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2-MATCHINGS AND 2-COVERS OF HYPERGRAPHS

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§ 0. Hypergraphs

By a hypergraph $\mathscr{H}$ we mean a finite collection of non-empty finite sets; "collection" means that the same finite set may occur more than once. The elements of a hypergraph are called *edges*; the elements of edges are called *vertices* (this way we exclude "isolated vertices". i.e. vertices not occurring in edges; these vertices would be insignificant in our discussion). The set of vertices of the hypergraph $\mathscr{H}$ will be denoted by $V(\mathscr{H})$.

We need some operations defined on hypergraphs. *Removing an edge* means removing this edge (and, of course, all points which become isolated by this). *Removing a vertex* $x\in V(\mathscr{H})$ means the removal of all edges containing this vertex. *Doubling* a vertex x means the addition of a new vertex x' and of a new edge $E'=E-\{u\}\cup\{x'\}$ for every $E\in\mathscr{H}$, such that $x\in E$. Finally, *cutting off* a vertex x can be carried out if $\{x\}\notin\mathscr{H}$; then it results in the hypergraph $\{E-\{x\}: E\in\mathscr{H}\}$.

A hypergraph obtained from $\mathscr{H}$ by removing edges (points) is called a *partial (induced partial) hypergraph.* The hypergraph obtained by removing the point x is denoted by $\mathscr{H}-x$.

§ 1. Matchings and covers of hypergraphs; results

Let $\mathscr{H}$ be a hypergraph. By a *k-matching of* $\mathscr{H}$ we mean a collection $\mathscr{N}$ of edges of $\mathscr{H}$ (the same edge of $\mathscr{H}$ may occur in $\mathscr{N}$ more than once) such that each point of $\mathscr{H}$ occurs in at most k members of $\mathscr{N}$. The *size* $\mathscr{N}$ of the k-matching $\mathscr{N}$ is the number of its edges.

A k-cover $\mathscr{T}$ is a collection of points of $\mathscr{H}$ such that each edge contains at least k of them. The number $\mathscr{T}$ of elements of $\mathscr{T}$ is called the *size* of $\mathscr{T}$.

By a *fractional matching* we mean a system of weights $n_E\geqq 0$ associated with the edges E of $\mathscr{H}$ such that $\sum_{E\ni x} n_E\leqq 1$ for each point x. Analogously, a *fractional cover* is a system of weights $t_x\geqq 0$ associated with the points x of $\mathscr{H}$ in such a way that $\sum_{x\in E} t_x\geqq 1$ for each edge E. The *size* of a fractional matching (cover) is $\sum_E n_E(\sum_x t_x)$.

If $\mathscr{N}$ is a k-matching then denoting by N_E the number of times E occurs in $\mathscr{N}$, $n_E=\frac{1}{k}N_E$ will be a fractional matching. We will denote this fractional matching by $\frac{\mathscr{N}}{k}$. If $\mathscr{N}_i$ is a k_i-matching ($i=1, 2$) then the collection $\mathscr{N}_1+\mathscr{N}_2$, in which an edge E occurs the number of times it occurs in $\mathscr{N}_1$ plus the number of times it occurs in $\mathscr{N}_2$, is a (k_1+k_2)-matching. Similar definitions and observations hold for covers.

We denote by $\nu_k(\mathscr{H})$ the maximum size of k-matching and by $\nu^*(\mathscr{H})$ the maximum size of a fractional matching. The minimum size of k-covers and fractional covers is denoted by $\tau_k(\mathscr{H})$ and $\tau^*(\mathscr{H})$, respectively. We remove the argument $\mathscr{H}$ if there is only a single hypergraph in consideration. Also we set $\tau_1=\tau$ and $\nu_1=\nu$.

It is clear from the definitions that

(1) $$\nu \leqq \frac{\nu_k}{k} \leqq \nu^*$$

and

(2) $$\tau^* \leqq \frac{\tau_k}{k} \leqq \tau$$

hold for every k. Moreover, the duality theorem of linear programming implies

(3) $$\nu^* = \tau^*$$

and other well-known results in linear programming imply that there is a k_0 such that $\nu_{k_0}=k_0\nu^*$, $\tau_{k_0}=k_0\tau^*$. Also, we trivially have $\nu_{k_1+k_2}\geqq \nu_{k_1}+\nu_{k_2}$ and $\tau_{k_1+k_2}\leqq \leqq\tau_{k_1}+\tau_k$. Hence it is easy to deduce that

$$\lim_{k\to\infty}\frac{\nu_k}{k}=\lim_{k\to\infty}\frac{\tau_k}{k}=\nu^*=\tau^*$$

and $\nu_{k_0\cdot s}=k_0\cdot s\cdot \nu^*$, $\tau_{k_0\cdot s}=k_0\cdot s\cdot\tau^*$.

Clearly $\nu(\mathscr{H})=\tau(\mathscr{H})$ implies that equality holds in (1) and (2) for every k. We can obtain inverses to this observation if we make similar assumption on certain "derived" hypergraphs.

In [4], [5] and [6] the following theorems were shown:[1]

THEOREM A. *If $\nu(\mathscr{H}')=\nu^*(\mathscr{H}')$ holds for every induced partial hypergraph $\mathscr{H}'$ of $\mathscr{H}$ then $\nu(\mathscr{H})=\tau(\mathscr{H})$.*

THEOREM B. *If $\tau(\mathscr{H}')=\tau^*(\mathscr{H}')$ holds for every partial hypergraph $\mathscr{H}'$ of $\mathscr{H}$ then $\nu(\mathscr{H})=\tau(\mathscr{H})$.*

THEOREM C. *If $\nu_2(\mathscr{H}')=2\nu(\mathscr{H}')$ holds true for each hypergraph $\mathscr{H}'$ arising from $\mathscr{H}$ by doubling and/or removing vertices then $\nu(\mathscr{H})=\tau(\mathscr{H})$.*

Also, the following sharpening of Theorem B is due to Berge.

THEOREM D. *If $\tau_2(\mathscr{H}')=2\tau(\mathscr{H}')$ holds true for each partial hypergraph $\mathscr{H}'$ of $\mathscr{H}$ then $\nu(\mathscr{H})=\tau(\mathscr{H})$.*

The aim of this paper is to discuss the following two theorems analogous to Theorems A—B, and give some applications:

THEOREM 1. *Suppose that $\nu_2(\mathscr{H}')=2\nu^*(\mathscr{H}')$ holds true for each hypergraph $\mathscr{H}'$ arising from $\mathscr{H}$ by doubling and/or removing certain vertices. Then $\nu_2(\mathscr{H})=\tau_2(\mathscr{H})$.*

THEOREM 2. *Suppose $\tau_2(\mathscr{H}')=2\tau^*(\mathscr{H}')$ holds true for each partial hypergraph of $\mathscr{H}$. Then $\nu_2(\mathscr{H})=\tau_2(\mathscr{H})$.*

[1] The first two of these are actually equivalent to certain results of FULKERSON on anti-blocking polyhedra; see [2].