Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 26 (3—4), (1975), 419—431.

ON THE ORDER OF CONVERGENCE OF A FINITE ELEMENT METHOD IN REGIONS WITH CURVED BOUNDARIES

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Finite element methods using elements with curved sides are frequently applied to obtain approximate solutions of elliptic boundary value problems in a bounded open plane region R (see, for example, [1]—[7]). The finite element method suggested by ZLÁMAL (see [7]) has the impotant advantage that it gives directly the first partial derivatives of the solution of the boundary value problem. In Zlámal's paper there are given error bounds for this method. In deriving the error bounds, however, it is assumed that the solution of the boundary value problem has square integrable third or fourth partial derivatives in R. These assumptions are satisfied, in general, only if the boundary of R is sufficiently smooth (see [8]—[9]). In the present paper we shall obtain error bounds for Zlámal's method without these assumptions.

1. Let R be a bounded open plane region whose boundary C consists of a finite number of piecewise-analytic simple closed curves. Denote by A_μ ($\mu=1, 2, \ldots, \nu$) the corners of C, i.e. those points on C, where distinct analytic curves meet.

We consider the equation

$$Lu(x, y) = g(x, y) \tag{1}$$

in R. Here

$$Lu \equiv \frac{\partial}{\partial x}\left[a(x, y)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial x}\left[b(x, y)\frac{\partial u}{\partial y}\right] + \frac{\partial}{\partial y}\left[b(x, y)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[c(x, y)\frac{\partial u}{\partial y}\right] - f(x, y)u.$$

Let the coefficients $a(x, y)$, $b(x, y)$, $c(x, y)$, $f(x, y)$ and the right-hand side $g(x, y)$ be analytic in an open region G containing the closure of R in its interior. Suppose that at all points of R

$$a\xi^2 + 2b\xi\eta + c\eta^2 \geqq \alpha(\xi^2+\eta^2) \qquad (\alpha = \text{const} > 0) \tag{2}$$

for all real ξ, η. Moreover, we assume that $f(x, y) \geqq 0$.

We consider two types of boundary conditions: the Dirichlet condition

$$u(x, y) = 0, \quad (x, y) \in C \tag{3}$$

and the Neumann condition

$$\frac{\partial u(x, y)}{\partial N} = 0, \quad (x, y) \in C, \tag{4}$$

where N is the conormal with direction cosines

$$\cos(N, x) = \frac{1}{E}[a\cos(n, x) + b\cos(n, y)], \; \cos(N, y) = \frac{1}{E}[b\cos(n, x) + c\cos(n, y)].$$

Here n is the outward normal to C and

$$E = \{[a\cos(n, x) + b\cos(n, y)]^2 + [b\cos(n, x) + c\cos(n, y)]^2\}^{\frac{1}{2}}.$$

At the corner A_μ we require that

$$\frac{\partial u}{\partial N_1} = \frac{\partial u}{\partial N_2} = 0, \tag{5}$$

where N_1 is the "left-hand conormal" and N_2 is the "right-hand conormal" to the boundary C.

Let Ω be an open region in the plane of R. Denote by $W_2^{(k)}(\Omega)$ the Hilbert space of all functions which together with their generalized derivatives up to the k^{th} order belong to $L_2(\Omega)$. The norm is given by

$$\|v\|_{k,\Omega}^2 = \sum_{j=0}^{k} |v|_{j,\Omega}^2, \quad \text{where} \quad |v|_{j,\Omega}^2 = \sum_{|i|=j} \|D^i v\|_{L_2(\Omega)}^2.$$

Here we use the notations

$$i = (i_1, i_2), \quad |i| = i_1 + i_2, \quad D^i v \equiv \frac{\partial^{|i|} v}{\partial x^{i_1} \partial y^{i_2}}.$$

It is well-known that under the above-mentioned assumptions the Dirichlet problem has a unique solution $u(x, y)$ which minimizes the functional

$$F(v) = \iint_R \left[a\left(\frac{\partial v}{\partial x}\right)^2 + 2b\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + c\left(\frac{\partial v}{\partial y}\right)^2 + fv^2 + 2gv \right] dx\, dy \tag{6}$$

in the space $\mathring{W}_2^{(1)}(R)$ (see, for example, [10]). Here $\mathring{W}_2^{(1)}(R)$ is the space of functions which we get by completing in the norm $\| \; \|_{1,R}$ the set of differentiable functions with compact support in R. If $f(x, y) \not\equiv 0$, then the Neumann problem has a unique solution[1] $u(x, y)$ which minimizes the functional (6) in the space $W_2^{(1)}(R)$. If $f(x, y) \equiv 0$ and $\iint_R g(x, y)\, dx\, dy = 0$, then the Neumann problem is solvable[2] and the solution is unique to within an additive constant. In this case we make the solution unique by requiring that

$$\iint_R u(x, y)\, dx\, dy = 0. \tag{7}$$

The function $u(x, y)$ minimizes the functional (6) in the space $W_2^{(1)}(R)$.

[1] We shall denote by $u(x, y)$ both the solution of the Dirichlet problem and the solution of the Neumann problem. It will always be clear what meaning of $u(x, y)$ is necessary to be taken.

[2] If $f(x, y) \equiv 0$ and $\iint_R g(x, y)\, dx\, dy \neq 0$, then the Neumann problem has no solutions.