least k totally covered vertices.

In the present paper we are going to give a structure of factorizable graphs. To sketch the train of thought of our investigations, we introduce two notions.

Consider the allowed edges of a factorizable graph G. The subgraph constituted by them consists of certain components. Considering the subgraph G_0 spanned by the vertices of such a component, this graph is factorizable and an edge of it is allowed if and only if it is allowed in G. Consequently, the allowed edges of G_0 form a connected subgraph of it. Such a graph will be called *elementary*.

Another notion to be introduced here is a *saturated* graph. A factorizable graph G is saturated if adding any new edge, the number of its factors increases, i.e. the new edge becomes allowed. One can saturate an arbitrary factorizable graph by adding new edges till this is possible without increasing the number of factors; these new edges will be forbidden in the obtained graph. Since, conversely, removing forbidden edges from a saturated graph we obtain a graph with the same factors, it is enough to describe the structure of saturated graphs. However, there can be points of view from which the saturating process is not easy to survey. Thus, where this is possible (for elementary graphs), we do not restrict ourselves to saturated graphs.

In §1 we investigate graphs with exactly one factor. To make our paper self-contained, we give a new proof of a result of KOTZIG [8] and then apply it to describe these graphs. The obtained results are special cases of the general theorems of §4, but they motivate the structure described in §4 and their proof being much simpler it seems to be worth to premise them.

In §2 we begin with a less elementary examination of factorizable graphs, based on the theorem of GALLAI and EDMONDS. This paragraph contains some lemmata.

In §3 elementary graphs are examined. Their structure is reduced to elementary bipartite graphs and to graphs from which removing any two points, there remains a