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FORMULAS RELATING THE AREA OF THE SURFACE OF A DOMAIN TO THE CURVATURE OF SURFACES FILLING THE DOMAIN

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ADLER [1] derived several formulas of the type described in the title for domains in Euclidean spaces of two or three dimensions. He also supplied a clever proof of a formula for planar domains which allows the family of curves to be fractured along a "fault". We think that his "unfractured" formulas are just integrated forms of the first variation formula for the area of a hypersurface. Such formulas can be given readily in any number of dimensions on any Riemannian manifold. It is not much more difficult to give a many-dimensional formula when "faults" are present. Formulas of this kind are most easily treated using the exterior differential calculus of E. Cartan. We follow the notation of FLANDERS [2].

Let M be an oriented smooth Riemannian manifold of dimension $n+1$ and suppose $W: M\to\mathbf{R}$ to be a smooth function. Set $\Omega=\{0\leqq W\leqq 1\}\subset M$ and suppose first of all that W has no singularities on $\bar{\Omega}$. Take local positively-oriented orthonormal frame fields $\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}$ such that $\mathbf{e}_{n+1}$ is perpendicular to the level surfaces $W=$constant. Let $\sigma_1, \ldots, \sigma_{n+1}$ be the dual solder one-forms with corresponding connection one-forms $\omega_{ij}(1\leqq i\leqq n+1,\ 1\leqq j\leqq n+1)$. Define g by $dW=g\sigma_{n+1}$. The non-singularity of W on $\bar{\Omega}$ is equivalent to the non-vanishing of g on $\bar{\Omega}$. Finally, let $*$ be the Hodge duality operator.

From $dW=g\sigma_{n+1}$, we get $*dW=g\sigma_1 \ldots \sigma_n$ (exterior product). Then

$$d*dW= dg(\sigma_1 \ldots \sigma_n)+g\sum_{i=1}^{n}(-1)^{i-1}\sigma_1 \ldots d\sigma_i \ldots \sigma_n.$$

Introduce the second fundamental form coefficients (a_{ik}) by the equation

$$\omega_{i,n+1}=\sum_{k=1}^{n}a_{ik}\sigma_k.$$

The first Cartan structure equation is

$$d\sigma_i=\sum_{j=1}^{n+1}\omega_{ij}\sigma_j.$$

We want that term in $d\sigma_i$ containing σ_i and σ_{n+1} since all others contribute 0 to the sum for $d*dW$. Because ω_{ij} does depend on all the σ's in general, we must take the term with $j=n+1$, and find the ith summand to be

$$(-1)^{i-1}\sigma_1 \ldots \sigma_{i-1}\left(\sum_{k=1}^{n}a_{ik}\sigma_k\right)\sigma_{n+1}\sigma_{i+1} \ldots \sigma_n=(-1)^{n-1}a_{ii}\sigma_1 \ldots \sigma_{n+1}.$$

Also, if $dg = \sum_{k=1}^{n+1} g_k \sigma_k$, then

$$(1) \qquad d*dW = \left[(-1)^n g_{n+1} + (-1)^{n-1} g \sum_{i=1}^{n} a_{ii}\right] \sigma_1 \ldots \sigma_{n+1}.$$

The element of area on a level surface is $\sigma_1 \ldots \sigma_n$, and

$$(2) \qquad d(\sigma_1 \ldots \sigma_n) = d\left(\frac{1}{g} g\sigma_1 \ldots \sigma_n\right) = (-1)^{n+1} \frac{g_{n+1}}{g} \sigma_1 \ldots \sigma_{n+1} + \frac{1}{g} d*dW.$$

Form the combination $\frac{1}{g}(1)+(2)$, cancel like terms, and get

$$d(\sigma_1 \ldots \sigma_n) = \left[(-1)^{n-1} \sum_{i=1}^{n} a_{ii}\right] \sigma_1 \ldots \sigma_{n+1}.$$

If we integrate this over Ω, the right-hand side gives

$$\int_{\Omega} (-1)^{n-1} \sum_{i=1}^{n} a_{ii}\, dV.$$

By Stokes' Theorem, the left-hand side integrates to

$$\int_{\partial\Omega} \sigma_1 \ldots \sigma_n = \text{area}_1 - \text{area}_0.$$

The quantity $(-1)^n \sum_{i=1}^{n}$ is to be identified with Adler's $K(P)$, and is the signed mean curvature. (If $n=2$, the minus sign makes up for the opposite orientations of e_2 and the Frenet normal.) Here we have Adler's formulas (1) and (2) proved with less computation and in greater generality.

It is interesting to note that the underlying differential (unintegrated) formula (2) has been used by pure and applied mathematicians for quite a long time. For example, it has been known to geodesists for about a century under the name of Bruns' Formula [5: *p.* 80].

To get the usual variational formula, take the interval [0, ε] instead of [0, 1]. Then

$$\text{area}_\varepsilon - \text{area}_0 = \int_{0 \leqq W \leqq \varepsilon} [(-1)^{n-1} \sum a_{ii}] \sigma_1 \ldots \sigma_{n+1}.$$

For small ε, $\{0 \leqq W \leqq \varepsilon\}$ is a product set, and Fubini's Theorem allows integration over the normal direction (the σ_{n+1} integral) first. Dividing by ε and letting $\varepsilon \to 0$, we get the usual formula for variation of surface area:

$$\frac{d}{d\varepsilon} \text{area}_\varepsilon|_{\varepsilon=0} = \int_{W=0} [(-1)^{n-1} \sum a_{ii}] \sigma_1 \ldots \sigma_n.$$

Now for the singularities. In the manifold M of dimension n, let C be a finite union of disjoint, embedded, compact, at most n-dimensional, smooth submanifolds