Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 27 (3—4), (1976), 267—280.

RELATIONS BETWEEN SUMMABILITY OF FUNCTIONS AND THEIR FOURIER SERIES

By

L.-E. PERSSON (Umeå)

1. Introduction and announcement of the results

Let the function f have period 1, let f be integrable on $[0, 1]$ and let $c_n = c_n(f)$, $n \in \mathbf{Z}$, be its complex Fourier coefficients. The sequence $(c_n^*)_0^\infty$ is the sequence $(|c_n|)_{-\infty}^\infty$ rearranged so that $c_0^* \geqq c_1^* \geqq c_2^* \geqq \dots$ and $f^*(x)$ denotes the function equimeasurable with $|f(x)|$ and nonincreasing.

STEIN [11, p. 490], has proved the following theorem.

THEOREM I. *Let α, β and γ be real numbers such that $-1 < \alpha \leqq \min(\beta - 2, 0)$ and $\gamma \leqq \beta$, Then*

$$\left(\sum_{n=1}^{\infty} (c_n^*)^\beta n^\alpha\right)^{1/\beta} \leqq K(\alpha, \beta, \gamma)\left(\int_0^1 (f^*(x))^\gamma x^{(\beta-\alpha-1)\gamma/\beta - 1}\, dx\right)^{1/\gamma}.$$

The notation $K(\alpha, \beta, \gamma)$ stands for a constant depending at most on α, β and γ. The theorem of Stein contains some well-known results of HAUSDORFF—YOUNG (see [14, Vol II, p. 101]). HARDY—LITTLEWOOD (see [14, Vol II, p. 123]), and PITT (see [9, p. 747]). Furthermore, WIK and the present author, [8, p. 298], have stated.

THEOREM II. *If α is a real number satisfying $-1 < \alpha < -1/2$, and if there exists a nonnegative and continuous function h on $[0, \infty[$ such that $h(t)a^t$ is an increasing function of t for some $a \in \mathbf{R}_+$, $1/h(x)$ is integrable and*

$$\int_0^1 |f(x)|^{-1/\alpha}\left(h(\log^+|f(x)|)^{-1-1/\alpha}\right) dx < \infty,$$

then

$$\sum_{n \neq 0} |c_n(f)|\,|n|^\alpha < \infty.$$

Our first purpose is to state a theorem containing Theorems I and II as special cases but first we need a definition. We say that a nonnegative function λ on $[1, \infty]$ belongs to the class $Q(a_0, b_0)$ if, for some real numbers $a < a_0$ and $b > b_0$, $\lambda(t)t^a$ is an increasing and $\lambda(t)\, t^b$ is a decreasing function of t.

THEOREM 1. *Let $\beta > 0$ and let λ be a nonnegative function on $[0, \infty[$ which is constant on $[0, 1]$.*

(a) *If $\lambda \in Q(1, 1-\beta/2)$ then*

$$\sum_{n=0}^{\infty} (c_n^*)^\beta \lambda(n) \leqq K(\beta, \lambda) \int_0^1 (f^*(x))^\beta x^{\beta-2} \lambda\left(\frac{1}{x}\right) dx. \tag{1.1}$$

(b) *If* $\lambda\in Q(1-\beta/2,\ 1-\beta)$ *and if* c_n *are complex numbers such that* $c_n\to 0$ *as* $n\to\pm\infty$ *and* $\sum_{n=0}^{\infty}(c_n^*)^\beta\lambda(n)<\infty$, *then there exists a function* f *with* c_n *as Fourier coefficients and such that*

$$\int_0^1 (f^*(x))^\beta x^{\beta-2}\lambda\left(\frac{1}{x}\right)dx \leqq K(\beta,\lambda)\sum_{n=0}^{\infty}(c_n^*)^\beta\lambda(n). \tag{1.2}$$

We note that

(i) if $\gamma\leqq\beta$ then

$$\Big(\int_0^1 (f^*(x))^\beta x^{\beta-\alpha-2}\,dx\Big)^{1/\beta} \leqq K\left(\sum_{l=1}^{\infty}(f^*(2^{-l}))^\beta 2^{l(1+\alpha-\beta)}\right)^{1/\beta} \leqq$$

$$\leqq K\left(\sum_{l=1}^{\infty}(f^*(2^{-l}))^\gamma 2^{l(1+\alpha-\beta)\gamma/\beta}\right)^{1/\gamma} \leqq K\Big(\int_0^1 (f^*(x))^\gamma x^{(\beta-\alpha-1)\gamma/\beta-1}\,dx\Big)^{1/\gamma}$$

and, thus, according to the fact that Theorem 1(a) can be applied with $\lambda(t)=t^\alpha$, $-1<\alpha<\beta/2-1$, we find that Theorem I holds in a wider range of parameters than stated.

(ii) by applying Theorem 1(a) with $\beta=1$ and $\lambda(t)=t^\alpha$, $-1<\alpha-1/2$, and by using our Theorem 7 we see that Theorem 1 generalizes Theorem II.

(iii) Theorem 1(a) does not hold in general for example when $\lambda(t)=t^\alpha,\alpha>\beta/2-1$ or $\alpha=\beta/2-1$, $0<\beta<2$ (see [10, Ch IV:6] or [14, Vol I, p. 225]) and Theorem 1(b) fails when $\lambda(t)=t^\alpha$, $\alpha<\beta/2-1$ or $\alpha=\beta/2-1, \beta>2$ (see [4, p. 41] or [14, Vol I, p. 215]).

THEOREM 2. *Let* $\beta>0$ *and let* λ *be a nonnegative function on* $[0,\infty[$ *which is constant on* $[0,1]$ *and* $\lambda\in Q(1,1-\beta)$.

(a) *If* f *is a nonnegative function on* $[0,1]$ *such that*

$$f^*(x)\leqq Ax^{-1}\int_0^x f(t)\,dt,$$

then (1.2) *holds.*

(b) *If* f *is nonnegative, even and nonincreasing in* $[0,1/2]$ *then* (1.1) *is satisfied.*

(c) *If* (c_n), $n\in\mathbf{Z}$, *are nonnegative and if*

$$c_n^*\leqq A\big(n^{-1}\sum_{|k|\leqq n}c_k\big)$$

then (1.1) *holds.*

(d) *If* $a_1\geqq a_2\geqq\ldots\to 0$ *and if* $f(x)=\sum_{n=1}^{\infty}a_n\cos 2\pi nx$ *then* (1.2) *is satisfied.*

In particular Theorem 2 contains the fact that if $\beta>0$ and if $\lambda\in Q(1,1-\beta)$ then the conditions

$$\int_0^1 (f^*(x))^\beta x^{\beta-2}\lambda\left(\frac{1}{x}\right)dx<\infty \quad\text{and}\quad \sum_1^{\infty}(c_n^*)^\beta\lambda(n)<\infty \tag{1.3}$$